# Chapter 7

# Absolute Value and Inequalities

## 7.1 Inequalities

### 7.1.1 Definition

An *inequality* is any statement where "$<$, $\leq$, $>$, or $\geq$" is used as the verb. The solution(s) to the inequality are those value(s) which make the statement true.

For example, the inequality $2x + 5 \geq 3$ has 12 as a solution, has 0 as a solution and has $-\frac{1}{2}$ as a solution. In fact, the entire solution set is $\{x \in \Re : x \geq -1\}$.

### 7.1.2 Algebra of Inequalities

The algebra for inequalities is not unlike that for equations, but extra care must be taken when dealing with multiplication or division by a negative value. In this last case, you must remember to change the sign of the inequality to its opposite. Below is a summary of the properties of inequalities.

1. If $a < b$, then $a + c < b + c$ and $a - c < b - c$.
2. If $a < b$ and $c$ is positive, then $ac < bc$ and $\dfrac{a}{c} < \dfrac{b}{c}$.
3. If $a < b$ and $c$ is negative, then $ac > bc$ and $\dfrac{a}{c} > \dfrac{b}{c}$.
4. If $a < b$ and if $a$ and $b$ are *both* positive or *both* negative, then $\dfrac{1}{a} > \dfrac{1}{b}$.
5. If $a < b$ and $b < c$, then $a < c$. (Transitive property)

Take a moment to use random real number values for $a$, $b$ and $c$ to test these properties until you are convinced that they are correct and you have a feel for why they work. Especially pay attention to Properties 3 and 4. Does Property 4 still hold if $a$ and $b$ have different signs? What is the relationship between $\frac{1}{a}$ and $\frac{1}{b}$ in this case?

These properties can be used to solve simple inequalities:

**Example 7.1.1.**

1. *If* $x+5<2$, *then* $x+5-5<2-5$. *So,* $x<-3$.
2. *If* $3x>12$, *then* $\frac{1}{3}\cdot 3x>\frac{1}{3}\cdot 12$. *So,* $x>4$.
3. *If* $\frac{-x}{2}<1$, *then* $-2\cdot\frac{-x}{2}>-2\cdot 1$. *So,* $x>-2$.

It is easy to apply properties 2 and 3 when the multipliers or divisors that we are using are constants. It is not as easy when we are dealing with variables or functions. For example, suppose we know that $a<b$, and we want to multiply both sides by $x$. Do we change the sign? Well, if $x>0$ (is positive), then we do not change the sign. But, if $x<0$ (is negative) then we must change the sign. Multiplication or division by an expression of unknown sign requires that we deal with both possibilities.

**Example 7.1.2.**

*Find the solution set for* $\dfrac{(x+5)}{(2x-1)}\leq 3$.

*We multiply both sides of* $\dfrac{(x+5)}{(2x-1)}\leq 3$ *by* $(2x-1)$ *and consider two cases:*

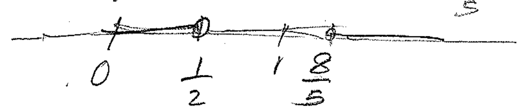
$$(x+5)\leq 3(2x-1) \quad \textit{if} \quad (2x-1)>0$$

*and* $$(x+5)\geq 3(2x-1) \quad \textit{if} \quad (2x-1)<0.$$

*Simplifying these statements, we get:*

| | |
|---|---|
| $x+5\leq 6x-3$ *if* $2x>1$ | $x+5\geq 6x-3$ *if* $2x<1$ |
| $-5x\leq -8$ *if* $x>1/2$ | $-5x\geq -8$ *if* $x<1/2$ |
| $x\geq \frac{8}{5}$ *if* $x>1/2$ | $x\leq \frac{8}{5}$ *if* $x<1/2$. |

If $x>\frac{1}{2}$ then it's $>\frac{8}{5}$
If $x<\frac{1}{2}$ then it's $<\frac{8}{5}$



*The solution to the inequality then is* $x\geq\frac{8}{5}$ *or* $x<\frac{1}{2}$.

Notice in Example 7.1.2 that the assumptions (the "if" statements) do play an important role in the final answer. They are not just a formality.

The solution to an inequality most often involves intervals of real numbers. In Chapter 1 we discussed three ways to present interval solutions. That was a long time ago, so we review quickly. The solution for Example 7.1.2 can be expressed:

- using algebaic notation $x<\frac{1}{2}$ *or* $x\geq\frac{8}{5}$
- using interval notation $(-\infty,\frac{1}{2})\cup[\frac{8}{5},\infty)$
- by graphing on a number line

We would NOT write "$x<\frac{1}{2}$ *or* $x\geq\frac{8}{5}$" as "$\frac{8}{5}\leq x<\frac{1}{2}$." This last expression implies that $\frac{8}{5}<\frac{1}{2}$. When the number line graph of a solution is disconnected then we can't write the solution algebraically in a single expression. It will always be multiple expressions joined by "*or*" statements.

There are, of course times when we do write expressions where the variable is "sandwiched" between two expressions. This is just a way to shortcut the writing of an "*and*" expression.

"$x > 4$ <u>and</u> $x < 10$" can be written as "$4 < x < 10$." The implication that $4 < 10$ causes no problem.

The expression $-2 \le \frac{x+2}{-3} < 5$ is just a shortcut way of writing $-2 \le \frac{x+2}{-3}$ *and* $\frac{x+2}{-3} < 5$. In all of these expressions in order to solve for $x$ we need to multiply both sides by $-3$ and then subtract 2. So, we can do this all at once using the shortcut notation.

**Example 7.1.3.**

*Solve for x:* $-2 \le \frac{x+2}{-3} < 5$

$$-2 \le \frac{x+2}{-3} < 5$$

$$6 \ge x+2 > -15 \qquad \textit{(multiply by negative number so change sign)}$$

$$4 \ge x > -17$$

*For good form we write our solution:* $-17 < x \le 4$. *Note that this would be one connected interval on the number line.*

We emphasize that shortcut expressions and algebra operations on them are only for expressions that are joined by "*and*" (that is those which are a single interval on a number line). This is not done for disjoint "*or*" expressions like we have with problems that must be broken into cases.

**Example 7.1.4.**

*Find the solution set for* $(2x-3) > \frac{2x^3-6}{x^2}$.

*We notice immediately (yes, immediately!) that* $x \ne 0$.

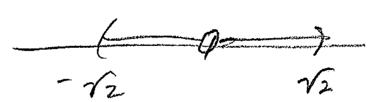
*Then we notice that* $x^2$ *is always positive. Thus, we can multiply both sides by* $x^2$ *without having to change the inequality sign.*

$$(2x-3) > \frac{2x^3-6}{x^2}$$

$$x^2(2x-3) > 2x^3-6$$

$$2x^3-3x^2 > 2x^3-6$$

$$-3x^2 > -6$$




$x^2 < 2$ *Sign change! Why?*

*Since x cannot be zero, our final solution is:* $(-\sqrt{2}, 0) \cup (0, \sqrt{2})$.

It's time for a one more reminder about notation. We could have expressed the answer to Example 7.1.4 by "$-\sqrt{2} < x < 0$ *or* $0 < x < \sqrt{2}$." We DO NOT say "$x < \pm\sqrt{2}$." This last expression makes no sense.

With inequalities, it is easy to make a mistake of sign somewhere in the process of finding the solution. It is always a good idea to check your answers with some "test" points. For instance, in Example 7.1.4 above, we should check (in the *original* problem) one number larger than $\sqrt{2}$, one number smaller than $-\sqrt{2}$, one number between $-\sqrt{2}$ and 0 and one number between 0 and $\sqrt{2}$. If we check the numbers $100, -100, -1$ and 1, the first two yield false statements and the second two yield true statements (check this). This is what we want. Our solution set represents all of the values for $x$ that make the statement true.

### 7.1.3 Problem Solving

The solution to the inequality $f(x) > 0$ is all of the values of $x$ that make the statement $f(x) > 0$ true. In other words, we are finding all of the values of $x$ for which $f(x)$ is positive. From a graphing perspective, we are finding all of the $x$ values where the graph of $f$ is above the $x$-axis.

**Important Idea 7.1.1.**
*If $a \cdot b > 0$, then $a$ and $b$ have the same sign.*
*If $a \cdot b < 0$, then $a$ and $b$ have different signs.*

Important Idea 7.1.1 is not startling news, but the concept is critical for what we do next. Many inequality problems can be rewritten so that they are in the form $f(x) > 0$. If the function $f$ can then be factored, we can determine where $f(x) > 0$ by investigating the signs of the individual factors for certain values of $x$. We need only find which values of $x$ yield the same (or different) signs for the factors, and then we will know for which values of $x$ the function $f$ is positive (or negative).

