Part I

SOME PRE-CALCULUS TOPICS

In the first four sections we review some topics which you have probably seen before in high school math. These topics are not to be taught to you as part of this course. The student is expected to read the sections and do the exercises. Class time can be used for answering questions and clarifying concepts and definitions. Part I should be finished in four or five class sessions.

If you find that you are seriously struggling with these sections, you would probably do well to take Math 108 prior to taking a calculus course. Be warned that it is not possible to succeed in calculus without a good working knowledge of high school algebra.

1 Real Numbers, Calibrated Lines, Axes

This section is about numbers. It will help you recall things you learned before coming to college. You should read it carefully and do the exercises at the end of the section.

Real Numbers:

In this course we will only be dealing with Real Numbers. Examples of real numbers are: $1, 0, -3, \frac{4}{5}, \pi, .\overline{8}, -\sin 41^\circ, \sqrt{17}$. Essentially, a real number is any number that *can be* written as a decimal. Some of the numbers in the list of examples are not written as decimals, but they all *could* be. The symbol for the set of all real numbers is the fancy $\mathbb{R}$.

One non-real number that you have probably encountered in the past is the Imaginary Number i, defined to be equal to $\sqrt{-1}$. This imaginary number has no part in this course. For this course the square root of any negative number will be considered as "undefined."

Infinity (∞) is not a real number. We will be discussing infinity, but not as a number. "Infinity" is our symbol to represent the concept of unboundedness. We often say that the set of real numbers "goes to infinity." What we mean is that no matter how large a real number one chooses, there is always another, larger, real number. Similarly, we use $-\infty$ to indicate unboundedness in the negative direction. The set of real numbers is unbounded in both directions, so we could write the set $\mathbb{R}$ equivalently as $(-\infty, \infty)$. Notice that we use parentheses here. Using a bracket "[" or "]" would be inappropriate because that would suggest that $-\infty$ or ∞ is *included* in the set of real numbers. Again, $\pm\infty$, are not real numbers; they are symbols used to indicate unboundedness.

For the remainder of the text, when we write "number", we mean "real number".

Subsets of Real Numbers:

There are some special sets of numbers: integers, rational numbers and irrational numbers.

The numbers $0, \pm 1, \pm 2, \pm 3, \cdots$ are *integers*. These are the counting numbers, their negatives, and zero.

A number that can be written as the ratio of two integers (the denominator not zero) is called a *rational number*. Examples of rational numbers are: $\frac{7}{4}$, $.5 = \frac{1}{2}$, $.\overline{37} = \frac{37}{99}$. Any integer, n is a rational number because it can be written as $\frac{n}{1}$.

An *irrational number* is a number which is not rational. It cannot be written as a ratio of integers. Examples of irrational numbers: $\sqrt{2}$, π, $\sqrt[5]{41}$.

In the exercises at the end of the section you will be asked to change some decimal expressions into fractions of the form integer/integer. Thus, you will be showing that these numbers meet the definition of rational number. You will also be asked to begin with rational numbers and find their equivalent decimal expansions. After completing the exercises, without a calculator, you should have a good idea why the set of rational numbers consists precisely of those numbers whose decimal expressions eventually contain a pattern of digits that repeats indefinitely. You will have discovered the interesting fact: *The decimal representation of every rational number is a repeating decimal, and every repeating decimal represents some rational number.* The immediate consequence of this statement, is: *Those numbers whose decimal representations do not contain an indefinitely repeated pattern are the irrational numbers.* (Recall that the full decimal expansion of 3.151 is $3.15100000\cdots$ or $3.151\overline{0}$, so the "terminating" decimal 3.151 is considered to be a repeating decimal.)

When completing the exercises you will also discover another interesting fact: *Sometimes there are two different decimal representations for the same rational number.* This does not happen with all rational numbers, however, and it never happens with irrational numbers.

It is expected that you can change fractions to decimals and change terminating decimals to fractions without the use of a calculator. An example is provided below in case you are "rusty" at changing repeating decimals to fractions:

Example 1.1. Change $.\overline{57}$ into its rational form.

Suppose we call the desired number n. That is, $n = .\overline{57}$. Then:

$$100 \times n = 57.\overline{57}$$

$$100n - n = 57.\overline{57} - .\overline{57}$$

$$99n = 57$$

$$n = \frac{57}{99}$$

Numbers on a line:

Draw a horizontal line. Pick a point and label it 0. Pick a point to the right of 0 and label it 1. Now label the points twice as far to the right, three times as far to the right etc. by $2, 3, \cdots$. To the left of 0 label the mirror image points $-1, -2, -3, \cdots$. This associates a point on the line with every integer. Now proceed to associate a point on the line with every rational number in the obvious way: for example $2\frac{1}{3}$ is the label of the point between 2 and 3 which is half as far from 2 as from 3.

What about the irrational numbers? For that we need to think in terms of decimals. Recall that a decimal is *repeating* if the same finite sequence of digits is repeated infinitely often from some point on; e.g. $43.1246464646\ldots$ or $43.12\overline{46}$.

So, once you have used every rational number (repeating decimal) to label a point on the line, you can (sort of) see how to squeeze in all the irrationals, since an irrational is closely approximated by chopping off the decimal digits from some point onward and replacing them with $\overline{0}$: the further out you chop, the better the approximation.

The process of associating each number with a point on the line is called *calibrating* the line. This is all done much more precisely in higher math courses.

Putting 1 to the right of 0 rather than the left is a convention, but everyone follows this convention. If the line is vertical the convention is to put 1 above 0.

Section 1 - Exercises (answers follow)

1. Without using a calculator, write $\frac{2}{3}$, $\frac{6}{11}$ and $\frac{11}{6}$ as repeating decimals.

2. Again without a calculator, write $\frac{1}{11}$, $\frac{2}{11}$, $\frac{3}{11}$, and $\frac{4}{11}$ as repeating decimals. Look at the pattern of your answers. What do you expect $\frac{5}{11}$ to be? What about $\frac{6}{11}$, $\frac{9}{11}$, $\frac{10}{11}$?

3. Write $\frac{1}{9}$, $\frac{2}{9}$, $\frac{3}{9}$, and $\frac{4}{9}$ as repeating decimals. What do you expect $\frac{7}{9}$ to be? The pattern suggests an answer for $\frac{9}{9}$. Of course we know that $\frac{9}{9} = 1 = 1.0000\cdots$, so the number 1 has at least two decimal representations, one of which is eventually all 0's and the other of which is eventually all 9's. It is a fact that no fraction has more than two decimal representations and that the only way a fraction can have two decimal representations (rather than one) is if one of them is eventually all 0's and the other is eventually all 9's. It's a doable but challenging problem to figure out why this is true.

4. Using a computer or calculator, try to write $\frac{23}{17}$ as a repeating decimal. Unless you use a computer or a calculator that gives many decimal places you won't see the answer. But if you do it by hand using long division you'll get the answer in a short time. Have a race with a friend to see who gets it first.

5. On the basis of these division exercises can you figure out a general rule which will tell you, for a given fraction, the maximum number of decimal places that could be needed to get the repeater in the repeating decimal?

6. Using your answer to problem 4, write $\frac{23}{1700}$ as a repeating decimal.

7. Change the following decimals into fractions: .75, 45.024, $.\overline{85}$, $3.2\overline{85}$, $.3\overline{857}$.

8. Decide whether each statement below is TRUE or FALSE.

   (a) An irrational number is a real number.

   (b) ∞ is an irrational number.

   (c) Between any two rational numbers is another rational number.

   (d) Between any two rational numbers is an irrational number.

   (e) Between any two rational numbers is an integer.

   (f) Between any two irrational numbers is a rational number.

   (g) The number $34.\overline{9}$ sits right next to the number 35 on a number line.

   (h) $\frac{0}{0} = 1$

   (i) $\frac{0}{0} = 0$

   (j) $\frac{1}{0} = \infty$

   (k) $\infty - \infty = 0$

   (l) All unbounded sets are open intervals.

   (m) All closed intervals are bounded sets.

(n) All irrational numbers have exactly one decimal representation.

(o) Integers are the only numbers that have two decimal representations.

(p) $\frac{\sqrt{5}}{3}$ is a rational number.

(q) If a number is not irrational, then it must be a rational number.

