

Focus Problems

Sec 18 p. 148 #2a, c, #3c

Sec 20 p. 168 #5e, f

Sec 21 p. 174 #3

Sec 22 p. 185 #2b, c, e, #4, #5, #9a

Sec 14 p. 121 #5, #6, #9, #11

Sec 18 p. 148 #2a, c, #3c

#2a $f(x) = \frac{1}{3}x^3 - 4x + 6$; $f(0) = 6$ (y-int)
 \downarrow \nearrow ends

Critical pts: $f'(x) = x^2 - 4 = 0$ at $x = \pm 2$ crit. pts

FDT: $f'(-3) = 5 > 0$, $f'(0) = -4 < 0$, $f'(3) = 5 > 0$
 $\overbrace{\quad \quad \quad}^{\text{f incr}} \quad \overbrace{\quad \quad \quad}^{\text{f decr}} \quad \overbrace{\quad \quad \quad}^{\text{f incr}}$

f increases on $(-\infty, -2) \cup (2, \infty)$

f decreases on $(-2, 2)$

SDT: $f''(x) = 2x$, $f''(-2) = -4 < 0$, conc. down, local max
 $f''(2) = 4 > 0$, conc. up, local min

POT?: $f''(x) = 2x = 0$ at $x = 0$.

$f''(-1) = -2 < 0$, $f''(1) = 2 > 0$

concave
down

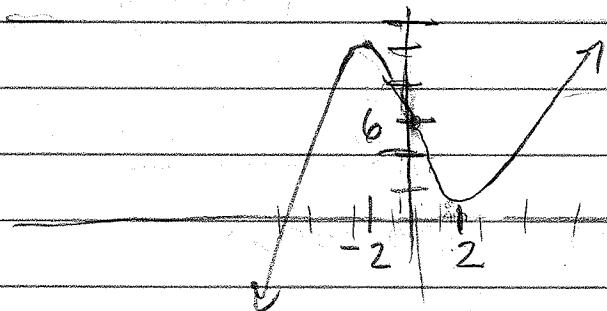
$$\begin{array}{c|c} & x=0 \\ \hline \text{concave down} & \text{concave up} \end{array}$$

POT

Note - $f''(x) = 0$ doesn't necessarily indicate there's a POI (recall $y = x^4$), but we can be confident where a local max and min occur on either side of the value; it's a POI.

Graph Mark the obvious points (y -int, local max, min, end behavior) and lightly sketch with a pencil before finalizing.

local max $f(-2) = \frac{1}{3}(-2)^3 - 4(-2) + 6 = \frac{34}{3} = 11\frac{1}{3}$
local min $f(2) = \frac{1}{3}(2)^3 - 4(2) + 6 = \frac{2}{3}$



The POI is also the y -int for this fcn.

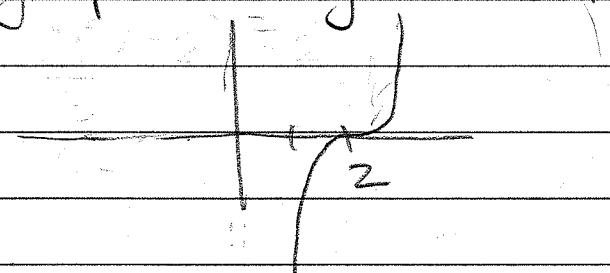
Scaling the graph is sometimes tricky. Don't spend too much time on it. I'll provide graph paper.

#2c $f(x) = (x-2)^3$

$f(0) = -8 \quad (\text{y-int})$

 $\leftarrow \rightarrow$ (ends)

In fact, this is an

easy graph — $y = x^3$ shifted right 2 units

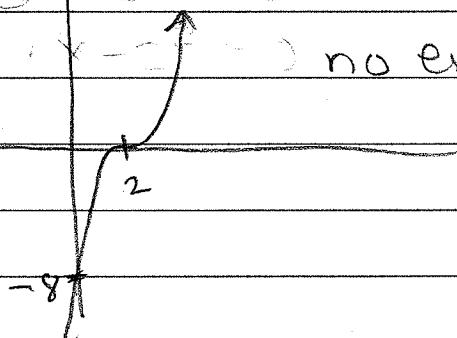
Supposing we don't see that (or even if we do) we proceed with the derivative tests.