**Example 7.1.5.**
*Solve for $x$: $x^2 - 3x - 10 > 0$.*

$x^2 - 3x - 10 = (x+2)(x-5)$*, so we really need to know when* $(x+2)(x-5) > 0$.

*The factor $(x+2)$ is positive when $x > -2$ and is negative when $x < -2$.*
*The factor $(x-5)$ is positive when $x > 5$ and is negative when $x < 5$.*

*The product $(x+2)(x-5)$ is positive when either both factors are negative or both factors are positive:*
$[x > -2$ *and* $x > 5]$ *or when* $[x < -2$ *and* $x < 5]$.

*Thus the solution is $(-\infty, -2) \cup (5, \infty)$.*
*Choose test numbers and check this solution.*

It can be useful on more complicated problems to write the factor information in table format. This is done below for Example 7.1.5 and will be used for further examples.

| $x$ | $(-\infty,-2)$ | $\{-2\}$ | $(-2,5)$ | $\{5\}$ | $(5,\infty)$ | Divide the domain into the appropriate intervals |
|---|---|---|---|---|---|---|
| $(x+2)$ | $-$ | 0 | $+$ | $+$ | $+$ | Sign for this factor in each interval |
| $(x-5)$ | $-$ | $-$ | $-$ | 0 | $+$ | Sign for this factor in each interval |
| $(x+2)(x-5)$ | $+$ | 0 | $-$ | 0 | $+$ | * |

*The bottom row of information (the solution) is obtained by looking at the product of the signs of the factors above it. In the first column, for example $(-\infty,-2)$ has two negative factors, so the product is positive. The zeros are included in the table because for some problems, an inequality of the form "$\leq 0$" or "$\geq 0$" may be desired.

Once we have the table of information, we can easily solve several problems. Find the solution sets for: $f(x) < 0$, $f(x) \geq 0$, $f(x) = 0$, and $f(x) \neq 0$.

When creating the table, the first step is to divide the domain of $f$ into "appropriate" intervals. How do we do this? We can see in Example 7.1.5 that the divisions were made at the same places where each factor changed from positive to negative. Those places are exactly the places where the factors are each equal to zero. In problems where the domain is not $\Re$ we also must include as divison points those values of $x$ where the function is undefined.

**Example 7.1.6.**
*Solve for x:* $\frac{7-2x}{x+9} \leq 0$.

*We notice that the function is undefined at* $x = -9$.
*Our division points are where* $(7-2x) = 0$ *(which is* $x = \frac{7}{2}$*) and* $x = -9$.

| $x$ | $(-\infty, -9)$ | $\{-9\}$ | $(-9, \frac{7}{2})$ | $\{\frac{7}{2}\}$ | $(\frac{7}{2}, \infty)$ |
|---|---|---|---|---|---|
| $(7-2x)$ | + | + | + | 0 | − |
| $(x+9)$ | − | 0 | + | + | + |
| $(7-2x)/(x+9)$ | − | *undef* | + | 0 | − |

*For the solution this time we want the quotient to be negative or zero, so the solution is* $(-\infty, -9) \cup [\frac{7}{2}, \infty)$.

**Example 7.1.7.**
*Solve for x:* $x^4 \geq 81x^2$.

*We need to rewrite this problem into the form* $f(x) \geq 0$ *and then factor.*
$x^4 \geq 81x^2$
$x^4 - 81x^2 \geq 0$.
$x^2(x^2 - 81) \geq 0$.
$x^2(x-9)(x+9) \geq 0$.
*The division points for our table are* 0 , 9 , −9.

| $x$ | $(-\infty, -9)$ | $\{-9\}$ | $(-9, 0)$ | $\{0\}$ | $(0, 9)$ | $\{9\}$ | $(9, \infty)$ |
|---|---|---|---|---|---|---|---|
| $x^2$ | + | + | + | 0 | + | + | + |
| $(x-9)$ | − | − | − | − | − | 0 | + |
| $(x+9)$ | − | 0 | + | + | + | + | + |
| $x^2(x-9)(x+9)$ | + | 0 | − | 0 | − | 0 | + |

*The solution is* $(-\infty, -9] \cup [0] \cup [9, \infty)$.

With a little practice you will not find it necessary to write out the entire table each time. It will be sufficient to find the division points, note the intervals, and then check each interval with a "test" value from that interval. In Example 7.1.7 we found intervals of $(-\infty, -9)$ ,$(-9, 0)$ ,$(0, 9)$ and $(9, \infty)$. We could choose −10 as a "test" value for the first interval. At $x = -10$, $x^2$ is positive, $(x-9)$ is negative and $(x+9)$ is negative. Think: $+ \cdot - \cdot - = +$, so this interval is part of the solution set. We are doing EXACTLY the same thing that we do with the table, but doing it mentally. After checking each interval, check the division points themselves to see if they should be included in the solution set. One hint: choose your "test" values so that the evaluation of the factors will be somewhat simple. Zeros and Ones and extremely large($\pm$) numbers are often good. You don't care about the exact value of each factor, only whether it is positive or negative.

**Example 7.1.8.**
*Solve for x:* $\dfrac{-x}{x+8} > 3$.

*One possibility for solving this problem is to multiply both sides by $(x+8)$ and then solve the two resulting inequalities (same method that was used in Example 7.1.2). We will use an alternative approach by rewriting the inequality into the form $f(x) < 0$ and solving by looking at a factor table. We will write out the entire table because it is hard to communicate to the reader if we do the interval testing mentally.*

*We first subtract 3 from both sides and then simplify to arrive at*

$$0 < \frac{-x}{x+8} - 3 = \frac{-x-3(x+8)}{x+8} = \frac{-x-3x-24}{x+8} = \frac{-4(x+6)}{x+8}.$$

*Finally, we divide both sides by the $-4$, but we must be careful to change the sign:* $0 > \dfrac{x+6}{x+8}$*. Now the problem is straightforward:*

| $x$ | $(-\infty,-8)$ | $\{-8\}$ | $(-8,-6)$ | $\{-6\}$ | $(-6,\infty)$ |
|---|---|---|---|---|---|
| $(x+6)$ | $-$ | $-$ | $-$ | 0 | $+$ |
| $(x+8)$ | $-$ | 0 | $+$ | $+$ | $+$ |
| $(x-6)/(x+8)$ | $+$ | *undef* | $-$ | 0 | $+$ |

*The solution set is the interval $(-8,-6)$.*

**Example 7.1.9.**
*Solve for x:* $\dfrac{x}{x+2} - \dfrac{4}{1-x} \geq 1$.

*We need to get this inequality into the form $f(x) \leq 0$ or $f(x) \geq 0$.*

$$\frac{x}{x+2} - \frac{4}{1-x} \geq 1$$

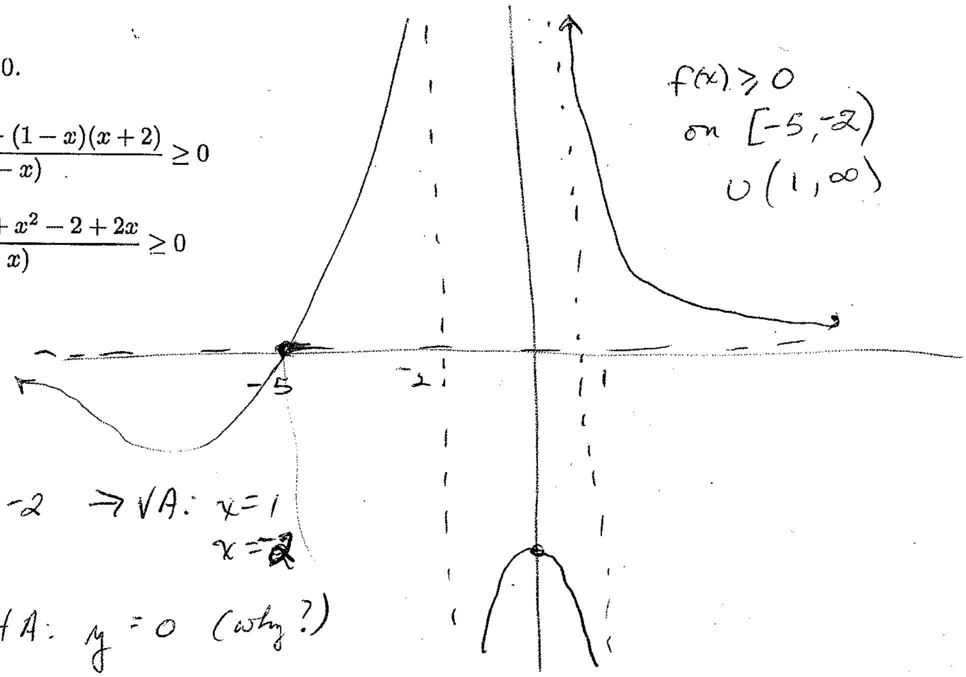
$$\frac{x}{x+2} - \frac{4}{1-x} - 1 \geq 0.$$