Section 1 - Answers

1. $.\overline{6}$, $.\overline{54}$, $1.8\overline{3}$

2. $.\overline{09}$, $.\overline{18}$, $.\overline{27}$, $.\overline{36}$, $.\overline{45}$, $.\overline{54}$, $.\overline{81}$, $.\overline{90}$

3. $.\overline{1}$, $.\overline{2}$, $.\overline{3}$, $.\overline{4}$, $.\overline{7}$, $.\overline{9}$

4. $1.\overline{3529411764705882}$

5. Hint: When you do the long division, how many *possible* remainders are there for each subtraction step?

6. $0.01\overline{3529411764705882}$

7. $\frac{3}{4}$, $\frac{45024}{1000} = \frac{5628}{125}$, $\frac{85}{99}$, $\frac{3253}{990}$, $\frac{3854}{9990} = \frac{1927}{4995}$

8. (a) True

   (b) False. Irrational numbers are real. ∞ is not a real number.

   (c) True

   (d) True

   (e) False. Example: There is no integer between $\frac{1}{3}$ and $\frac{1}{2}$.

   (f) True

   (g) False. These two numbers are equal. They are the same point on the line.

   (h) False. Division by zero is never allowed.

   (i) False. Division by zero is never, never, allowed.

   (j) False. Division by zero is never, never, never, allowed.

   (k) False. You cannot use the arithmetic operations for real numbers on numbers that are not real.

   (l) False. Example: The set of integers is an unbounded set, but isn't an interval at all.

   (m) True

   (n) True

   (o) False. Any terminating decimal can be written in two ways. For example: $2.75 = 2.74\overline{9}$

   (p) False. A rational number is a ratio of *integers*. $\sqrt{5}$ is not an integer.

   (q) True

2 Functions and Algebra

Truly understanding functions and their notation is essential for success in this course. In this section we discuss functions and then, at the end of the section, review a few algebra topics that have historically been pitfalls for calculus students.

Here are some equations in the two variables x and y:

- $y = \sqrt{x^2 + 7} - 4$
- $y = -6$
- $x^2 + y^4 = y + 3$
- $x^2 + y^2 = 9$
- $x = 5^x$

The symbol "=" is a verb meaning "is the same as" or "equals." A statement of the form "something = something else" is called an *equation.* In the first equation, above, y is on the left of the = symbol, and only x's and constants are on the right. In the second equation the same is true (even though there are no x's on the right). But the other three equations are not of this form. To put this in different words, the first two equations look like $y =$ formula in x, or more briefly $y = f(x)$, while the last three are not of that form. Whenever an equation has the form $y = f(x)$ we say that *y is a function of x.* It's convenient that in English the words "function" and "formula" start with the same letter. Usually people read the line $y = f(x)$ as saying that y is a function of x.

There is nothing sacred about the letters x and y. If we see $t = 9u - \frac{1}{3}u^3 + 6\pi$ we say that t is a function of u. Or, in the case $x = 7y - 4$, we see that x is given as a function of y.

This is how the functions used in this course will usually be given[2].

Often, we want to give a name to a particular function (or formula). We might give the name $h(x)$ to the formula $\sqrt{x^2 - 7} - 4$, and write $h(x) = \sqrt{x^2 - 7} - 4$. Or we might use $U(x)$ for $x^2 + x^4$, writing $U(x) = x^2 + x^4$. Then if we want to "plug in" a particular number in place of the variable x, say 5, we get $h(5) = \sqrt{5^2 - 7} - 4$ or $U(5) = 5^2 + 5^4$. These can be simplified to $h(5) = 3\sqrt{2} - 4$ and $U(5) = 650$, but that's not the point we are trying to bring out here. Rather, it is function notation that you need to get straight, because it will be used all through the course.

More abstractly, if $f(x)$ is a function and a specific value, say a, is used for x, then $f(a)$ is the called the *value* of f at a.

Domain:

When studying a function $f(x)$ in a particular problem, one often needs to be clear on which numbers x are to be permitted as "plug-ins" in the formula $f(x)$. This set of numbers x is called the *domain* of the function $f(x)$ and is denoted by D_f. The domain of $f(x)$ will always be found in one of three ways:

[2]The abstract definition of a function isn't terribly helpful for what we will be studying, and we omit it.

Error / Complement needed

1. D_f may be the *natural domain* of the formula $f(x)$; this means: D_f is the set of all numbers x for which the formula $f(x)$ makes sense. For example the natural domain of $h(x)$, above, is the closed interval $[-\sqrt{7}, \sqrt{7}]$, because if x is not in that interval the formula would involve the square root of a negative number. The natural domain of the function $U(x)$ is the set of all numbers, since the formula makes sense for any x.

2. It may be that the natural domain makes sense mathematically, but in the physical or real-life problem under consideration some of those allowable values of x are silly. For example, you know that the formula for the area of a circle of radius r is $A(r) = \pi r^2$. The natural domain for this function $A(r)$ is the set $\mathbb{R}$ of all numbers, since any number can be squared and then multiplied by π. But who ever heard of a circle of negative radius? So if the problem is about areas of circles it would be understood, even if not explicitly stated, that the domain D_A is the set of non-negative numbers (numbers ≥ 0) rather than the set of all numbers. More generally, if common sense tells you that the natural domain is too big to be useful in your problem, go with common sense in identifying the domain for your problem.

3. Or the poser of a problem may specify the domain explicitly for you. For example, *find the maximum value of the function* $f(x) = x^2$ *if the domain is the closed interval* $[3, 4]$. In this problem the domain D_f is $[3, 4]$. (By the way, what is the answer?)

In summary: it is always (*yes, always*) a good idea to write down the domain of the function you are considering before you tackle a problem. The domain D_f is the appropriate set of numbers x to be considered in connection with the function $f(x)$ in your problem. Remember: D_f cannot be bigger than the natural domain, but it may be smaller.

Example 2.1. Find the domain for each of the following functions.

1. $f(x) = \frac{3+x}{x-2}$
   Answer: $D_f = (-\infty, 2) \cup (2, \infty)$ because the x cannot take on any value that would make the denominator equal to zero.

2. $g(x) = \sqrt{(4-x)}$
   Answer: $D_g = (-\infty, 4]$ because the square root of a negative number does not exist (remember that we are only working in $\mathbb{R}$). We can include $x = 4$ because $\sqrt{4-4}$ does exist. $\sqrt{0} = 0$.

3. $C(p) = 1.25p$ where C represents the cost to buy p slices of pizza.
   Answer: D_C is the set of non-negative integers. We assume here that in real life the vendor doesn't sell fractions of slices.

4. $K(s) = 125s$ where K represents the number of calories in an 8 oz. can of soda.
   Answer: $D_K = [0, \infty)$. Here, fractional values of s do make sense.

5. $h(x) = x^2 + 3x - 7$ where $-1 < x \leq 8$
   Answer: $D_h = (-1, 8]$. Here the domain was explictly given.

Graphs of Equations and Functions:

We are usually interested in the graphs of functions, but let's begin with the more general idea of the graph of an equation which has two variables x and y. The graph of an equation is a picture in the xy-plane which shows information about the equation in visual form. As an example, let's think about the graph of the equation $x^2 + y^4 = y + 3$. The point $(\sqrt{3}, 1)$ is on the graph because when you plug in those values for x and y you get a true statement: both sides come out to 4. The point $(2, \frac{1}{2})$ is not on the graph because $4 + \frac{1}{16} \neq \frac{1}{2} + 3$. In general the *graph* of an equation is the set of points in the plane such that when the first coordinate of the point is plugged in for x and the second coordinate of the point is plugged in for y you get a true statement. (Remember $=$ is a verb and $a = b$ means that a and b are the same.)

Now consider a particular function, say, $y = 8x^3 - 6$. Here, the domain is all numbers. The graph of this function is the set of all points $(x, 8x^3 - 6)$. For example, $(0, -6)$ is on the graph. Can you find 10 more points on this graph? Can you draw the graph of the function $y = x^2$? Surely you have seen this parabola before.

Piecewise Defined Functions:

One type of function that occurs frequently in real life, but which you may not have studied previously, is the piecewise defined function. This is a function whose domain can be thought of as broken into pieces. Each piece of the domain has its own "rule" for finding the function values (y values). Some examples of piecewise defined functions are below.

Example 2.2.