Crit. pt: $f'(x) = 3(x-2)^2 = 0$ when $x=2$

FDT $f'(0) = 12 > 0$, $f'(3) = 3 > 0$ \rightarrow f increases
 f incr 2 f incr everywhere

SDT $f''(x) = 6(x-2) \leq 0$ at $x=2$ — Critical pt might be inflection pt

$f''(0) < 0$, $f''(3) > 0$ \rightarrow concave down \rightarrow concave up at $x=2$

SketchPOI $[x=2]$ 

no extreme pts.

$$\#3c. \quad f(x) = 2x^3 - 4x^2 + 2 \quad f(0) = 2 \quad y\text{-int}$$

↖ ↗ ends

Critical pts: $f'(x) = 6x^2 - 8x$

$$0 = 2x(3x - 4)$$

$$x=0, 4/3$$

FDT: $f'(-1) = 14$, $f'(1) = -2$, $f'(2) = 8$

f incr 0 decr $4/3$ incr

SDT: $f''(x) = 12x - 8$, $f''(0) = -8 < 0$

concave down, local max

$$f''(4/3) = 12(4/3) - 8 = 8 > 0$$

concave up, local min

$$f'' = 12x - 8 = 0 \text{ at } x = 8/12 = 2/3$$

Clearly $x = 2/3$ is a POI, but we.

Show by SDT anyway:

$$f''(0) = -8, \quad f''(1) = 4$$

c. down $2/3$ c. up.

↓
POI $x = 2/3$

local max $f(0) = 2(0)^3 - 4(0)^2 + 2 = 2$

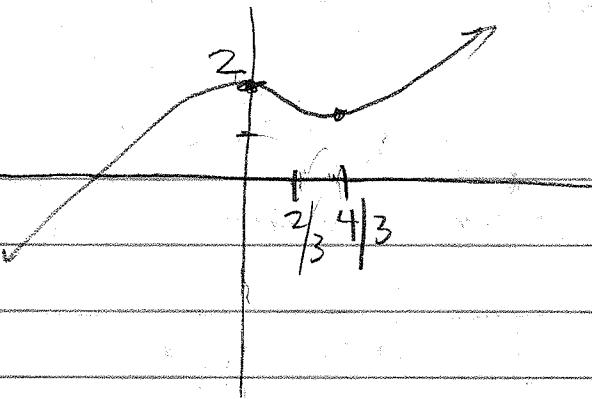
local min $f(4/3) = 2(4/3)^3 - 4(4/3)^2 + 2$
 $= 34/27 = 17/27$

→ (0, 2) (same as y-int)

($4/3, 17/27$)

(3)

#3c



See 20 p. 168 #5e, f.

5e. $f(x) = \frac{3x}{2x^2 - x - 1} = \frac{3x}{(2x+1)(x-1)}$

Dom: $x \neq -\frac{1}{2}, 1$ (from denom)y-int: $f(0) = 0$ (also, root)VA: $x = -\frac{1}{2}, x = 1$ (eqns)HA: deg numerator < deg denominator
so x-axis is the HA: $y = 0$

$$f'(x) = \frac{3(2x^2 - x - 1) - 3x(4x - 1)}{(2x^2 - x - 1)^2} = \frac{-6x^2 - 3}{(2x^2 - x - 1)^2}$$

Crit pt $f' = -6x^2 - 3 = 0$
 $x^2 = -3/6 \rightarrow$ no solution

Since $f' \neq 0$, the only crit pt would be where f' DNE. But these are the same as where f DNE.

Inspect around these values (the x of the VA) :

FDT

Notice $f'(x) = \frac{-6x^2 - 3}{\text{square}} = \frac{-(6x^2 + 3)}{\text{square}}$

$f'(-1) = \frac{-}{+} < 0$, $f'(0) = \frac{-}{+} < 0$, $f'(2) = \frac{-}{+} < 0$

$f \text{ decr.}$ at $-1/2$ $f \text{ decr}$ at $1/2$ $f \text{ decr}$

Apparently, f is decreasing on all intervals.

What about concavity?