$$\frac{x(1-x)-4(x+2)-(1-x)(x+2)}{(x+2)(1-x)} \geq 0$$

$$\frac{x-x^2-4x-8-x+x^2-2+2x}{(x+2)(1-x)} \geq 0$$

$$\frac{-2x-10}{(x+2)(1-x)} \geq 0$$

$$\frac{-2(x+5)}{(x+2)(1-x)} \geq 0.$$



$$\frac{x+5}{(x+2)(1-x)} \leq 0.$$

| $x$ | $(-\infty,-5)$ | $\{-5\}$ | $(-5,-2)$ | $\{-2\}$ | $(-2,1)$ | $\{1\}$ | $(1,\infty)$ |
|---|---|---|---|---|---|---|---|
| $x+5$ | $-$ | 0 | $+$ | $+$ | $+$ | $+$ | $+$ |
| $(x+2)$ | $-$ | $-$ | $-$ | 0 | $+$ | $+$ | $+$ |
| $(1-x)$ | $+$ | $+$ | $+$ | $+$ | $+$ | 0 | $-$ |
| $\frac{(x+5)}{(x+2)(1-x)}$ | $+$ | 0 | $-$ | *undef* | $+$ | *undef* | $-$ |

*So, the solution set is* $[-5,-2) \cup (1,\infty)$.

## 7.2 The Absolute Value Function

You have most likely dealt with the idea of *absolute value* before. We think of the absolute value of a number as just the magnitude (size) of the number, or the number without the sign. Another way to think of it is that we take the number and always make it positive. So if the number is already positive we just keep it the same, but if the number is negative we change it to positive. We will expand this idea beyond just numbers to functions.

### 7.2.1 Definitions and Properties

#### Definitions

We want to take our idea of the absolute value of a number and write it formally in mathematical language. In fact we have three definitions that all work for us.

The first definition expresses absolute value as a piece-wise defined function. This definition is directly consistent with the idea of keeping a positive number the same, but changing a negative number to positive.

**Definition 7.2.1.** *Absolute Value Definition A*
*For any real number x, the absolute value of x, written as* $|x|$, *is defined to be:*

$$|x| = \begin{cases} -x & if \quad x < 0 \quad (i.e., \; if \; x \; itself \; is \; negative) \\ x & if \quad x \geq 0 \quad (i.e., \; if \; x \; itself \; is \; non-negative) \end{cases}$$

Using this definition we get: $|6| = 6$ and $|-5| = -(-5) = 5$.

The second definition uses the square root function to algebraically turn negatives into positives. You recall that the square root function is indeed a function, having only one solution, a positive one. So, $x = \sqrt{9} = 3$, not $\pm 3$. We take advantage of this to write our second definition of absolute value.

**Definition 7.2.2.** *Absolute Value Definition B*
*For any real number x, the absolute value of x, written as* $|x|$, *is defined to be:*

$$|x| = \sqrt{x^2}.$$

Using this definition we get: $|6| = \sqrt{6^2} = \sqrt{36} = 6$ and $|-5| = \sqrt{(-5)^2} = \sqrt{25} = 5$.

The third definition gives us an entirely different way to look at the concept of absolute value. This definition gives a geometric interpretation. This third definition uses the idea that distance is always positive.

**Definition 7.2.3.** *Absolute Value Definition C*
*For any real number $x$, the* <u>*absolute value of $x$*</u>*, written as $|x|$, is defined to be:*

$$|x| = \text{the distance on a number line between the origin, 0, and the number } x.$$

Using this definition, we think of $|6|$ as the distance between 0 and 6 on the number line, which is 6, and we think of $|-5|$ as the distance between 0 and $-5$ on the number line, which is 5.

**Comprehension Check 7.1.**

1. *Review each of the three definitions, using values $x = \frac{3}{2}$, $x = 0$ and $x = -\sqrt{7}$ so that you become very familiar with the definitions and understand how they are consistent with our understanding of the meaning of "absolute value."*

2. *We often speak of the absolute value "function". Explain why $f(x) = |x|$ is a function. What is the domain of $f(x) = |x|$?*

3. *Recall (or look back in Chapter 3) the definition of "equivalent functions". Are the functions described by each of the definitions equivalent?*

4. *On a set of coordinate axes, plot some points for the function $f(x) = |x|$ until you are certain how the graph looks. It should resemble a "V" composed of two half-lines meeting at the origin. What are the equations of those half-lines? How would you describe that graph using a piecewise defined function? Compare your answer to Definition A.*

You may wonder why we need three different ways to express the absolute value function. We don't. Some problems can more easily be solved, however, using one definition over the other two. It's nice to have more than one tool available to get the job done.

## Properties

Below are some properties of the absolute value function. Take a moment to "test" these properties by substituting random real numbers (positive and negative) in for $a$ and $b$ until you are comfortable with why they work.

**Important Idea 7.2.1.**

1. $|a| = |-a|$
2. $|a|^2 = a^2$
3. $|ab| = |a||b|$
4. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

In Important Idea 7.2.1 we see some ways of handling the absolute value of products and quotients. There are no rules there for sums or differences.

The following statements are *NOT* true: $\begin{cases} |a+b| = |a| + |b| \\ |a-b| = |a| - |b| \end{cases}$

It is easy to find numbers $a$ and $b$ such that $|a+b| \neq |a|+|b|$. For example, for $a = 3$ and $b = -2$ we have $|a+b| = |3-2| = |1| = 1$, but $|a|+|b| = |3|+|-2| = 3+2 = 5$. Similarly, one can see that $|a-b| = |a| - |b|$ does not hold always. Choose several number combinations for $a$ and $b$ and check them in the various expressions. With sufficient experimenting and observing you should be able to justify the conclusions stated in Important Idea 7.2.2, which we call the Triangle Inequalities.

**Important Idea 7.2.2.** *The Triangle Inequalities*

$$|a+b| \leq |a| + |b| \qquad |a-b| \geq |a| - |b|$$

In calculus, and other places, we are very often interested in the absolute value of a difference. It is often helpful to think in terms of a geometric (from Definition C) interpretation of the expression $|a-b|$.

**Important Idea 7.2.3.**
*The value $|a-b|$ is the distance between the numbers $a$ and $b$ on a number line. Note also that* $|a-b| = |b-a|$

To illustrate Important Idea 7.2.3 we choose $a = -5$ and $b = 2$. We calculate $|(-5)-(2)| = |-7| = 7$. Certainly the distance between the numbers $-5$ and 2 on a number line is 7. Choose some other numbers to "test" this idea yourself until you are convinced of its validity.

Notice that this idea is consistent with Definition C, using value $b = 0$.

Since the distance from $a$ to $b$ is the same as the distance from $b$ to $a$ it is reasonable to conclude that $|a-b| = |b-a|$. A proof for this is trivial when using Property 3: $|a-b| = |-1(b-a)| = |-1||b-a| = |b-a|$.

## 7.2.2 Equations Involving Absolute Value

When dealing with equations that involve absolute value, usually the easiest thing to do is try to rewrite the equation into some equivalent equation(s) that do not have absolute value signs in them. We can use the definitions to do this.

In Definition A we see that $|x| = -x$ if $x$ is negative, and $|x| = x$ if $x$ is non-negative. So, if we are given "$|x| = c$" (for some non-negative constant $c$) , we can rewrite this into a statement that has no absolute value signs: "$-x = c$ if $x < 0$, or $x = c$ if $x \geq 0$".

**Example 7.2.1.**
*The statement "$|x| = 2$" becomes "$-x = 2$ or $x = 2$." We would then simplify this to $x = \pm 2$.*

We expand our interpretation of these definitions beyond just having numeric input into having functional input. The corresponding definition idea for functions is $|f(x)| = -f(x)$ if $f(x)$ is negative, and $|f(x)| = f(x)$ if $f(x)$ is non-negative. So, if we have "$|f(x)| = c$", then we can rewrite it to "$-f(x) = c$, if $f(x) < 0$, or $f(x) = c$ if $f(x) \geq 0$."

