

9/16 Sec 8 DifferentiationDef

The derivative of a fun. $f(x)$ on its domain is the limit of its difference quotient as $h \rightarrow 0$:

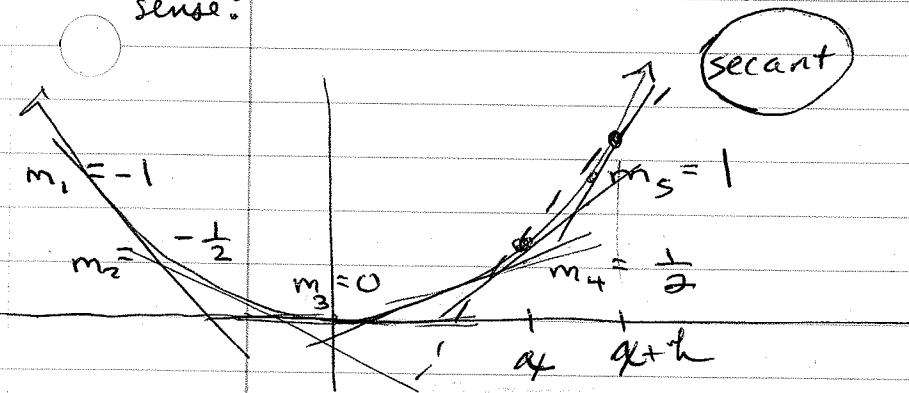
$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

For ex:

$f(x) = \frac{1}{x}$,
 $f'(x)$ at
 $x=0$
makes no
sense!

A derivative only exists where the function is defined!! So, we are careful to examine the natural domain.

That's why we say "the derivative of $f(x)$ for $x \in \text{dom } f(x)$ ".



ex $f(x) = \frac{1}{2}x^2$

on \mathbb{R} (the natural domain of $f(x)$)

Looking at various tangent lines to the curve, we get a sense of what the slope is, how it changes as we consider different x values.

If we just took the slope of the secant line between x and $x+h$, we'd have found the average rate of change of $f(x)$. This is not too useful, since it doesn't paint a picture

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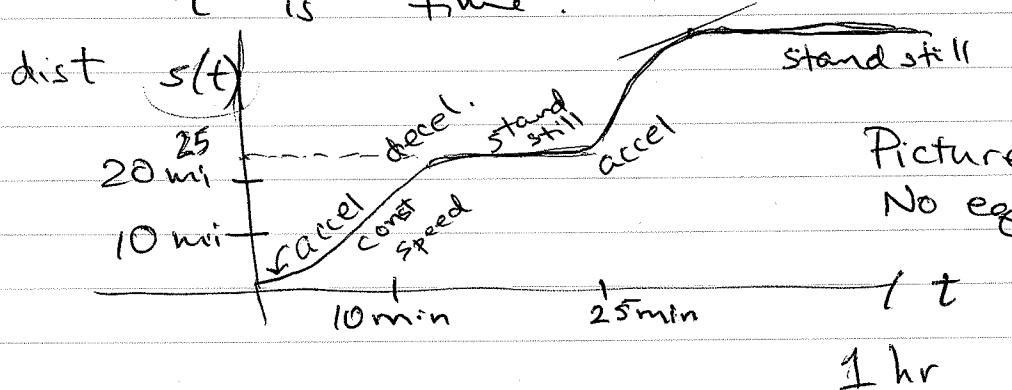
... paint a picture
of the fun's behavior at each
 x in the interval $(x, x+h)$.

Interval
of
1 hour
time
 t

For example, a vehicle can cover
40 miles in, say, 1 hour, so its
avg. speed is $\frac{40 \text{ mi}}{\text{hr}}$.

But at ~~any~~^t moment (time ~~t~~) in that
hour, it could have been at a stand-
still, or it might have been going
60 mi/hr. A graph of dist vs time
would give us the picture of the
vehicle's motion. Better, an eqn.
of dist as a fun. of time would
model the motion.

Distance $s(t)$ is the displacement fun
where s is dist. traveled and
 t is time.



Picture only.
No equation.

The derivative $s'(t) = \lim_{h \rightarrow 0} \frac{s(t+h) - s(t)}{h}$

for any time t in the interval $(0, 1)$

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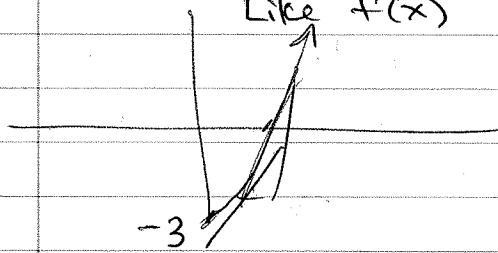
dist "s" as a fun of "t"

So another ex, this time with an egn.

$$s(t) = t^2 + 2t - 3 \quad t \geq 0 \text{ in min}$$

Like $f(x)$

s is dist



$$s(0) = -3 \text{ ft}$$

$$s(1) = 0 \text{ ft}$$

$$s(2) = 5 \text{ ft}$$

Interest: How fast does the object move at any instant, i.e., at any give time t.