1. A car rental company charges \$270 per week to rent a compact car. The first 300 miles driven are "free." If more than 300 miles are driven, the company charges an additional 55 cents per mile. A function that describes the cost, C, in dollars, to rent a car driven m miles in one week is given by:

$$C(m) = \begin{cases} 270 & 0 \leq m \leq 300 \\ 270 + .55(m - 300) & 300 < m \end{cases}$$

2. In 2011, the federal income tax owed by a single person with a taxable income of \$100,000 or more is figured by the following function. T represents the tax due. I represents the taxable income.[3]

$$T(I) = \begin{cases} .28I - 6,383 & 100,000 \leq I \leq 174,400 \\ .33I - 15,103 & 174,400 < I \leq 379,150 \\ .35I - 22,686 & I > 379,150 \end{cases}$$

Tax \$21,917 → 42,449

Tax \$42,449 →

3.

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ -1 & x \text{ is irrational} \end{cases}$$

[3]From www.irs.gov

4.

$$g(x) = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

In the first example, the domain of the function is $[0, \infty)$. The domain is split into two pieces: $[0, 300]$ and $(300, \infty)$ For each piece, the function value (y value, or $C(m)$ value) is calculated by a different rule. For values of m in $[0, 300]$ the rule simply assigns the function value 270 (dollars). For m in $(300, \infty)$, the rule calculates the function value using the mathematical formula $270 + .55(m - 300)$.

Does this make sense? What should it cost if the renter only drives 70 miles? $C(70) = 270$ dollars. What should it cost if the renter drives 400 miles? $C(400) = 270 + .55(400 - 300) = 270 + 55 = 325$ dollars. What is $C(450)$? $C(45)$? Answers: \$352.50, \$270.

In the second example, how much tax is owed if the taxable income is \$200,000? $T(200,000) = 200,000 \times .33 - 15,103 = 50,897$ What is T(1,000,000)? What does it mean? Answer: \$327,314 is the tax owed for a taxable income of \$1,000,000

In the third example, what is $f(4)$? Since 4 is a rational number, $f(4) = 1$. What is $f(\pi)$? Since π is irrational, $f(\pi) = -1$. What is $f(\frac{1}{2})$? $f(-10)$? Answers: 1, 1

Try some values in the last function. You should recognize $g(x)$ as one way to express the absolute value function.

The domain for a piecewise defined function is very easy to determine because it is given directly. You only need to look in the right column of the function and see the possible values for the independent variable. $D_C = [0, \infty)$, $D_T = [100,000, \infty)$, and $D_f = \mathbb{R}$.

It is emphasized here that a piecewise defined function *is* a function. For each independent variable there is only one function value. The function may use multiple ways to find values, but only one way is appropriate for any given domain element.

Function Operations: Functions can be added, subtracted, multiplied, divided (being careful not to divide by zero) and composed. We review those operations and notations with the following example:

Example 2.3. Suppose $f(x) = \frac{x}{x^2+3}$ and $g(x) = \sqrt{x}$. Then:

$$(f+g)(x) = f(x) + g(x) = \tfrac{x}{x^2+3} + \sqrt{x}$$
$$(f-g)(x) = f(x) - g(x) = \tfrac{x}{x^2+3} - \sqrt{x}$$
$$(f \cdot g)(x) = f(x) \cdot g(x) = \tfrac{x}{x^2+3} \cdot \sqrt{x} = \tfrac{x\sqrt{x}}{x^2+3}$$
$$\left(\tfrac{f}{g}\right)(x) = \tfrac{f(x)}{g(x)} = \frac{\frac{x}{x^2+3}}{\sqrt{x}} = \frac{x}{(x^2+3)\sqrt{x}}$$
$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = \tfrac{\sqrt{x}}{(\sqrt{x})^2+3}$$

Certainly the subtraction and division operations are not, in general, commutative. $(f-g)(x) \neq (g-f)(x)$ and $\left(\frac{f}{g}\right)(x) \neq \left(\frac{g}{f}\right)(x)$. The composition of functions is not commutative

either. $(f \circ g)(x) \neq (g \circ f)(x)$. Using the functions from Example 2.3 we get:
$(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x^2+3}\right) = \sqrt{\frac{1}{x^2+3}}$. This clearly not the same as $(f \circ g)(x)$.

When algebraically combining functions, you must be careful about the domain of the newly created function. When you look at $\left(\frac{f}{g}\right)(x)$ and $(f \circ g)(x)$ in Example 2.3 you might be tempted to automatically simplify the expressions on the far right. You may do so, but you must do so correctly.

The expression $\dfrac{x}{(x^2+3)\sqrt{x}}$ is not defined at $x = 0$. So the domain for $\left(\frac{f}{g}\right)(x)$ cannot include zero. The expression $\dfrac{\sqrt{x}}{x^2+3}$ however, IS defined at $x = 0$. So, to write $\left(\frac{f}{g}\right)(x) = \dfrac{\sqrt{x}}{x^2+3}$ is not correct. You must write "$\left(\frac{f}{g}\right)(x) = \dfrac{\sqrt{x}}{x^2+3}$, if $x \neq 0$."

A similar domain issue occurs when dealing with simplifying $(f \circ g)(x)$. $(\sqrt{x})^2 = x$ *only* for $x \geq 0$. If x is negative, $(\sqrt{x})^2$ does not exist. To simplify $(f \circ g)(x)$, then you must be sure that the domain is clear. "$(f \circ g)(x) = \frac{1}{x+3}$ if $x \geq 0$."

ERROR

$(f \circ g)(x) = \dfrac{\sqrt{x}}{(\sqrt{x})^2 + 3} = \dfrac{\sqrt{x}}{x+3}$

We are now ready for the algebra review.

Exponents: It is assumed that you are familiar with basic exponent rules and are proficient in using them, at least for positive integer exponents. This section is to remind you how to interpret other exponents and to reinforce the idea that the rules for other exponents are essentially the same as those for positive integer exponents.

1. $a^0 = 1$ for all numbers a, EXCEPT $a = 0$. 0^0 is not defined.
2. $a^{-n} = \dfrac{1}{a^n}$, EXCEPT when $a = 0$ because a denominator can never be zero. Two immediate consequences of this rule are:
   (a) $a^{-1} = \dfrac{1}{a}$
   (b) $\dfrac{1}{a^{-n}} = a^n$
   (c) $\dfrac{a^{-2}}{b^{-3}} = \dfrac{b^3}{a^2}$
   (d) $\dfrac{-a^{-2}}{b^2} = \dfrac{-1}{a^2b^2}$
3. $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$. Of course a cannot be negative if n is even. When $m = 1$, this rule simplifies to $a^{\frac{1}{n}} = \sqrt[n]{a}$.
4. $(a^m)^n = a^{mn}$

Example 2.4. Here are some examples for using these rules.

1. $\left(\frac{2}{5}\right)^{-1} = \frac{1}{2/5} = \frac{5}{2}$
2. $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$
3. $\sqrt[4]{\sqrt[3]{x}} = \left(x^{\frac{1}{3}}\right)^{\frac{1}{4}} = x^{\frac{1}{12}} = \sqrt[12]{x}$
4. $\dfrac{x^2y^{-5}z}{x^{-4}yz^3} = x^6y^{-6}z^{-2} = \dfrac{x^6}{y^6z^2}$

Be Careful with Parentheses:

1. $ab^n = a \cdot b^n$

2. $(ab)^n = a^n \cdot b^n$

3. $(a+b)^n \neq a^n + b^n$

You might think that these are obvious, but don't be insulted that they are here. Many an error has been made when an expression like -5^2 is equated to 25. While it is true that $-5 \cdot -5 = 25$, it is not true that -5^2 means $-5 \cdot -5$. To correctly write $-5 \cdot -5$, one would need parentheses: $(-5)^2$. The correct evaluation of -5^2 is $-5 \cdot 5 = -25$.

There are even more frequent abuses of the third rule. When $n = 2$, there isn't much problem. You would never think to write $(a+b)^2 = a^2+b^2$ because you know to "FOIL" the $(a+b)^2$. However, when n is a value other than 2 there is a sorry eagerness to "distribute" the power through the parentheses. Sometimes the n is disguised as a root so the crime is not so obvious.

Here are some typical errors involving parentheses and exponents.

ERROR: $\sqrt{a^2+b^2} = a+b$ No! This is saying $(a+b)^{\frac{1}{2}} = a^{\frac{1}{2}} + b^{\frac{1}{2}}$

ERROR: $\left(\frac{1}{a}+\frac{1}{b}\right)^{-1} = a+b$ No! This is saying $(a+b)^{-1} = a^{-1}+b^{-1}$

ERROR: $\sqrt[3]{x^3+8} = x+2$ No! What is this saying?