**Example 7.2.2.**

*The statement "$|x+3| = 2$" becomes:*

| | | | | |
|---|---|---|---|---|
| $-(x+3) = 2$ | *if* $(x+3) < 0$ | *or* | $x+3 = 2$ | *if* $(x+3) \geq 0$ |
| $x+3 = -2$ | *if* $x < -3$ | | $x = -1$ | *if* $x \geq -3$ |
| $x = -5$ | *if* $x < -3$ | | $x = -1$ | |
| $x = -5$ | | | | |

*In each case, the value obtained for $x$ met the requirements of the "if..." statement, so the answers are valid. Verify that these values of $x$ are solutions to the original problem.*

We do not always need to specifically involve the "if..." statements when solving problems. To be rigorous, we should do it, but from a practical standpoint it is sufficient to check our final answers in the original problem. We will sometimes find that our answers are invalid (extraneous) and need to be rejected. If you do not carry the "if..." statements through your work you must check your answers in the original equation or risk serious error.

An alternative, and probably easier, way to handle Example 7.2.2 is by using the Definition C interpretation of absolute value found in Important Idea 7.2.3

**Example 7.2.2 Revisited**

*The statement $|x+3| = 2$ is the same algebraically as $|x-(-3)| = 2$. We interpret this as "the distance on a number line between $x$ and $-3$ is 2". Certainly, then $x = -1$ or $x = -5$.*

**Example 7.2.3.**

*Solve for $x$: $|4x-8| = 6$*

*Method 1: $-(4x-8) = 6$ or $4x-8 = 6$. This leads to solutions $x = \frac{1}{2}$ or $x = \frac{7}{2}$. Both solutions are valid in the original equation.*

*Method 2: $|4x-8| = 6 \implies |4||x-2| = 6 \implies |x-2| = \frac{6}{4} = \frac{3}{2}$. Now use the geometric interpretation: the difference between $x$ and 2 on the number line is $\frac{3}{2}$. So the solutions are $x = \frac{1}{2}$ or $x = \frac{7}{2}$.*

**Example 7.2.4.**

*Solve for $x$: $|x^2+x-2| = 4$.*

| | | |
|---|---|---|
| $x^2+x-2 = 4$ | *or* | $-(x^2+x-2) = 4$ |
| $x^2+x-6 = 0$ | | $-x^2-x+2 = 4$ |
| $(x+3)(x-2) = 0$ | | $-x^2-x-2 = 0$ |
| $x = -3$ *or* $x = 2$ | | $x = \frac{1\pm\sqrt{1-8}}{-2}$ *(quadratic formula)* |
| | | *no real solution* |

*We check the two real solutions in the original problem and keep them both, so $x = -3$ or $x = 2$.*

We are now ready to go yet another step forward. We go beyond the constant $c$, to equations of the form: $|f(x)| = g(x)$.

We rewrite $|f(x)| = g(x)$ into "$-f(x) = g(x)$ or $f(x) = g(x)$." To be precise we should add the corresponding "if..." statements "if $f(x) < 0$ and $g(x) \geq 0$" and "if $f(x) \geq 0$ and $g(x) \geq 0$." Notice that we have a new condition. Since $g(x)$ is equal to an absolute value expression, the values for $g(x)$ must be non-negative. Therefore we cannot accept any answer for $x$ that would make $g(x)$ negative. Now it can be really tedious to keep track of all of the underlying (though critical) assumptions and we simply check our answers in the original equation. Keep the valid answers; pitch the rest.

**Example 7.2.5.**
*Solve for* $x$. $|2x-5| = 4-x$.

| | | |
|---|---|---|
| $2x-5=4-x$ | *or* | $-(2x-5)=4-x$ |
| $2x=9-x$ | | $-2x+5=4-x$ |
| $3x=9$ | | $5=4+x$ |
| $x=3$ | | $1=x$. |

*We must check both answers in the original equation. They are both valid, so* $x=3$ *or* $x=1$ *are the solutions.*

**Example 7.2.6.**
*Solve for* $x$. $|x^2-3| = 2x$.

| | | |
|---|---|---|
| $x^2-3=2x$ | *or* | $-(x^2-3)=2x$ |
| $x^2-2x-3=0$ | | $-x^2+3=2x$ |
| $(x-3)(x+1)=0$ | | $0=x^2+2x-3$ |
| $x=3$ *or* $x=-1$ | | $0=(x+3)(x-1)$ |
| | | $x=-3$ *or* $x=1$. |

*When we check our solutions in the original problem we see that only* $x=3$ *and* $x=1$ *are valid. A look at the original equation tells us why. Since* $2x$ *is equal to an absolute value expression,* $2x$ *must be non-negative. Any negative value of* $x$ *would make* $2x$ *negative, and therefore would have to be rejected.*

We now consider the case where our equation has an absolute value sign enclosing the functions on both sides of the equation: $|f(x)| = |g(x)|$. To solve this we incorporate the definition of absolute value repeatedly. We know that the values inside the absolute value signs can be either positive or negative. So, if we list all of the possible cases we could have when we rewrite the equation without absolute value signs we get:

1. $f(x) = g(x)$
2. $-f(x) = g(x)$
3. $f(x) = -g(x)$
4. $-f(x) = -g(x)$

Looking closely we notice that the first and fourth equations are algebraically the same. The second and third equations are also the same. So instead of having four cases to deal with, we really only have two.

**Example 7.2.7.**
*Solve for* $x$. $|3x+2| = |6-x|$.

| | | |
|---|---|---|
| $3x+2=6-x$ | *or* | $-(3x+2)=6-x$ |
| $4x=4$ | | $-3x-2=6-x$ |
| $x=1$ | | $-8=2x$ |
| | | $-4=x$. |

*We check and verify that both answers are valid in the original problem.*

Another way to handle equations of the form $|f(x)| = |g(x)|$ uses $|a|^2 = a^2$ (Property 2 from Important Idea 7.2.1) to eliminate the absolute value signs. As with any squaring operation we have to be careful to not introduce extraneous solutions. We are in the habit of checking answers to absolute value problems anyway so that is not a problem.

**Example 7.2.7 Revisited**

*Solve for x.* $|3x+2| = |6-x|$.

$$|3x+2| = |6-x|$$
$$|3x+2|^2 = |6-x|^2$$
$$9x^2+12x+4 = 36-12x+x^2$$
$$8x^2+24x-32=0$$
$$8(x^2+3x-4)=0$$
$$8(x+4)(x-1)=0$$
$$x=-4 \quad \text{or} \quad x=1$$

*We check and verify that both answers are valid in the original problem.*

This method is practical only when the squaring operation does not make a difficult problem to solve. Example 7.2.8 below does not lend itself to the squaring method.

When we have an equation that involves more than one absolute value sign *and* includes terms that cannot be incorporated into any of the absolute value expressions the situation is more complicated. We have to consider all of the equations that could occur when applying absolute value Definition A to each of the absolute value expressions. This is the situation in Example 7.2.8. In this example we show not only all of the cases but also their corresponding assumptions. Look at the assumptions so that you can see what is really happening in the problem. In practice, however, one would solve all of the equations and then test the answers rather than keep track of the assumptions.

**Example 7.2.8.**

*Solve for x.* $|x-5| = |2x+6| - 1$.

*Note that we cannot simply "move" the "−1" into an absolute value expression. We must deal with each expression, using Definition A. So, our equation has four possibilities:*

$$
\begin{array}{lll}
 & (x-5) = (2x+6)-1 & \textit{if } [(x-5)\geq 0 \textit{ and } (2x+6)\geq 0] \\
\textit{or} & (x-5) = -(2x+6)-1 & \textit{if } [(x-5)\geq 0 \textit{ and } (2x+6)< 0] \\
\textit{or} & -(x-5) = (2x+6)-1 & \textit{if } [(x-5)< 0 \textit{ and } (2x+6)\geq 0] \\
\textit{or} & -(x-5) = -(2x+6)-1 & \textit{if } [(x-5)< 0 \textit{ and } (2x+6)< 0].
\end{array}
$$

*Simple algebra yields the following four equivalent statements:*

$$
\begin{array}{lll}
 & (x-5) = (2x+6)-1 & \textit{if } [x\geq 5 \textit{ and } x\geq -3] \\
\textit{or} & (x-5) = -(2x+6)-1 & \textit{if } [x\geq 5 \textit{ and } x< -3] \\
\textit{or} & -(x-5) = (2x+6)-1 & \textit{if } [x< 5 \textit{ and } x\geq -3] \\
\textit{or} & -(x-5) = -(2x+6)-1 & \textit{if } [x< 5 \textit{ and } x< -3].
\end{array}
$$

*Further solving of each equation:*

$$
\begin{array}{llll}
 & x=-10 & \textit{if } x\geq 5 & (\textit{inconsistent situation, so invalid solution}) \\
\textit{or} & x=-\frac{2}{3} & \textit{impossible "if"} & (\textit{"if" never satisfied, so invalid solution}) \\
\textit{or} & x=0 & \textit{if } -3\leq x<5 & \\
\textit{or} & x=-12 & \textit{if } x<-3. &
\end{array}
$$

*So, the only valid solutions are* $x=0$, $x=-12$.