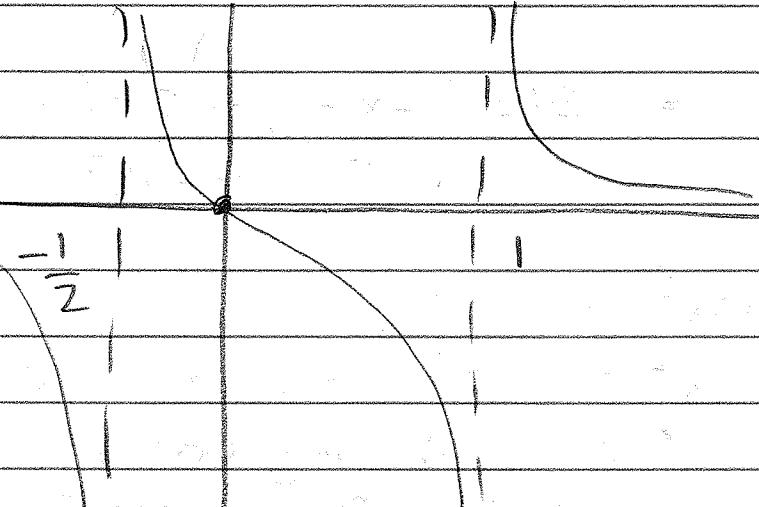
SDT + POT

$$f'' = \frac{-12x(2x^2 - x - 1) + (6x^2 + 3)(2)(2x^2 - x - 1)(4x - 1)}{\text{square}}$$

$$= \frac{-24x^3 + (2x^2 + 12x + \dots)}{\text{square}}$$

This is a nightmare

But graph it
from just
knowing it
decreases
everywhere
and the
asymptotes
and y-int.



P. 168 See 20

(4)

#3f $f(x) = \frac{4x+3}{x-2}$, VA: $x=2$, $f(2) = -\frac{3}{2}$
HA: $y=4$, $y=mx+b$

Deg top = Deg bottom, so HA = ratio of lead coeff.

$$f'(x) = \frac{4(x-2) - (4x+3)(1)}{(x-2)^2}$$

$$f'(x) = \frac{-11}{(x-2)^2} < 0 \text{ for all } x \in \text{dom.}$$

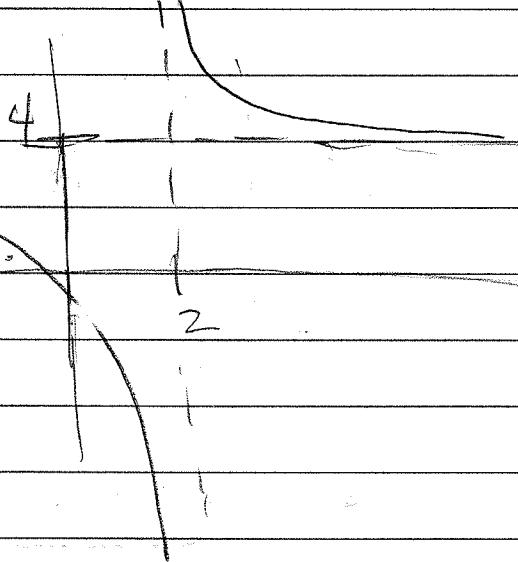
Write
 f' as
 $-11(x-2)^{-2}$

$$f''(x) = +22(x-2)^{-3} = \frac{+22}{(x-2)^3}$$

$$f''(0) < 0, f''(3) > 0$$

Concave down \downarrow + concave up \uparrow

So f is decreasing and concave up for all $x \in \text{Domain}$.



Sec 21 #3

$$f(x) = \frac{x^2}{1+x^2}$$

Dom: \mathbb{R}
HA: $y = 1$ (ratio of lead coeffs)

$$f'(x) = \frac{2x(1+x^2) - x^2(2x)}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$$

Crit pt $f'(x) = 0$ at $x=0$ crit pt.