A Reminder about Even Roots:

$\sqrt{a}$ is a positive number. It is the positive number whose square is a. For example, $\sqrt{16} = 4$. It is incorrect to say: $\sqrt{16} = \pm 4$.

The Absolute Value Function:

Since the result of a square root problem must be positive, we must be mindful when evaluating $\sqrt{a^2}$. Suppose $a = 3$. Then $\sqrt{3^2} = \sqrt{9} = 3$. So $\sqrt{a^2} = a$. But suppose $a = -3$. Then we get $\sqrt{(-3)^2} = \sqrt{9} = 3$. We do NOT have $\sqrt{a^2} = a$. We have $\sqrt{a^2} = |a|$.

We will be using the function $f(x) = |x|$ several times in this course. From what we have seen here in Section 2 we can justify the following equivalent ways of defining the function. One definition uses square roots, the other a piecewise expression.

Definition 2.1. *The* <u>*absolute value*</u> *of x is defined to be:*

$$f(x) = |x| = \sqrt{x^2} = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

Finding Roots (Zeroes) of Functions:

In this course we will often have to find the roots of functions. Here "roots" (sometimes called the "zeroes" of a function) refers to the values of x which make $f(x)$ equal zero. Finding roots can be made simpler by using the facts :

$$\text{If } a \cdot b = 0, \text{ then } a = 0 \text{ or } b = 0$$

$$\text{If } \frac{a}{b} = 0, \text{ then } a = 0. \qquad \text{Reminder: } b \neq 0$$

So, if we can take an expression and factor it (or in the case of fractions factor the numerator), then we can set each factor equal to zero and solve. Here are some examples:

Example 2.5. Find the roots of the following functions:

1. $f(x) = x^2 + 3x + 2$
   Solution: $f(x) = (x+1)(x+2)$, so $x + 1 = 0$ or $x + 2 = 0$. $x = -1$ or $x = -2$.
2. $g(x) = \dfrac{2x-5}{x^2-7}$
   Solution: $2x - 5 = 0$. $x = \frac{5}{2}$
3. $h(x) = \dfrac{x^2-9}{x+3}$
   Solution: $x^2 - 9 = (x+3)(x-3) = 0$, so $x + 3 = 0$ or $x - 3 = 0$. $x = -3$ or $x = 3$. However, the domain of this function does not include $x = -3$, so $x = 3$ is the only root.
4. $F(x) = (x+2)^{\frac{1}{2}} + \frac{1}{2}(x+2)^{-\frac{1}{2}}$
   Solution: $F(x) = (x+2)^{-\frac{1}{2}}[(x+2)^1 + \frac{1}{2}] = (x+2)^{-\frac{1}{2}}(x + \frac{5}{2}) = 0$. So, $x + \frac{5}{2} = 0$. $x = -\frac{5}{2}$
   Notice that we do not have $x = -2$ as a solution. It is not in the domain of the function.

Example 2.6. Solve the following equations for x.

1. $x^3 = x$
   Solution:
   $$\begin{aligned} x^3 &= x \\ x^3 - x &= 0 \\ x(x^2-1) &= 0 \\ x(x+1)(x-1) &= 0 \end{aligned}$$
   So, $x = 0$, $x = -1$, or $x = 1$ Notice that we do not begin by "canceling" an x from both sides. If we did that, we would have $x^2 = 1$ and not find the root $x = 0$.
2. $x^2 - 4x - 5 = 7$
   $$\begin{aligned} x^2 - 4x - 5 &= 7 \\ x^2 - 4x - 12 &= 0 \\ (x+2)(x-6) &= 0 \end{aligned}$$
   So, $x = -2$ or $x = 6$. Notice that we did not factor the left side immediately. If we did that, we would have $(x+1)(x-5) = 7$. This is not useful because we cannot conclude: "$x + 1 = 7$ or $x - 5 = 7$." The fact "If $a \cdot b = 0$ then $a = 0$ or $b = 0$" only works for zero.

Section 2 - Exercises (answers follow)

1. Specify the domain of the given function.

   (a) $f(x) = x^3 - 3x^2 + 2x + 5$

   (b) $y = \dfrac{2x-4}{2x+5}$

   (c) $f(x) = \dfrac{4x+2}{x^2}$

   (d) $g(t) = \sqrt{t^2+4}$

   (e) $f(x) = -\sqrt{\dfrac{5}{x^2+6}}$

   (f) $f(x) = \sqrt{\dfrac{-5}{x^2+6}}$

   (g) $f(x) = -\sqrt{\dfrac{5}{x+6}}$

2. For the following functions find the domain and all roots.

   (a) $f(x) = \sqrt{2x-7}$

   (b) $f(x) = \sqrt{5-x}$

   (c) $f(x) = \frac{x^2+x-2}{x^2+7x+10}$

   (d) $f(x) = \frac{x^2+2}{2x+1}$

   (e) $f(x) = \frac{x^2+3x}{x}$

   (f) $f(x) = \sqrt[3]{\frac{x-2}{x+6}}$

   (g) $f(x) = \sqrt{\frac{x}{x+1}}$

   (h) $f(x) = \sqrt{\frac{x-2}{x+6}}$

   (i) $f(x) = \frac{x^2+2x-15}{x-3}$

   (j) $f(x) = \sqrt{16-x^2} - \frac{12}{\sqrt{16-x^2}}$

3. For piecewise defined function f, find: domain, $f(0)$, $f(1)$, $f(5)$

   $$f(x) = \begin{cases} 2x+2, & x < 1, \\ 4x, & 1 < x < 3, \\ \frac{3+x}{3-x}, & x > 3. \end{cases}$$

   Challenge: Does this function have any roots? If so, what are they?

4. Given $f(x) = \frac{3}{x+1}$ and $g(x) = \frac{x+2}{x-1}$.
   Find the following functions and their domains: $(f+g)(x)$, $(fg)(x)$, $\left(\frac{f}{g}\right)(x)$.

5. Given $f(x) = 2 - 3x^2$ and $g(x) = x - 1$.
   Find: $(f \circ g)(x)$, $(g \circ f)(x)$, $(g \circ g)(x)$ and $(f \circ f)(2)$.

6. Given $f(x) = 1 - x$ and $g(x) = \begin{cases} 2x & x < 0 \\ x^2 & x \geq 0 \end{cases}$

   Find: $(f \circ g)(4)$, $(f \circ g)(-4)$, $(g \circ f)(4)$ and $(g \circ f)(-4)$.

7. For each function F, find two functions f and g such that $F = (f \circ g)$.
   Do not use the trivial $f(x) = x$ or $g(x) = x$.

   (a) $F(x) = \frac{3}{x+5}$

   (b) $F(x) = \sqrt{x^2 + x - 2}$

8. $32^{\frac{4}{5}}$ can be written as $\sqrt[5]{32^4}$ or as $(\sqrt[5]{32})^4$.
   Which expression is easier to evaluate? Evaluate $32^{\frac{4}{5}}$ without a calculator.

9. Without a calculator, evaluate the following:
   (a) 17^0 (b) $8^{-\frac{1}{3}}$ (c) $4^{\frac{3}{2}}$ (d) $100^{\frac{1}{2}} - 64^{\frac{1}{2}}$ (e) $(100-64)^{\frac{1}{2}}$ (f) -3^2
   (g) $\sqrt{25}$ (h) $\sqrt{-9}$

10. Change the following to exponential form (eliminate the radical sign). Simplify.
    (a) $\sqrt[3]{x^5}$ (b) $(\sqrt[5]{2x})^3$ (c) $\left(\sqrt{\dfrac{x}{y^3}}\right)^5$ (d) $\dfrac{x}{\sqrt[5]{x^3}}$ (e) $\sqrt[6]{\sqrt[3]{x^4}}$

11. Change the following to radical form:
    (a) $x^{\frac{1}{3}}$ (b) $-x^{\frac{1}{2}}$ (c) $(-x)^{\frac{1}{2}}$ (d) $x^{\frac{9}{5}}$ (e) $-3x^{\frac{2}{3}}$ (f) $2(xy)^{-\frac{3}{4}}$