Can you see why this problem would not be good candidate for the squaring method of solution used in "Example 7.2.7 Revisited?"

We have not discussed yet the usefulness of absolute value definition $B$. It is more often used when solving problems that involve taking square roots than it is for solving absolute value problems. It helps us to keep the notation "clean."

**Example 7.2.9.**
*Solve for x.* $(x-3)^2 = 25$.

$(x-3)^2 = 25$ —— Alternately → $x - 3 = \pm\sqrt{25}$

$\sqrt{(x-3)^2} = \sqrt{25}$

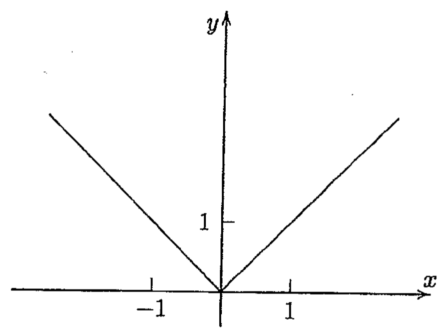
$|x-3| = 5$

$x = 8$ *or* $x = -2$

*Both solutions are valid in the original equation.*

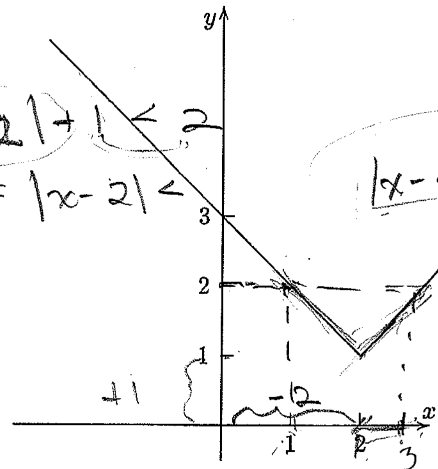
### 7.2.3 The Graph of the Absolute Value Function

In Comprehension Check 7.1 you were asked to draw the graph of $f(x) = |x|$. The sketch should have looked like this:

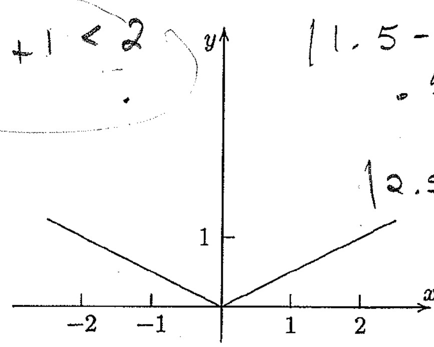



$y = |x|$

This is a good graph to add to your easily sketched, readily recognizable, "Family of functions." It can be shifted and scaled using the transformations that we learned in Chapter 4.



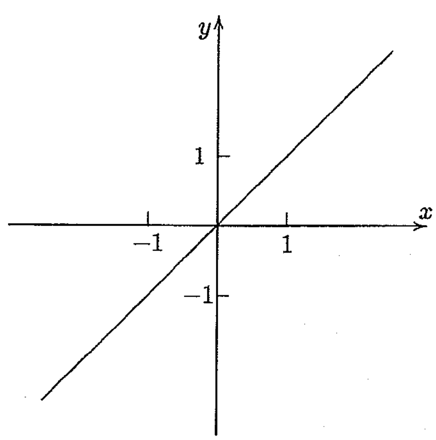

$y = |x - 2| + 1$



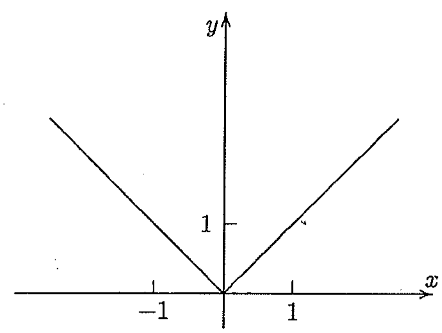

$y = \frac{1}{2}|x|$

When absolute value signs surround any function, $|g(x)|$, it has the effect of taking all of the $y$-values of $g$ and making them all non-negative.

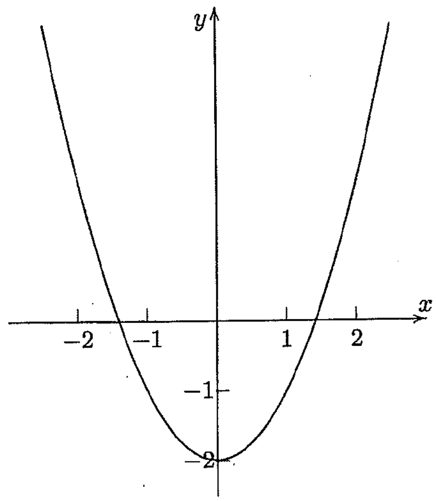
Consider the graphs below. Notice how the absolute value signs do not affect those $y$-values that are already positive or zero. However, all negative $y$-values are reflected about the $x$-axis and become positive.



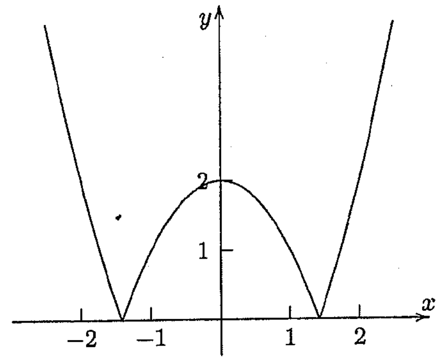

$y = x$



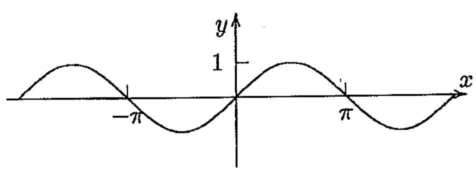

$y = |x|$



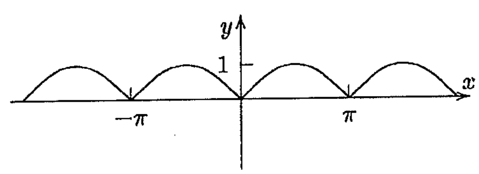

$y = x^2 - 2$




$y = |x^2 - 2|$




$y = \sin x$




$y = |\sin x|$

## 7.3 Absolute Value and Inequalities Combined

**Important Idea 7.3.1.**

1. *If $a > 0$, then $|x| < a$ if and only if $-a < x < a$.*
2. *If $a > 0$, then $|x| > a$ if and only if $[x < -a \text{ or } x > a]$.*

There are two ways to understand the statements in Important Idea 7.3.1. Both are useful. One should not be ignored in favor of the other.

A geometric interpretation is an extension of the geometric definition of absolute value (Definition 7.2.3). We think of $|x|$ as the distance from 0 to $x$ on a numberline. So $|x| < a$ would mean all numbers $x$ whose distance from 0 on a numberline is less than $a$. This would certainly be all of the numbers between $-a$ and $a$, in other words: $-a < x < a$. Similarly, $|x| > a$ would be the set of all numbers whose distance from 0 on the numberline is greater than $a$. Those numbers would

*So, our solution is:*

$$|x^3 - 5x^2 + 4x| = \begin{cases} -(x^3 - 5x^2 + 4x) & if \quad x < 0 \\ x^3 - 5x^2 + 4x & if \quad 0 \leq x \leq 1 \\ -(x^3 - 5x^2 + 4x) & if \quad 1 < x < 4 \\ x^3 - 5x^2 + 4x & if \quad x \geq 4 \end{cases}$$

## 7.4 Exercises

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**Problems for Section 7.1** 7.1 + 7.2 tonite 10/21

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**Problem 1.** Solve. Graph your solutions on a number line.