FDT $f'(-1) < 0, f'(1) > 0$

f deer. < 0 f iner.

$$\Sigma DT \quad f''(x) = \frac{2(1+x^2)^2 - 2x(2)(1+x^2)(2x)}{(1+x^2)^4}$$

$$f'' = \frac{2(1+x^2) - 8x^2}{(1+x^2)^3} = \frac{2 - 6x^2}{(1+x^2)^3} = 0 \text{ at } x = \pm\sqrt{\frac{1}{3}}$$

In case you need detail, $2 - 6x^2 = 0$

$$2 = 6x^2$$

$$\frac{2}{6} = x^2$$

$$x = \pm\sqrt{\frac{2}{6}} = \pm\sqrt{\frac{1}{3}}$$

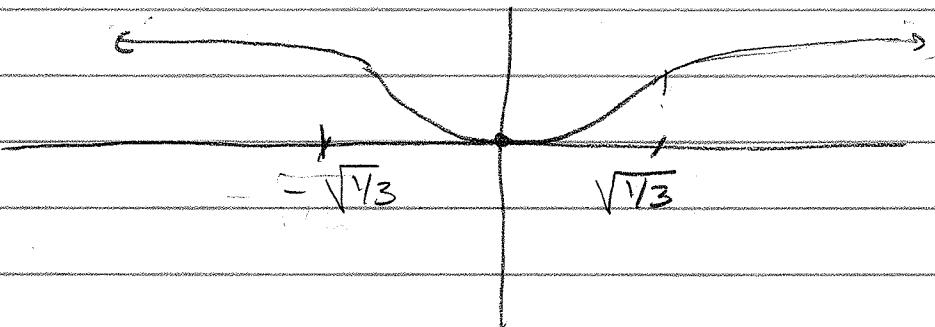
$$f''(-\sqrt{\frac{1}{3}}) < 0, f''(0) = 2 > 0, f''(\sqrt{\frac{1}{3}}) < 0$$

c. down $-\sqrt{\frac{1}{3}}$ c. up $\sqrt{\frac{1}{3}}$ c. down.

POI

P. 174. #3 Sketch

(5)



P. 185 See 22

#4

$$C(x) = 5x \quad P(x) = x(20-x)$$

$$5(20-x) - (20x - x^2) = 100 - 5x - 20x + x^2$$

$$P(x) = 100 - 25x + x^2$$

$$P'(x) = -25 + 2x = 0$$

$$x = 12.50$$

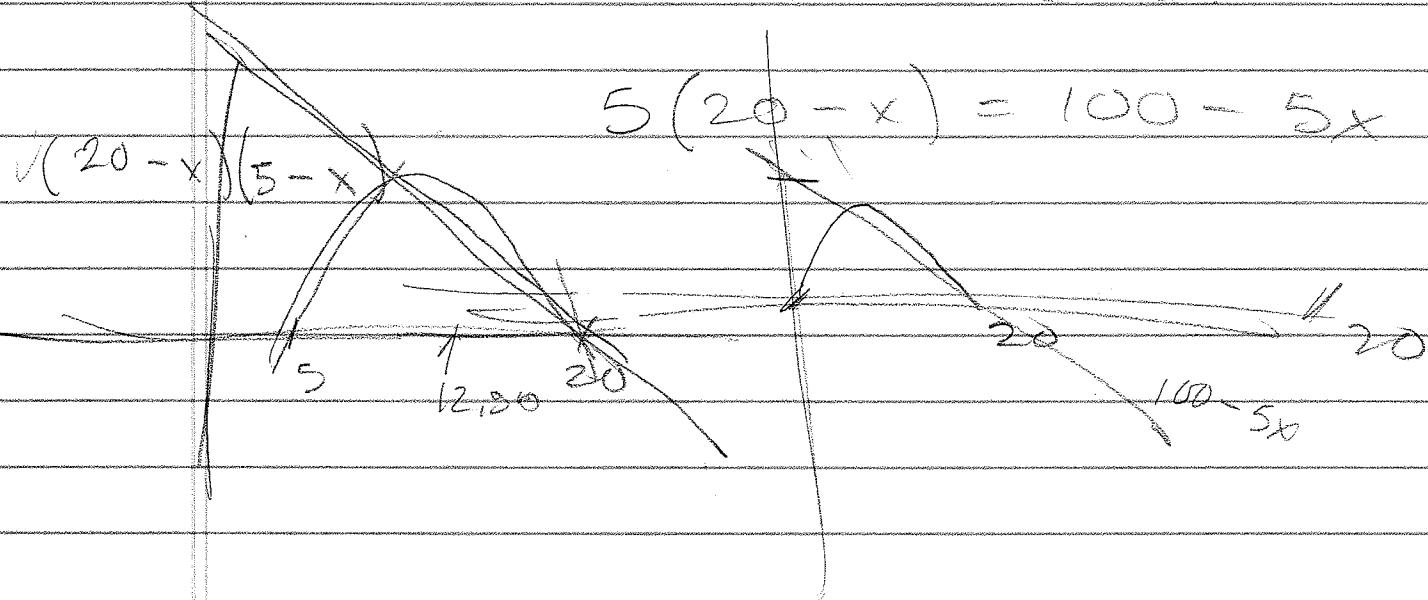
Does $x = 7.5$ represent a max?