12. For which values of x is each of the following defined?
    (a) $\sqrt{x}$ (b) $\sqrt{-x}$ (c) $\sqrt{x^2}$ (d) $\frac{1}{\sqrt{x}}$ (e) $\sqrt{x-6}$ (f) $\sqrt{6-x}$ (g) $\sqrt[3]{x}$

13. Which expressions, if any, are equivalent to $\sqrt{(-x)^5}$?
    (a) $x^{-\frac{5}{2}}$ (b) $(-x)^{\frac{5}{2}}$ (c) $-x^{\frac{2}{5}}$ (d) $(-x)^{\frac{2}{5}}$ (e) $-\sqrt{x^5}$ (f) $\sqrt{-x^5}$ (g) $(\sqrt{-x})^5$

14. Rewrite into exponential form with only positive exponents. Simplify.
    (a) $\left(x^{\frac{1}{2}}\right)^{-\frac{1}{3}}$ (b) $\left(\dfrac{3x}{y}\right)^{-2}$ (c) $x^{\frac{1}{2}}x^{-\frac{2}{3}}$ (d) $\sqrt{x^{-7}}$ (e) $\left(\dfrac{a^{-2}}{b^{-2}} + \dfrac{b^{-2}}{a^{-1}}\right)^{-1}$

    (f) $\left(\dfrac{x^{m^2}}{x^{2m-1}}\right)^{\frac{1}{m-1}}$ where m is a constant and $m > 1$

15. Rewrite into radical form. Simplify as much as possible.
    (a) $\left(\dfrac{x^6 y}{z^3}\right)^{\frac{1}{2}}$ (b) $\left(\dfrac{x^2 + y^2}{x^4}\right)^{\frac{1}{2}}$

Section 2 - Answers

1. (a) $\mathbb{R}$

   (b) $(-\infty, -\frac{5}{2}) \cup (-\frac{5}{2}, \infty)$

   (c) $(-\infty, 0) \cup (0, \infty)$

   (d) $\mathbb{R}$

(e) $\mathbb{R}$

(f) $\emptyset$ (No real numbers are valid in this expression)

(g) $(-6, \infty)$

2. (a) Domain: $[\frac{7}{2}, \infty)$ Roots: $\frac{7}{2}$
   (b) Domain:$(-\infty, 5]$ Roots: 5
   (c) Domain:$(-\infty, -5) \cup (-5, -2) \cup (-2, \infty)$ Roots: 1
   (d) Domain:$(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$ Roots: none
   (e) Domain:$(-\infty, 0) \cup (0, \infty)$ Roots: -3
   (f) Domain:$(-\infty, -6) \cup (-6, \infty)$ Roots: 2
   (g) Domain:$(-\infty, -1) \cup [0, \infty)$ Roots: 0
   (h) Domain:$(-\infty, -6) \cup [2, \infty)$ Roots: 2
   (i) Domain:$(-\infty, 3) \cup (3, \infty)$ Roots: -5
   (j) Domain:$(-4, 4)$ Roots: $2, -2$

3. Domain: $(-\infty, 1) \cup (1, 3) \cup (3, \infty)$ $f(0) = 2$, $f(1)$ does not exist, $f(5) = -4$ Roots: -1

4. $(f+g)(x) = \frac{x^2+6x-1}{x^2-1}$, $D_{f+g} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
   $(fg)(x) = \frac{3x+6}{x^2-1}$, $D_{fg} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
   $\left(\frac{f}{g}\right)(x) = \frac{3x-3}{x^2+3x+2}$, $D_{f/g} = (-\infty, -2) \cup (-2, -1) \cup (-1, 1) \cup (1, \infty)$

5. $(f \circ g)(x) = -3x^2 + 6x - 1$ $(g \circ f)(x) = -3x^2 + 1$ $(g \circ g)(x) = x - 2$ $(f \circ f)(2) = -298$

6. $(f \circ g)(4) = -15$ $(f \circ g)(-4) = 9$ $(g \circ f)(4) = -6$ $(g \circ f)(-4) = 25$

7. Answers are not unique. Possible answers are:
   (a) $f(x) = \frac{3}{x}$; $g(x) = x + 5$ (b) $f(x) = \sqrt{x}$; $g(x) = x^2 + x - 2$

8. 16

9. (a) 1 (b) $\frac{1}{2}$ (c) 8 (d) 2 (e) 6 (f) -9 (g) 5 only (h) Does not exist.

10. (a) $x^{\frac{5}{3}}$ (b) $(2x)^{\frac{3}{5}}$ (c) $x^{\frac{5}{2}}y^{-\frac{15}{2}}$ (d) $x^{\frac{2}{5}}$ (e) $x^{\frac{2}{9}}$

11. (a) $\sqrt[3]{x}$ (b) $-\sqrt{x}$ (c) $\sqrt{-x}$ (d) $\sqrt[5]{x^9}$ or $(\sqrt[5]{x})^9$ (e) $-3\sqrt[3]{x^2}$ or $-3(\sqrt[3]{x})^2$
    (f) $\dfrac{2}{\sqrt[4]{(xy)^3}}$ or $\dfrac{2}{(\sqrt[4]{xy})^3}$

12. (a) $x \geq 0$ (b) $x \leq 0$ (c) $\mathbb{R}$ (d) $x > 0$ (e) $x \geq 6$ (f) $x \leq 6$ (g) $\mathbb{R}$

13. b, f, g

14. (a) $\dfrac{1}{x^{\frac{1}{6}}}$ (b) $\dfrac{y^2}{9x^2}$ (c) $\dfrac{1}{x^{\frac{1}{6}}}$ (d) $\dfrac{1}{x^{\frac{7}{2}}}$ (e) $\dfrac{a^2b^2}{a^3+b^4}$ (f) x^{m-1}

15. (a) $\dfrac{|x|^3\sqrt{y}}{z\sqrt{z}}$ Note: x can be negative; z cannot be negative. (b) $\dfrac{\sqrt{x^2+y^2}}{x^2}$

3 Polynomials and Rational Functions

In this section we review the definitions of two important types of functions: polynomial functions and rational functions. Then we look at some specific examples of these fuctions, focusing primarily on linear functions.

Definition 3.1. *A polynomial is an expression of the form:*

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a non-negative integer and the a_i are real number constants with $a_n \neq 0$.

The degree of the polynomial is the value of n. Thus a polynomial of degree 0 can be thought of as a non-zero real number. The requirement "non-zero" here can be inconvenient, so we hereby declare that the real number 0 is a polynomial which does not have a degree.

Example 3.1. Some examples of polynomials, and their degrees are:

1. $x^5 + 3x^4 - x + 12$ degree: 5
2. $\frac{1}{3}x^9 + \sqrt{5}x^2 - \pi$ degree: 9
3. $2x$ degree: 1
4. 7 degree: 0
5. $x^2 - x^3 + x$ degree: 3
6. $(x^2+4)(x^4+x^3-1)$ degree: 6
7. $\dfrac{3x^4+6x}{-2}$ degree: 4

Some examples of expressions that are *not* polynomials are:

1. $3\sqrt{x}+7$
2. $\dfrac{x^3+5x^2}{x-7}$
3. $3^x + 3^{x-1} + 9$
4. $\sin(x)$

A *polynomial function* is simply a function whose "rule" for calculating the y value is a polynomial. The distinction between a polynomial and a polynomial function is not of consequence here and we will use the terms interchangeably. The domain of any polynomial is $\mathbb{R}$.