(a) $3x + 1 \geq 2 + x$ (b) $-1 < 2 - \frac{x}{3} \leq 1$ (c) $x^2 - x - 6 > 0$

(d) $(x+3)(x-2)^2(x-1)^4 < 0$ (e) $\frac{2x}{x-2} > 0$ (f) $\frac{2}{x} \leq \frac{x}{2}$

**Problem 2.** Solve. Write solutions in interval notation.

(a) $-1 < \frac{3-x}{2} \leq 1$ (b) $x^3 + 2x^2 - 4x - 8 \leq 0$ (c) $\frac{1}{x} \geq \frac{1}{x+3}$

(d) $\frac{x}{2} - \frac{8x}{3} + \frac{x}{4} > \frac{23}{6}$ (e) $x^2 - 2x + 1 \leq 0$

**Problem 3.** Solve. Write the solutions in algebraic notation.

(a) $-6x + 3 > x + 5$ (b) $x^4 - 16 < 0$ (c) $\frac{2x}{x-2} > 1$

(d) $\frac{-x^2 + x}{x+2} > -x + 3$ (e) $\frac{x+12}{x+2} - 3 \geq 0$ (f) $2^x(x-1) < 0$

**Problem 4.** Find the domain for each of the following functions:

(a) $f(x) = \sqrt{x^2 + 4x + 3}$ (b) $\sin^{-1}(1 - x^2)$

**Problem 5.** For what non-negative integers $n$ is it true that $\sum_{i=1}^{n} i < 465$?

**Problem 6.** Your friend Cletus wrote: "$\frac{x}{3x+8} > \frac{x-2}{5}$, and so $5x > (x-2)(3x+8)$." Please explain to him (kindly) why this is not correct.

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**Problems for Section 7.2**

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**Problem 1.** Find numbers $a$ and $b$ to show that $|a + b| \neq |a| + |b|$.

**Problem 2.** Find numbers $a$ and $b$ to show that $|a - b| \neq |a| - |b|$.

**Problem 3.** For what values of $x$ does $|3x-4|=3x-4$?
For what values of $x$ does $|3x-4|=-(3x-4)$?

**Problem 4.** Clive (Cletus' twin brother) wrote: "$|x-5|=x+5$." Explain to him (also kindly) why this is not correct.

**Problem 5.** Solve each of the following:

(a) $|2x|=x+1$ (b) $|-3x+6|=9x$ (c) $|2x-5|=9$

(d) $x-|x|=1$ (e) $|x-10|=x^2-10x$ (f) $\left|\dfrac{x+3}{2x-1}\right|=2$

(g) $|x^2+x|=|x-15|$ (h) $|1-2x|=3+|x+5|$ (i) $|x|^2+|x|-12=0$

**Problem 6.** Simplify:

(a) $\dfrac{|x+2|}{x+2}$ for $x\neq-2$ (b) $\dfrac{|x^2-4|}{|3x+6|}$ for $x\neq-2$

**Problem 7.** Express the following using absolute value:

(a) The distance between $x$ and 3 is 12.
(b) $x$ is four units away from 7.
(c) The distance between $2x$ and $-4$ is 1.
(d) $x$ is six units from the origin.

**Problem 8.** Graph the following: (a) $y=|x-3|+2$ (b) $y=|x^3-1|$ (c) $y=|-x|$
What does this last graph tell you about the function $f(x)=|x|$ concerning even/odd/neither?

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**Problems for Section 7.3**

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**Problem 1.** Express each of the following as an inequality statement involving absolute value:

(a) "$x$ is less than 6 units from 4 on a number line."
(b) "$y$ is at least 8 units from $-1$ on a number line."
(c) "$z$ is no more than 5 units from the origin of a number line."
(d) $-3\leq x\leq 3$ (e) $-4<x<10$
(f) $x<-4$ or $x>4$ (g) $x<2$ or $x>10$

**Problem 2.** Solve for $x$. Express your answer in algebraic notation. For (a) and (b) also sketch your solution on a number line.

(a) $|x+7|<5$ (b) $|1-2x|>5$ (c) $\left|1-\dfrac{2x}{3}\right|\leq 1$ (d) $\left|\dfrac{4-5x}{2}\right|\geq 1$

(e) $|\cos x-243|\leq-2$ (f) $|x|>x+1$ (g) $3|x+2|-5>10$ (h) $|x-1|>|3x-5|$

(i) $\dfrac{1}{|2x+7|}\leq\dfrac{1}{4}$ (j) $\left|\dfrac{x+3}{x-1}\right|\leq 3$ (k) $|x+3|+|x-2|>6$ (Hint: Check four cases).

**Problem 3.** Rewrite as a piecewise function: $f(x)=|x^2-4x-21|$ See Example 7.3.9.

**Problem 4.** Suppose $f(x)=x^2-x$ and $g(x)=x^2+3x-8$. For which values of $x$ is the vertical distance between the graphs of $f$ and $g$ less than 4?

## 7.5 Answers to Exercises

### Answers for Section 7.1 Exercises

**Answer to Problem 1.**

(a) $x \geq \frac{1}{2}$

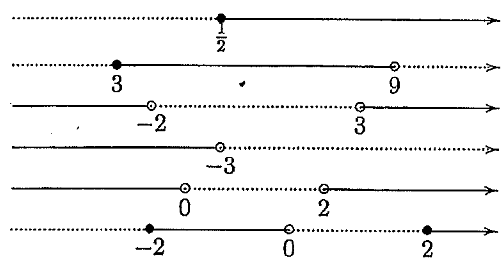
(b) $3 \leq x < 9$

(c) $x < -2$ or $x > 3$

(d) $x < -3$

(e) $x < 0$ or $x > 2$

(f) $-2 \leq x < 0$ or $x \geq 2$




**Answer to Problem 2.**

(a) $[1,5)$ (b) $(-\infty,2]$ (c) $(-\infty,-3)\cup(0,\infty)$ (d) $(-\infty,-2)$ (e) $[1]$

**Answer to Problem 3.**

(a) $x < -\frac{2}{7}$ (b) $-2 < x < 2$ (c) $x < -2$ or $x > 2$
(d) $x < -2$ (e) $-2 < x \leq 3$ (f) $x < 1$

**Answer to Problem 4.**

(a) $(-\infty,-3]\cup[-1,\infty)$ (b) $[-\sqrt{2},\sqrt{2}]$

**Answer to Problem 5.**
$1 \leq n \leq 29$

**Answer to Problem 6.**
We don't know the sign of $(3x+8)$ so we can't be sure which inequlity sign is valid.

### Answers for Section 7.2 Exercises

**Answer to Problem 1.**
Answers will vary.

**Answer to Problem 2.**
Answers will vary.

**Answer to Problem 3.**
(a) $x \geq \frac{4}{3}$ (b) $x \leq \frac{4}{3}$.

**Answer to Problem 4.**
$$|x-5| = \begin{cases} x-5 & if \quad x \geq 5 \\ -x+5 & if \quad x < 5 \end{cases}$$
$(x+5)$ is not always non-negative.

**Answer to Problem 5.**

(a) $1, -\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $-2, 7$ (d) no solution (e) $-1, 10$
(f) $-\frac{1}{5}, \frac{5}{3}$ (g) $-5, 3$ (h) $-\frac{7}{3}, 9$ (i) $-3, 3$

**Answer to Problem 6.**

(a) $\dfrac{|x+2|}{x+2} = \begin{cases} -1 & if \quad x < -2 \\ 1 & if \quad x > -2 \end{cases}$ (b) $\dfrac{|x^2-4|}{|3x+6|} = \dfrac{|x-2|}{3}$ if $x \neq -2$

**Answer to Problem 7.**

(a) $|x-3| = 12$ (b) $|x-7| = 4$ (c) $|2x+4| = 1$ (d) $|x| = 6$

**Answer to Problem 8.**
Graphs not shown. (b) even

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**Answers for Section 7.3 Exercises**

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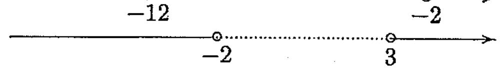
**Answer to Problem 1.**

(a) $|x-4| < 6$ (b) $|y+1| \geq 8$ (c) $|z| \leq 5$ (d) $|x| \leq 3$
(e) $|x-3| < 7$ (f) $|x| > 4$ (g) $|x-6| > 4$

**Answer to Problem 2.**

(a) $-12 < x < -2$ 


(b) $x < -2$ or $x > 3$ 


(c) $0 \leq x \leq 3$ (d) $x \leq \frac{2}{5}$ or $x \geq \frac{6}{5}$ (e) no solutions

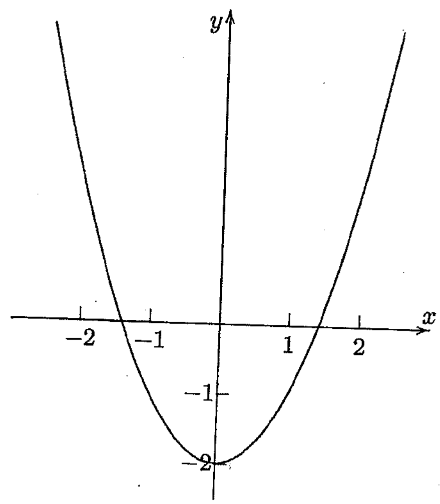
(f) $x < -\frac{1}{2}$ (g) $x < -7$ or $x > 3$ (h) $\frac{3}{2} < x < 2$