By SDT, $P''(x) = -2$, negative for all x

so $P(x)$ is concave down at $x = 7.5$,

so it must be an absolute max

$$5(20-x) = 100 - 5x$$





P-121 #5

Known $c = 50\text{ ft}$

$\frac{da}{dt} = ?$ Find $\frac{da}{dt}$ at $b = 30\text{ ft}$

Known $\frac{db}{dt} = 10\text{ ft/sec}$

$$a^2 + b^2 = c^2 \quad \text{Pythag relation}$$

$$a^2 + 30^2 = 50^2$$

$$a = \sqrt{50^2 - 30^2} = 40\text{ ft}$$

Differentiate implicitly

with respect to t each term:

$$\frac{d}{dt}(a^2 + b^2 = 50) \quad \begin{array}{l} \text{Note that } c = 50 \\ \text{is substituted} \\ \text{first!} \end{array}$$

$$2a \frac{da}{dt} + 2b \frac{db}{dt} = 0$$

Fill in

what is known: $2(40) \frac{da}{dt} + 2(30)(10) = 0$

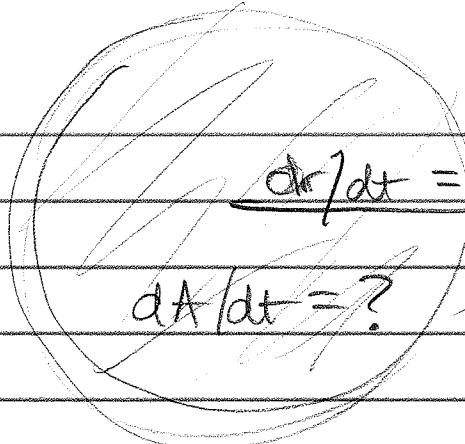
Solve

$$\text{for } \frac{da}{dt}: \quad 80 \frac{da}{dt} = -600$$

$$\frac{da}{dt} = -\frac{600}{80} = -\frac{15}{2} \text{ ft/sec}$$

That is, the ladder is sliding down
(negative) at $7\frac{1}{2}\text{ ft/sec}$.

#6 a)



$$\frac{dr}{dt} = 1.5 \text{ ft/sec}$$

$\frac{dA}{dt} = ?$ ~ Find $\frac{dA}{dt}$ at $r = 30 \text{ ft}$

Begin with relation $A = \pi r^2$

Differentiate implicitly with respect to t :

$$\left| \frac{dA}{dt} = 2\pi r \frac{dr}{dt} \right|$$

Fill in what you know:

$$\frac{dA}{dt} = 2\pi(30)(1.5) = 190\pi \frac{\text{ft}^2}{\text{sec}}$$

b) Find $\frac{dA}{dt}$ at $t = 2 \text{ min. (120 sec)}$

This is a bit more indirect. We know that $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. To find $\frac{dA}{dt}|_{t=2 \text{ min. (i.e. 120 sec)}}$ we need to find r at that time.

Since $\frac{dr}{dt} = 1.5 \text{ ft/sec}$,

$$\text{then } r = 120 \text{ sec} \cdot \left(\frac{1.5 \text{ ft}}{\text{sec}} \right) = 180 \text{ ft}$$

$$\#6b. \text{ Thus, } \frac{dA}{dt} \Big|_{t=2\text{ min}} = \frac{dA}{dt} \Big|_{r=180 \text{ ft}} \\ = 2\pi (180 \text{ ft}) (1.5 \text{ ft}^2/\text{sec}) \\ = 540\pi \text{ ft}^2/\text{sec}$$

#9 $N(p) = p^2 + 5p + 900$ people visiting ER
as a func. of population p .