You have probably learned that a polynomial of degree n has at most n roots. When we study graphing in Section 19 the reason for this should become clear. real

Definition 3.2. *A rational expression is a ratio of polynomials where the degree of the denominator is at least one.*

Example 3.2. Some examples of rational expressions are:

$$\frac{x^3+x-4}{x^2-x} \qquad 6(x+3)^{-2}$$

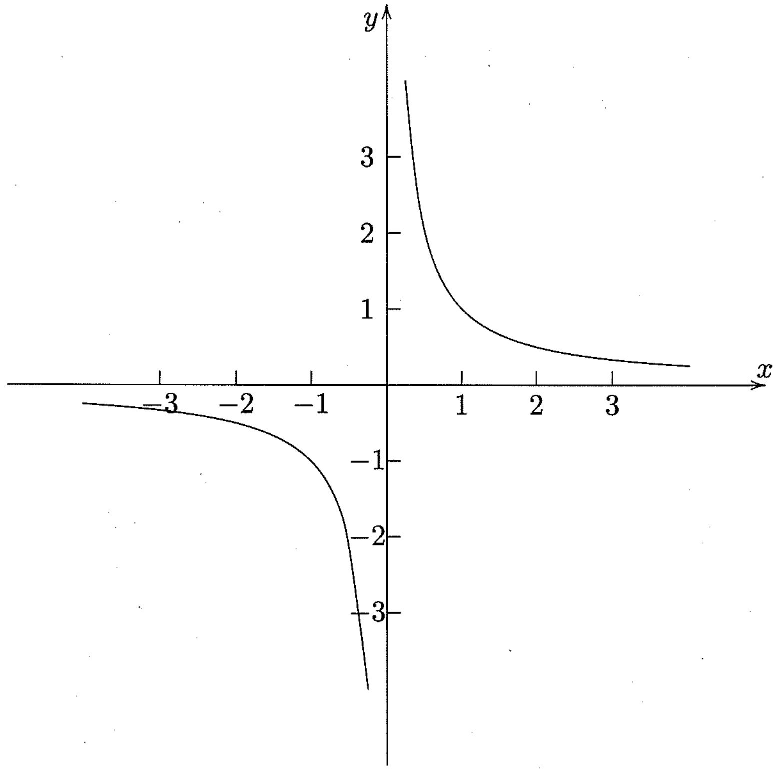
As above, we will use the term "rational *function*" to mean a function that is the ratio of two polynomials: $f(x) = \dfrac{P(x)}{Q(x)}$, where P and Q are polynomials.

The domain of f is all values of x for which $Q(x) \neq 0$. Since the denominator is a polynomial, and the number of roots of a polynomial is at most the degree of the polynomial, the domain of a rational function is $\mathbb{R}$ with finitely many points removed.

The Rational Function $f(x) = \dfrac{1}{x}$

In this course we will frequently use the rational function $f(x) = \frac{1}{x}$ as an example. It will be helpful to you to be familiar with its properties and graph. It should be clear that the domain of the function is $(-\infty, 0) \cup (0, \infty)$. The range of the function is also $(-\infty, 0) \cup (0, \infty)$. This will become more evident as you think about the possible y values of the function. What are some of the ordered pairs that make up this function? Here are a few: $(1,1)$, $(2, \frac{1}{2})$, $(10, \frac{1}{10})$, $(-1,-1)$, $(-2, -\frac{1}{2})$, $(-10, -\frac{1}{10})$, $(\frac{1}{2}, 2)$, $(\frac{1}{10}, 10)$, $(-\frac{1}{2}, -2)$, $(-\frac{1}{10}, -1)$.

Plot these points on an evenly calibrated set of axes. Drawing a graph is not simply a matter of plotting a few points and then playing "dot-to-dot." Consider your function. Does it make sense that when x is positive, y must also be positive? and when x is negative, y must be negative? Can you see why y can never be zero? This means that there are no x-intercepts. When x is positive, can you justify the fact that the bigger x gets, the smaller y gets? and the smaller x gets, the bigger y gets? What is the situation when x is negative? Use these observations to justify the way that you sketch your graph. It should look like this:



$$y = \frac{1}{x}$$

Quadratic Functions

Definition 3.3. *A quadratic function is a polynomial function of degree 2. So, the general form of a quadratic is $a_2x^2 + a_1x + a_0$, or more commonly written as $ax^2 + bx + c$, where $a \neq 0$.*

Example 3.3. Here are some examples of quadratic functions:

1. $f(x) = x^2 + 3x + 2$
2. $f(x) = 3x^2 + 2x - 8$
3. $f(x) = x^2 - 2x$
4. $f(x) = x^2 - 3x + 1$
5. $f(x) = 2x^2 + 5$

The graph of a quadratic function is a parabola.

Quadratic functions are nice because it is always possible to find all of the roots of a quadratic. In the first three quadratic functions above we can find the roots by factoring and setting each factor equal to zero:

1. $0 = x^2 + 3x + 2 = (x + 2)(x + 1)$, so $x = -2$ or $x = -1$

2. $0 = 3x^2 + 2x - 8 = (3x-4)(x+2)$, so $x = \frac{4}{3}$ or $x = -2$

3. $0 = x^2 - 2x = x(x-2)$, so $x = 0$ or $x = 2$

The last two quadratic functions in Example 3.3 do not factor. To find the roots for these functions, we use the Quadratic Formula:

$$\text{If } ax^2 + bx + c = 0 \text{ (where } a \neq 0\text{) then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{3.1}$$

This formula isn't magic. It's algebra. If you divide across the equation by the (non-zero) number a it becomes

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Factoring the left hand side we get

Discuss this with example $(x - r_1)(x - r_2) = 0$

$$\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = 0.$$

So, you see that the process of "finding factors" is just a way of bypassing the Quadratic Formula when you can spot the factors.

As a polynomial of degree 2, a quadratic function can have at most two roots. If the radicand[4] of the quadratic formula is negative, then the function has no roots. If the radicand is positive, there are two roots. If the radicand is zero, then both factors are the same, so there is only one distinct root.

Caution: Ocassionally an overzealous student will attempt to use the quadratic formula on functions like $f(x) = x^3 + x - 7$. This does not work. Even though there are only three terms, the degree of this polynomial is 3, not 2. It is not a quadratic function. The quadratic formula applies only to quadratic functions.

Linear Functions and Equations

Eqn of x + y-axes.

Definition 3.4. *A linear function is a polynomial function of degree* 1 *or degree* 0*. The constant function* $f(x) = 0$ *is also considered a linear function (recall that it is a polynomial which does not have a degree). So, the general form of a linear function is* $a_1x + a_0$ *(degree* 1 *when* $a_1 \neq 0$*) or just* a_0 *(degree* 0*). More commonly, a linear function is written as* $mx + b$*, where* $m \neq 0$ *in the degree* 1 *case, and* $m = 0$ *in the degree* 0 *case. A degree* 0 *linear function is, of course, just a constant function, such as* $y = b$ *or* $y = 26$*. The graph of a linear function is a non-vertical (straight) line*[5].

Here we recall some ideas about lines (from analytic geometry) that you will have seen before coming into this course.

The most general equation of a line is:

$$px + qy + r = 0 \text{ where } p, q, r \text{ are constants.} \tag{3.2}$$

[4] "Radicand" refers to the expression under a radical. In this case it is $b^2 - 4ac$.

[5] We always mean a "line" to be straight. We use the word "curve" otherwise.

Draw axes in the plane. A line is *vertical* if it is parallel to the y-axis. A vertical line has equation $x = k$; here k is the point on the x-axis where the line crosses the x-axis. As its equation indicates, it is exactly the set of ordered pairs (x, y) where the x-coordinate is always the number k. So, some of the points are $(k, 3)$, $(k, -1)$, $(k, \sqrt{5})$, etc. Clearly, this is not the graph of a function.

When equation 3.2 has $p = 1$, $q = 0$ and $r = -k$, we have exactly the equation $x = k$.

All other lines are *non-vertical.* They have slopes. The *slope* of the non-vertical line joining two different points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. The denominator is not zero because on a non-vertical line, no two points can have the same x coordinate.[6]

Any non-vertical line will cross the y-axis at some number, b. The number b is the *y-intercept* of the line.

When Equation 3.2 has $p = m$ (the slope), $q = -1$ and $r = b$ (the y=intercept), then we have $mx - y + b = 0$. This easily becomes the more familiar equation of a non-vertical line:

$$y = mx + b \tag{3.3}$$

This is called the *slope-intercept* form of the linear equation.

There is another way of writing the equation of a non-vertical line which we'll find useful. The equation of the line through the specific point (x_1, y_1) with slope m is

$$y - y_1 = m(x - x_1). \tag{3.4}$$

This is called the *point-slope* form of the linear equation. This line is the graph of the linear function $f(x) = mx + b$.

When Equation 3.2 has $p = m$, $q = -1$ and $r = y_1 - mx_1$ then, with minor algebraic manipulations, we have the linear equation in point-slope form.

All of equations 3.2, 3.3 and 3.4 are interchangeable when writing the equation of a function whose graph is a non-vertical line.

In short, the equation $px + qy + r = 0$ describes all linear equations, both vertical and non-vertical (vertical when $q = 0$ and $p \neq 0$ and non-vertical when $q \neq 0$). The familiar $y = mx + b$ and the useful $y - y_1 = m(x - x_1)$ do not apply to vertical lines because for a vertical line, m has no meaning.