(i) $x \leq -\frac{11}{2}$ or $x \geq -\frac{3}{2}$ (j) $x \leq 0$ or $x \geq 3$ (k) $x < -\frac{7}{2}$ or $x > \frac{5}{2}$

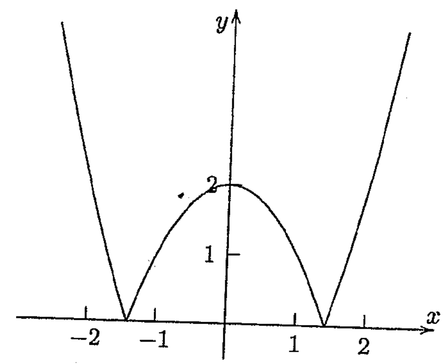
**Answer to Problem 3.**

$$|x^2-4x-21| = \begin{cases} x^2-4x-21 & if \quad x \leq -3 \\ -x^2+4x+21 & if \quad -3 < x < 7 \\ x^2-4x-21 & if \quad x \geq 7 \end{cases}$$

**Answer to Problem 4.**
$1 < x < 3$

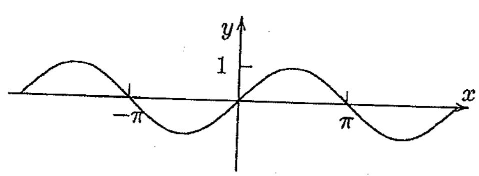



$y = x^2 - 2$

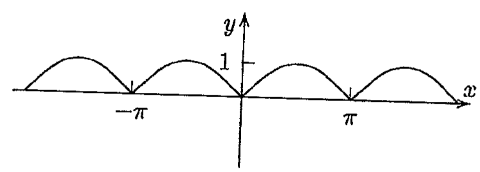



$y = |x^2 - 2|$

memorize




$y = \sin x$




$y = |\sin x|$

## 7.3 Absolute Value and Inequalities Combined

**Important Idea 7.3.1.**

1. *If $a > 0$, then $|x| < a$ if and only if $-a < x < a$.*
2. *If $a > 0$, then $|x| > a$ if and only if $[x < -a$ or $x > a]$.*

There are two ways to understand the statements in Important Idea 7.3.1. Both are useful. One should not be ignored in favor of the other.

A geometric interpretation is an extension of the geometric definition of absolute value (Definition 7.2.3). We think of $|x|$ as the distance from 0 to $x$ on a numberline. So $|x| < a$ would mean all numbers $x$ whose distance from 0 on a numberline is less than $a$. This would certainly be all of the numbers between $-a$ and $a$, in other words: $-a < x < a$. Similarly, $|x| > a$ would be the set of all numbers whose distance from 0 on the numberline is greater than $a$. Those numbers would be the positive numbers greater than $a$ and the negative numbers less than $-a$, in other words: $[x < -a \text{ or } x > a]$.

The second interpretation is an algebraic one. If $|x| < a$ then we are saying that if we ignore the sign, the *size* of the number $x$ is less than $a$. Certainly those would be the positive numbers less than $a$ and the negative numbers greater than $-a$, and of course 0. This combines to $-a < x < a$. On the other hand the set of numbers whose *size* is greater than $a$ would be the positive numbers greater than $a$ and the negative numbers less than $-a$. So $|x| > a$ means exactly $[x < -a \text{ or } x > a]$.

Notice that when the sign is "<" we get a single interval result. When the sign is ">" we get a disjoint set. This should make sense. For "greater than" we are looking at extreme values. For "less than" we are looking to set a boundary.

Geometrically then, we could interpret $|x| < 4$ as being all of the numbers on the number line whose distance from the origin is less than 4. This is the interval $(-4, 4)$. Algebraically we think of it as all of the numbers whose *size* is less than 4. This translates to $-4 < x < 4$.

Given $|x| > 4$ we think of all of those numbers on the number line whose distance from the origin is greater than 4. These would be the numbers on the ends of the number line, those lying beyond 4 to the right and beyond $-4$ to the left. This is the set $(-\infty, -4) \cup (4, \infty)$. Algebraically we think of $|x| > 4$ as all of the numbers whose *size* is greater than 4. This would be $[x < -4 \text{ or } x > 4]$.

Notice that if we combine the sets from $|x| < 4$ and $|x| > 4$ we get $(-4,4) \cup (-\infty,-4) \cup (4,\infty)$, which is the entire set of real numbers, except for the numbers $-4$ and 4. It may seem obvious, but it is worth the observation that $|x| < 4$ and $|x| > 4$ and $|x| = 4$ are disjoint (non-overlapping) sets which together encompass all of $\Re$. Again, you can think of this two ways. Geometrically, we think that the entire number line is covered by the union of these sets. Algebraically we think that every real number is included in one, and only one, of these sets. In other words, for every real number, its absolute value must be less than 4, equal to 4 or greater than 4.

In the previous section's study of absolute value we interpreted $|x - c|$ geometrically to mean "the distance between $x$ and $c$ on the number line." So, $|x+3| = 5$ meant that the distance between $x$ and $-3$ on the number line is 5. We solved that to say that $x$ had to be either 2 or $-8$. We carry this geometric interpretation to inequalities.

**Important Idea 7.3.2.**

1. *For $a > 0$, $|x - c| < a$ represents the set of numbers $x$ whose distance from $c$ on the number line is less than $a$.*
2. *For $a > 0$, $|x - c| > a$ represents the set of numbers $x$ whose distance from $c$ on the number line is greater than $a$.*

Our algebraic interpretations are directly applied from Important Idea 7.3.1: $|x - c| < a$ means $-a < (x - c) < a$, and $|x - c| > a$ translates to $[(x - c) < -a \text{ or } (x - c) > a]$.

**Example 7.3.1.**

*Solve for $x$: $|x - 3| < 8$*

*Geometric solution: We want the set of all numbers $x$ whose distance from 3 on the number line is less than 8. So, beginning at 3 we mentally go a distance 8 in either direction to get the solution* *set* $(-5, 11)$.

*Algebraic solution* $|x-3| < 8 \implies -8 < (x-3) < 8 \implies -5 < x < 11$.

**Comprehension Check 7.2.**

1. *Express in words the geometric interpretation of* $|x+2| > 6$ *and then solve for* $x$.
2. *Express algebraically* $|x+2| > 6$ *and then solve for* $x$.
3. *Express as an inequality that uses absolute value: "The set of all numbers* $x$ *whose distance from* 6 *on the number line is at least* 12*".*
4. *Express as an inequality that uses absolute value: "The set of all numbers* $x$ *whose distance on the number line is within* 4 *units of* $-9$*".*

We will not always have expressions as simple as "$(x-c)$" inside our absolute value signs. Any function of $x$ could be there. Algebraically, we still manipulate this using the idea of Important Idea 7.3.1:

For $a > 0$, $|f(x)| < a$ if and only if $-a < f(x) < a$.
For $a > 0$, $|f(x)| > a$ if and only if $f(x) < -a$ *or* $f(x) > a$.

In the study of calculus there are several places where mathematicians are interested in how close together the graphs of two functions are. For instance, if they want to find the values of $x$ for which the graphs of functions $f$ and $g$ are within some vertical distance $c$ of each other, they will set up the equation $|f(x) - g(x)| < c$. We know that the geometric interpretation of this equation is "the set of numbers $x$ for which the distance between $f$ and $g$ is less than $c$."

**Example 7.3.2.**

*Solve for* $x$ *and give a geometric interpretation:* $|2-5x| < 9$.

*We solve this algebraically:*

$$\begin{gathered} |2-5x| < 9 \\ -9 < 2-5x < 9 \\ -11 < -5x < 7 \\ \frac{11}{5} > x > -\frac{7}{5} \end{gathered}$$

*(note the sign change in the last step as we multiply by a negative number)*
*So, the solution set is all* $x$ *in the interval* $(-\frac{7}{5}, \frac{11}{5})$.

*For a geometric interpretation we can think of* $f(x) = 2$ *and* $g(x) = 5x$ *and so say that the graphs for the equations* $y = 5x$ *and* $y = 2$ *are within nine (vertical) units for any* $x$ *in the interval* $(-\frac{7}{5}, \frac{11}{5})$.

Actually, our geometric interpretation is not unique. We could think of $f(x) = 2 - 5x$ and $g(x) = 0$ and say that the graphs for $y = 2-5x$ and $y = 0$ (the $x$-axis) are within nine (vertical) units for any $x$ in the interval $(-\frac{7}{5}, \frac{11}{5})$.