Find $\frac{dN}{dt}$	given $\frac{dp}{dt} = 1.2$ (keeping units in terms of 1000's)
	$P = 20$ (20,000)

$$\left. \frac{dN}{dt} = \frac{dN}{dp} \cdot \frac{dp}{dt} = 2p \frac{dp}{dt} + 5 \frac{dp}{dt} \right.$$

Fill in what you know:

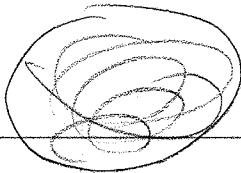
$$\left. \frac{dN}{dt} \right|_{P=20} = 2(20)(1.2) + 5(1.2)$$

$$= 54 \text{ people/yr}$$

- will visit the ER
when $P = 20$

6 yrs
48000
44000

11.



Melting spherical snowball

$$V_{\text{sphere}} = \frac{4}{3} \pi r^3$$

Find $\frac{d(\text{diam})}{dt}$ (the rate of diameter decrease)
at

if $\frac{dV}{dt} = 1 \text{ cm}^3$ and when diam = 10 cm
min

Express V in terms of diam $d = 2r$

$$V = \frac{4}{3} \pi \left(\frac{d}{2}\right)^3 = \frac{4\pi d^3}{24} = \frac{\pi d^3}{6}$$

$$\frac{dV}{dt} = \frac{3\pi d^2}{6} \cdot \frac{d(\text{diam})}{dt}$$

Fill in $\frac{1 \text{ cm}^3}{\text{min}} = \frac{3\pi}{6} (10 \text{ cm})^2 \frac{d(\text{diam})}{dt}$
known: $\frac{1 \text{ cm}^3}{\text{min}}$ $\frac{3\pi}{6} (10 \text{ cm})^2$ $\frac{d(\text{diam})}{dt}$

Solve for $\frac{d(\text{diam})}{dt} = \frac{1 \text{ cm}^3/\text{min}}{(100\pi/2) \text{ cm}^2}$

$\frac{d(\text{diam})}{dt} :$ $\frac{1}{(100\pi/2) \text{ cm}^2}$

$$\frac{d(\text{diam})}{dt} = \boxed{\frac{1}{50\pi} \frac{\text{cm}}{\text{min}}}$$

Sec 13 Implicit Differentiation

#3 Find eqn of line tangent to $y^2 - x^2 = 16$
at $(2, 2\sqrt{5})$:

$$2y \frac{dy}{dx} - 2x = 0 \quad \text{where } \frac{dy}{dx} = \text{slope } m$$

$$\frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}$$

Substitute pt. $\frac{dy}{dx} = \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}} = m$

Use pt-slope form of line:

$$y - y_1 = m(x - x_1)$$

$$y - 2\sqrt{5} = \frac{1}{\sqrt{5}}(x - 2)$$

or

$$y = \frac{1}{\sqrt{5}}x - \frac{2}{\sqrt{5}} - 2\sqrt{5}$$

$$= \frac{1}{\sqrt{5}}x - \left(\frac{2}{\sqrt{5}} + 2\sqrt{5}\right)$$

$$y = \frac{1}{\sqrt{5}}x + \frac{8}{\sqrt{5}}$$

#4 Same for $x^2 - xy + y^3 = 8$ at $(0, 2)$

$$\text{Slope: } 2x - \left(1y + x \frac{dy}{dx}\right) + 3y^2 \frac{dy}{dx} = 0$$

$$2x - y - x \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 0$$

$$2x - y = (x - 3y^2) \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x - y}{x - 3y^2} \rightarrow m = \frac{2(0) - 2}{0 - 3(2^2)}$$

$$\rightarrow m = \frac{-2}{-12} = \frac{1}{6}$$

$$\text{Eqn. } y - 2 = \frac{1}{6}(x - 0)$$

$$\boxed{y = \frac{1}{6}x + 2}$$

#5. $y + x\sqrt{y} = 8$ at $(2, 4)$

$$\frac{1}{x} \frac{dy}{dx} + \left(1 \cdot \sqrt{y} + x \cdot \frac{1}{2} y^{-\frac{1}{2}} \frac{dy}{dx}\right) = 0$$

$$\sqrt{y} = -\frac{dy}{dx} - \frac{x}{2y^{\frac{1}{2}}} \frac{dy}{dx}$$