Here are some other things to recall from high school math:

1. If the slope of a line is positive, the line is increasing (going up) when viewed left-to-right in the xy plane.

2. If the slope of a line is negative, the line is decreasing (going down) when viewed from left-to-right in the xy plane.

3. A horizontal line has slope 0, a vertical line has no slope, and these lines are perpendicular to each other.

[6] By the same reasoning, vertical lines do not have a slope. Any two points on a vertical line *will* have the same x coordinate. So the denominator of the slope fraction would be zero. The fraction is then meaningless.

4. If a non-horizontal, non-vertical line has slope m ($m \neq 0$) the lines perpendicular to it all have slope $-\dfrac{1}{m}$.   Question: Why must we insert ($m \neq 0$)?

5. Parallel lines have the same slope.

Vocabulary

During our study of economic applications we will be using some terms that you might not have seen in high school.

Suppose you decide to make some money by selling cold drinks on a hot day. You buy a cooler (\$8), ice (\$3) and a variety of canned beverages (100 cans at 25 cents each, for a total of \$25). You sell your drinks, charging 75 cents for each can. Suppose you successfully sell all 100 cans. For simplicity, we will suppose that you live in Delaware where there is no sales tax.

Cost – This is the amount of money that the seller, vendor, manufacturer, etc. has to spend to make and market the product. This is the expense that you had for purchasing the cooler, ice and drinks (\$36). Often cost will include a *fixed cost* that doesn't change regardless of how many items are being produced for sale, and a *variable cost* that does depend on that quantity. For your beverage enterprise the fixed cost is the expense for the cooler and ice (\$11). Your variable cost is the cost of the cans of drink that you buy (\$25). because the # of cans is not fixed

Price – This is the amount of money that is charged by the seller for each item. Your price is 75 cents.

* *Demand* [7] – This is the quantity of items sold. Demand is sometimes called *Quantity sold.* Your demand is 100 (cans).

Revenue – This is the total money that the seller receives from the customers. Revenue = (Price × Demand). Since you sold all of your drinks, your revenue is $.75 \times 100 = \$75$.

Profit – This is the amount of money that the seller has after all of the costs are paid.
* Profit = Revenue − Cost. Your business had a profit of $75 - 36 = \$39$. A business is said to have a loss if the profit is negative (i.e., Cost > Revenue). A business is said to break even when the profit is zero (i.e., Cost = Revenue).

We will use these terms in the following examples so that you become familiar with them.

Example 3.4. If production of chairs has a fixed cost of \$25,000 and a per chair cost of \$200 then the *cost* of producing x chairs is the linear function

$$C(x) = 200x + 25,000.$$

$C(x)$ is a number of dollars, x is a number of chairs. We usually omit the units ("dollars" or "number of chairs") in doing the math but it's a good idea to keep them in mind and they should be written as part of the answer to a word problem.

If the manufacturer charges p dollars per item then the *revenue* from selling x items will be the linear function

$$R(x) = px$$

[7] This term can have several different meanings. In an economics course, make sure that you understand the definition of *demand* that the course is using.

P(x) profit fcn R(x) revenue fcn C(x) cost fcn

(Units: p dollars, $R(x)$ dollars, x is a number.) The manufacturer's *profit* from selling x chairs will be the linear function

$$* \quad P(x) = R(x) - C(x) = px - 200x - 25,000.$$

The *break even point* occurs when

$$\begin{aligned} px - 200x - 25,000 &= 0 \\ (p-200)x &= 25,000 \\ \text{or} \quad x &= \frac{25,000}{p-200} \end{aligned} \tag{3.5}$$

$p > 200$

$\uparrow p \quad \downarrow x$

That is, if you are going to sell at $\$p$ per item you need to sell this number of chairs to break even.

The last paragraph answered the question: How many chairs must be produced in order to break even if you charge a pre-determined price $\$p$ per chair? But the business problem might be different. Perhaps you are definitely going to produce 500 chairs. Then you would ask: How much should be charged per chair in order to break even? Now we must solve the profit function for p in terms of x rather than (as above) for x in terms of p.

$$\begin{aligned} px - 200x - 25,000 &= 0 \\ px &= 25,000 + 200x \\ p &= \frac{25,000}{x} + 200 \end{aligned}$$

and if $x = 500$, we get: $\quad p = \dfrac{25,000}{500} + 200 = 50 + 200 = 250$

Answer: Charge \$250 per chair to break even.

A Graphical Representation

Following is a graphical representation of what we have been doing concerning linear cost and revenue functions. What can we identify in the graph?

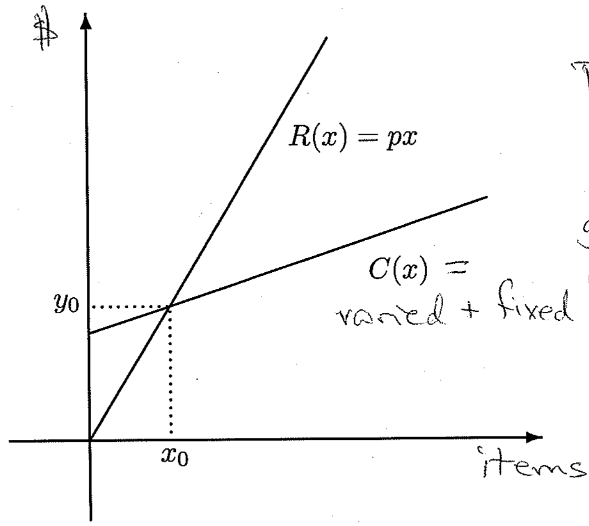
The lines $y = C(x)$ and $y = R(x)$ cross at a point (x_0, y_0). This is where the cost and revenue are equal. This is the break even point. The number x_0 is the number of items to be produced (sold) for breaking even. The number $y_0 = C(x_0) = R(x_0)$ is the cost, and also the revenue, for that number of items.

* For a fixed value of p, the function $R(x) = px$ is known. The number p is the slope of $R(x)$. When p is large, the graph is steep, so the intersection with $C(x)$ will occur closer to the y axis (the x coordinate is smaller). When p is smaller, the graph is less steep, so the intersection with $C(x)$ is further from the y axis (the x coordinate is larger). Recall that x represents the quantity sold. Does it make sense that when the price is higher you need to sell fewer items to break even, and vice-versa?

The revenue function has y-intercept zero. When $x = 0$ (no items are sold), the revenue will be zero.

The y-intercept for $C(x)$ represents the fixed cost. This is the expense incurred even when no product is made (when x=0).

When the cost function $C(x)$ is linear, the slope of the graph of $C(x)$ is called the *marginal cost*[8]. You can think of the marginal cost as the cost per item when fixed costs are ignored; it is the amount by which your cost increases each tme you produce an extra item.



$P(x) = R(x) - C(x)$
$0 = R(x) - C(x)$
gives x value (items sold) to break even.
i.e. $R(x) = C(x)$

Linear Cost and Revenue

Example 3.5. Sally has found a way to help finance her family's vacation at the beach. Sally pays her children 25 cents for each nice shell they find. Then she gets a vendor's license and she sells sea shells by the seashore. The license costs her \$350.

1. Write a linear cost function $C(x)$ to describe Sally's cost as a function of the number of shells (x) she buys from her children.

2. Sally sells her shells for \$1.35 each. Write a linear revenue function, $R(x)$ to describe this. How many shells must she sell for a profit of \$1,000? Profit as a fcn. of x

3. Suppose Sally's children are lazy and only bring her 100 shells. What price must she charge per shell in order to break even? Profit as a fcn. of p.

Solution:

1. Sally has a fixed cost of of \$350 for the license. She pays this even if she sells no shells. She has a per shell cost of \$.25 that she pays to obtain the shells. Thus, $C(x) = .25x + 350$, where x is the number of shells that she buys from her children and $C(x)$ is her total cost.

2. Sally sells her shells for \$1.35 each, so her revenue function is $R(x) = 1.35x$.
   Profit, $P(x) = R(x) - C(x) = 1.35x - (.25x + 350) = 1.10x - 350$.
   If $P(x) = 1,000$, we get: $1,000 = 1.10x - 350 \implies 1,350 = 1.1x \implies x = 1,227.\overline{27}$.
   So, she must sell 1,228 shells.

[8]Once we have studied the derivative in Section 8 we will have a way to find the marginal cost of non-linear functions that is consistent with the slope calculation here.