Or, we could even think of $f(x) = 2 - 3x$ and $g(x) = 2x$ and say that the graphs for $y = 2-3x$ and $y = 2x$ are within nine (vertical) units for any $x$ in the interval $(-\frac{7}{5}, \frac{11}{5})$.

All of these interpretations fit the form $|f(x) - g(x)| < c$ for $c = 9$.

*If $x > 0$, the last statement can be rewritten as:*

$$-3x < 1 - x \text{ AND } 1 - x < 3x$$
$$-2x < 1 \text{ AND } 1 < 4x$$
$$x > -\frac{1}{2} \text{ AND } x > \frac{1}{4}$$

*So, if $x > 0$ AND $x > -\frac{1}{2}$ AND $x > \frac{1}{4}$ we simplify to $x > \frac{1}{4}$ is part of the solution to the original problem.*

*If $x < 0$, we have similar algebra, but must change the sign when multiplying by $x$:*

$$-3x > 1 - x \text{ AND } 1 - x > 3x$$
$$-2x > 1 \text{ AND } 1 > 4x$$
$$x < -\frac{1}{2} \text{ AND } x < \frac{1}{4}$$

*So, if $x < 0$ AND $x < -\frac{1}{2}$ AND $x < \frac{1}{4}$ we simplify to $x < -\frac{1}{2}$ is the other part of the solution to the original problem.*

*Our solution set is $(-\infty, -\frac{1}{2}) \cup (\frac{1}{4}, \infty)$.*

**Example 7.3.6.**

*For what values of $x$ are the graphs of the functions $f(x) = \dfrac{x+1}{2}$ and $g(x) = \dfrac{2x-1}{3}$ within one vertical unit of each other?*

*We need to solve the equation $|f(x) - g(x)| < c$. We use $f$ and $g$ as defined above, and $c = 1$. We get: $\left|\dfrac{x+1}{2} - \dfrac{2x-1}{3}\right| < 1$ and need to solve for $x$.*

$$\left|\frac{x+1}{2} - \frac{2x-1}{3}\right| < 1$$
$$6 \cdot \left|\frac{x+1}{2} - \frac{2x-1}{3}\right| < 6 \cdot 1$$
$$|3(x+1) - 2(2x-1)| < 6$$
$$|-x+5| < 6$$

*So, the solution set is $(-1, 11)$.*

**Example 7.3.7.**

*Solve for $x$: $\left|\dfrac{4-5x}{2}\right| \geq 1$*

$$\left|\frac{4-5x}{2}\right| \geq 1$$
$$2 \cdot \left|\frac{4-5x}{2}\right| \geq 2 \cdot 1$$
$$|4-5x| \geq 2$$
$$4 - 5x \geq 2 \quad \text{or} \quad 4 - 5x \leq -2$$

$$-5x \geq -2 \quad or \quad -5x \leq -6$$
$$x \leq \frac{2}{5} \quad or \quad x \geq \frac{6}{5}$$

*So, the solution set is* $(-\infty, \frac{2}{5}) \cup (\frac{6}{5}, \infty)$

**Example 7.3.8.**

*Solve for x:* $|2x-7| > |2-3x|$

*Solution 1: This is true for* $x = \frac{2}{3}$, *so include* $\frac{2}{3}$ *in the solution set. Then, for* $x \neq \frac{2}{3}$, *rewrite as* $\frac{|2x-7|}{|2-3x|} > 1$ *and solve in the manner as Example 7.3.5.*

*Solution 2: We use the squaring technique:*

$$|2x-7| > |2-3x|$$
$$|2x-7|^2 > |2-3x|^2$$
$$4x^2 - 28x + 49 > 4 - 12x + 9x^2$$
$$-5x^2 - 16x + 45 > 0$$
$$(x+5)(-5x+9) > 0$$

| $x$ | $(-\infty,-5)$ | $\{-5\}$ | $(-5,\frac{9}{5})$ | $\{\frac{9}{5}\}$ | $(\frac{9}{5},\infty)$ |
|---|---|---|---|---|---|
| $(x+5)$ | $-$ | 0 | $+$ | $+$ | $+$ |
| $(-5x+9)$ | $+$ | $+$ | $+$ | 0 | $-$ |
| $(x+5)(-5x+9)$ | $-$ | 0 | $+$ | 0 | $-$ |

*So, the solution set is the interval* $(-5, \frac{9}{5})$.

There will be times, particularly in calculus, where it is important to know the absolute value of a function, $|f(x)|$. From the definition of absolute value we can get:

$$|f(x)| = \begin{cases} -f(x) & if \quad f(x) < 0 \\ f(x) & if \quad f(x) \geq 0 \end{cases}$$

So, determining $|f(x)|$ essentially amounts to determining for which values of $x$ the function $f$ is positive and for which values of $x$ the function is negative. Then we just apply the appropriate sign.

**Example 7.3.9.**

*Rewrite* $|x^3 - 5x^2 + 4x|$ *as a piecewise defined function.*

$$|x^3-5x^2+4x| = \begin{cases} -(x^3-5x^2+4x) & if \quad (x^3-5x^2+4x) < 0 \\ x^3-5x^2+4x & if \quad (x^3-5x^2+4x) \geq 0 \end{cases}$$
$$= \begin{cases} -(x^3-5x^2+4x) & if \quad x(x-1)(x-4) < 0 \\ x^3-5x^2+4x & if \quad x(x-1)(x-4) \geq 0 \end{cases}$$

*To simplify the inequalities, we use the table method. Our division values are* 0, 1 *and* 4.

| $x$ | $(-\infty,0)$ | $\{0\}$ | $(0,1)$ | $\{1\}$ | $(1,4)$ | $\{4\}$ | $(4,\infty)$ |
|---|---|---|---|---|---|---|---|
| $x$ | $-$ | 0 | $+$ | $+$ | $+$ | $+$ | $+$ |
| $(x-1)$ | $-$ | $-$ | $-$ | 0 | $+$ | $+$ | $+$ |
| $(x-4)$ | $-$ | $-$ | $-$ | $-$ | $-$ | 0 | $+$ |
| $x(x-1)(x-4)$ | $-$ | 0 | $+$ | 0 | $-$ | 0 | $+$ |

*So, our solution is:*

$$|x^3-5x^2+4x| = \begin{cases} -(x^3-5x^2+4x) & \text{if } x<0 \\ x^3-5x^2+4x & \text{if } 0\le x\le 1 \\ -(x^3-5x^2+4x) & \text{if } 1<x<4 \\ x^3-5x^2+4x & \text{if } x\ge 4 \end{cases}$$

## 7.4 Exercises

**Problems for Section 7.1**

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**Problem 1.** Solve. Graph your solutions on a number line.

(a) $3x+1\ge 2+x$ (b) $-1<2-\frac{x}{3}\le 1$ (c) $x^2-x-6>0$

(d) $(x+3)(x-2)^2(x-1)^4<0$ (e) $\frac{2x}{x-2}>0$ (f) $\frac{2}{x}\le\frac{x}{2}$

**Problem 2.** Solve. Write solutions in interval notation.

(a) $-1<\frac{3-x}{2}\le 1$ (b) $x^3+2x^2-4x-8\le 0$ (c) $\frac{1}{x}\ge\frac{1}{x+3}$

(d) $\frac{x}{2}-\frac{8x}{3}+\frac{x}{4}>\frac{23}{6}$ (e) $x^2-2x+1\le 0$

**Problem 3.** Solve. Write the solutions in algebraic notation.

(a) $-6x+3>x+5$ (b) $x^4-16<0$ (c) $\frac{2x}{x-2}>1$

(d) $\frac{-x^2+x}{x+2}>-x+3$ (e) $\frac{x+12}{x+2}-3\ge 0$ (f) $2^x(x-1)<0$

**Problem 4.** Find the domain for each of the following functions:

(a) $f(x)=\sqrt{x^2+4x+3}$ (b) $\sin^{-1}(1-x^2)$

**Problem 5.** For what non-negative integers $n$ is it true that $\sum_{i=1}^{n} i<465$?

**Problem 6.** Your friend Cletus wrote: "$\frac{x}{3x+8}>\frac{x-2}{5}$, and so $5x>(x-2)(3x+8)$." Please explain to him (kindly) why this is not correct.

**Problems for Section 7.2**

**Problem 1.** Find numbers $a$ and $b$ to show that $|a+b|\ne|a|+|b|$.

**Problem 2.** Find numbers $a$ and $b$ to show that $|a-b|\ne|a|-|b|$.