3. Here, we are given $x = 100$ and we want to find p such that $P(p) = 0$.
$0 = P(p) = R(p) - C(p) = 100p - (.25 \cdot 100 + 350) = 100p - 375.$
So, $p = \$3.75$

(1) $P(x) = 0 = R(x) - C(x) = 1.35x - (.25x + 350)$

$P(x) = 1.10x - 350$

Profit as a fcn. of items sold.

(2) $P(p) = 0 = R(p) - C(p) = (p)(100) - (.25(100) + 350)$

$P(p) = 100p - 25 - 350$

$P(p) = 100p - 375$

Profit as a fcn. of price charged

The earlier formulation of a line was

$px + qy + r = 0$

anticipating revenue + cost = breakeven 0

Section 3 - Exercises (answers follow)

1. For each of the following, decide whether the expression is a polynomial, a rational expression, or neither. If it is a polynomial, give the degree of the polynomial.

   (a) $\dfrac{6x^2+1}{6x-3}$

   (b) $3x^6 + 2x^4 - x + 2$

   (c) $\dfrac{x^2-4}{x+2}$

   (d) $\frac{1}{2}x^3 - 4x$

   (e) $\dfrac{\sqrt{x}}{\sqrt{x+1}}$

   (f) $(x+1)(x+2)(x^3+4)$

   (g) $x - x^3 - 8$

   (h) $2^x + x^2$

   (i) $\sqrt{2}x - \sqrt{3}$

   (j) 10

   (k) $x^{\frac{1}{2}} + x^{\frac{1}{3}} + x^{\frac{1}{4}} + 1$

2. Apply the Quadratic Formula to find the roots for each of the functions in Example 3.3.

3. Find the slope of the line that passes through each pair of points.

   (a) $(2,5)$ and $(1,3)$

   (b) $(4,5)$ and $(-1,-2)$

   (c) $(\frac{2}{3}, -\frac{1}{4})$ and $(\frac{1}{7}, \frac{1}{5})$

4. Find an equation for each line. use $y - y_1 = m(x - x_1)$

   (a) Through $(2,2)$ and $(-1,4)$

   (b) Through $(2,-4)$; $m = 3$

   (c) Through $(1,4)$ and parallel to the x axis

   (d) Through $(-1,4)$, parallel to $2x - y = 6$

   (e) Through $(-1,4)$ and perpendicular to $y = 2x + 6$

   (f) Through (a,b) with slope k $y - b = k(x - a) \rightarrow y = kx - ka + b$
   $= kx + (b - ak)$

   (g) x-intercept $-2/3$ and perpendicular to $x + y + 1 = 0$

5. Given that the point $(2,9)$ lies on the line $kx + 3y + 4 = 0$, find k.

   $k(2) + 3(9) + 4 = 0$

   $2k = -31$

   $k = -31/2$

6. A newsletter has fixed production costs of \$400 per edition and marginal printing and distribution costs of 40¢ per copy. It sells for 50¢ per copy.

   $C(x) = .40x + 400$, $R(x) = .50x$ $\Big\}$ $P(x) = R(x) - C(x)$

   (a) Write the cost, revenue, and profit functions.

   (b) What profit (or loss) results from the sale of 500 copies of the newsletter? $P(500) = -350$

   (c) How many copies should be sold in order to break even? $P(x) = 0 = R(x) - C(x)$

   $R(x) = C(x) \to .10x = 400 \to x = 4000$

7. Assume that each of the following can be expressed as a linear cost function. Find the cost function in each case.

   (a) Fixed cost \$150; 10 items cost \$300 to produce $C(p) = 300 = 10p + 150$, $p = 15$, $C(x) = 15x + 150$

   $C(x) = 15x + 150$

   (b) Marginal cost: \$100; 10 items cost \$2237 to produce

   $C(p) = 2237 = 10p + 100 \to 2137 = 10p$

8. What is the marginal cost in Problem 7a?

9. Your marginal cost to produce one item is \$2.50. Your total cost to produce 100 items is \$300, and you sell them for \$6 each.

   (a) Find the linear cost function for item production.

   (b) How many items must you produce and sell in order to break even?

   (c) How many items must you produce and sell to make a profit of \$500?

10. Each unit of a certain commodity sells for $p = 5x + 20$ cents when x units are produced. If all x units are sold at this price, express the revenue derived from the sales as a function of x.

11. A manufacturer has a monthly fixed cost of \$10,000 and a variable cost of \$.50/unit. Find a function C that gives the total cost incurred in the manufacture of x units/month.

12. Producing x desserts costs $C(x) = 7x + 21$; revenue is $R(x) = 14x$, where $C(x)$ and $R(x)$ are in dollars.

    (a) What is the break-even quantity?

    (b) What is the profit from 100 desserts?

    (c) How many desserts will produce a profit of \$500?

13. The sales of a company were \$20,000 in its third year of operation and \$55,000 in its fifth year. Let y denote sales in the xth year of operation. Assume that the points (x, y) all lie on a line.

    (a) Find the slope of the sales line, and give an equation for the line in the form $y = mx + b$.

    (b) Use your answer from part (a) to find out how many years must pass before the sales surpass \$200,000.

14. A manufacturer's total cost consists of a fixed cost of \$4,000 and a production cost of \$40 per unit. Express the total cost as a function of the number of units produced and draw the graph.

15. A car gets 30 miles to the gallon and has a 15 gallon tank. It starts the trip with x gallons in the tank. Write down a linear function $f(x)$ giving the number of miles it can go without needing more gas.

16. A thinking question: in the previous exercise what does $f(16.5)$ mean?

Section 3 - Answers

1. (a) rational (b) poly, degree 6 (c) rational (d) poly, degree 3
   (e) neither (f) poly, degree 5 (g) poly, degree 3 (h) neither
   (i) poly, degree 1 (j) poly, degree 0 (k) neither

2. 1. $x = \dfrac{-3 \pm \sqrt{9-8}}{2} \Longrightarrow x = -1$ or $x = -2$
   2. $x = \dfrac{-2 \pm \sqrt{4+96}}{6} \Longrightarrow x = \frac{4}{3}$ or $x = -2$
   3. $x = \dfrac{2 \pm \sqrt{4-0}}{2} \Longrightarrow x = 0$ or $x = 2$
   4. $x = \dfrac{3 \pm \sqrt{9-4}}{2} \Longrightarrow x = \dfrac{3+\sqrt{5}}{2}$ or $x = \dfrac{3-\sqrt{5}}{2}$
   5. No roots.

3. (a) 2 (b) $\frac{7}{5}$ (c) $-\frac{189}{220}$

4. (a) $y = -\frac{2}{3}x + \frac{10}{3}$ (b) $y = 3x - 10$ (c) $y = 4$ (d) $y = 2x + 6$
   (e) $y = -\frac{1}{2}x + \frac{7}{2}$ (f) $y = kx + (b - ak)$ (g) $y = x + \frac{2}{3}$

5. $k = -\frac{31}{2}$

6. (a) $C(x) = .4x + 400$ $R(x) = .5x$ $P(x) = .1x - 400$
   (b) loss of \$350
   (c) 4000

7. (a) $C(x) = 15x + 150$ (b) $C(x) = 100x + 1237$

8. 15

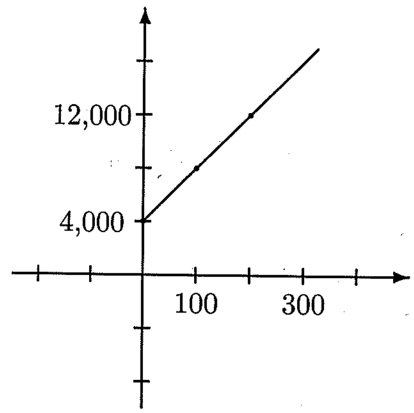
9. (a) $C(x) = 2.50x + 50$ (b) 15 items (rounded up) (c) 158 items (rounded up)

10. $R(x) = 5x^2 + 20x$

11. $C(x) = .50x + 10,000$

12. (a) 3 desserts (b) \$679 (c) 75 desserts (rounded up)

13. (a) $m = 17,500$
    (b) $y = 17,500x - 32,500$
    (c) $\frac{2325}{175}$ (approx. 13 years, $3\frac{1}{2}$ months)



$y = 40x + 4,000$

14. $y = 40x + 4,000$

15. $f(x) = 30x$

16. Hint: What does x represent in your function? Which values of x make sense in the problem?