Start with lim of DQ and then find a pattern, see a rule that is apparent in the answer, which we'll apply to any poly $x^n + x^{n-1} + \dots$

$$\star s(t+h) = (t+h)^2 + 2(t+h) - 3$$

$$\lim_{h \rightarrow 0} \frac{(t+h)^2 + 2(t+h) - 3 - (t^2 + 2t - 3)}{h}$$

$\nwarrow t+h - t$

$$= \lim_{h \rightarrow 0} \frac{2ht + h^2}{h} = \lim_{h \rightarrow 0} (2t + 2 + h)$$

$$= \lim_{h \rightarrow 0} 2t + \lim_{h \rightarrow 0} 2 + \lim_{h \rightarrow 0} h$$

$$= 2t + 2 + 0 = 2t + 2 = s'(t)$$

4

Take an inventory of the fens.

Given $s(t) = t^2 + 2t - 3$

found $s'(t) = 2t + 2$ (limit deal)

We see, perhaps, a rule for taking a derivative of a single term x^n

Aside $\lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} = ?$

requires a binomial coeff skill that we sadly lack now.

So, we'll just memorize the power rule for the $f'(x)$

when $f(x) = x^n$

$$f'(x) = nx^{n-1}$$

$$\text{Sec 16: } f(x) = x^n \rightarrow f'(x) = nx^{n-1}$$

$n \in \mathbb{R}$

$$\text{ex } f(x) = x^5 \Rightarrow f'(x) = 5x^4$$

so what is the slope of a line

tangent to $f(x) = x^5$ at $x = 1$?

$$f'(x) = 5x^4 \rightarrow f'(1) = 5(1)^4 = 5$$

Eqn. of line at $x_1 = 1, y_1 = 1$

tangent to $f(x) = x^5$

$$\text{Give } x_1 = 1, y_1 = 1 \quad y - y_1 = m(x - x_1)$$

$$y_1 = x_1^5 = 1^5 = 1 \quad y - 1 = 5(x - 1)$$

$$y = 5x - 5 + 1$$

$$\boxed{y = 5x - 4}$$

For problems that ask

"What is the eqn. of the line tangent to curve $f(x)$ at $x = a$?"

1. Find $f'(x)$.

2. Find $f'(a) = m$

3. Find $f(a) = y_1$

4. $y - y_1 = m(x - x_1)$

$$y - f(a) = f'(a) \cdot (x - a)$$

ex) $f(x) = \sqrt[3]{x} = x^{1/3}$

$$f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3} x^{-2/3}$$

What is eqn of line tangent to

$$f(x) = \sqrt[3]{x} \text{ at } x = -1$$

$$1. f'(x) = \frac{1}{3} x^{-2/3}$$

$$2. f'(-1) = \frac{1}{3} (-1)^{-2/3} = \cancel{\frac{1}{3} (-1)^{-2/3}}$$

$$= \frac{1}{3(-1)^{2/3}} = \frac{1}{3(-1)^{1/3})^2}$$

$$= \frac{1}{3(-1)^2} = \frac{1}{3} = m$$

$$3. x_1 = a = -1$$

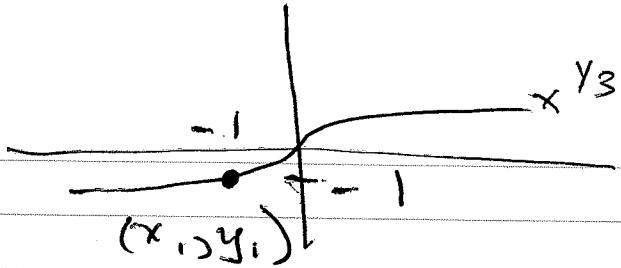
$$y_1 = f(x_1) = f(-1) = \sqrt[3]{-1} = -1$$

given as -1

$$4. y - y_1 = m(x - x_1)$$

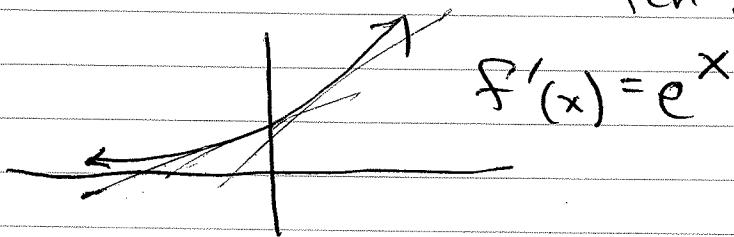
$$y - (-1) = \frac{1}{3}(x - -1)$$

$$y + 1 = \frac{1}{3}(x + 1)$$



Sec 10 - read all
- do #1-6, 8-10

Rule $f(x) = e^x$ exponential fn.



$$f(x) = e^x \quad (\text{constant})$$

$$f(2) = e^2 \quad (e \approx 2.718)$$

irrational

→ base for growth of human
many phenomena in nature

$$f'(2) = e^2 \quad \text{because}$$

$$f'(x) = e^x$$

The exponential fn is its own derivative.

$$e \in \mathbb{R}, \text{ so } f(x) = x^e$$

$$\text{then } f'(x) = ex^{e-1}$$

1. $\text{C}_6\text{H}_5\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_3$

2. $\text{C}_6\text{H}_5\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_3$

3. $\text{C}_6\text{H}_5\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_3$

4. $\text{C}_6\text{H}_5\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_3$

5. $\text{C}_6\text{H}_5\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_3$

($\text{C}_6\text{H}_5\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_3$) $_n$

1. $\text{C}_6\text{H}_5\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_3$

2. $\text{C}_6\text{H}_5\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_3$

3. $\text{C}_6\text{H}_5\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_3$

4. $\text{C}_6\text{H}_5\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_3$

5. $\text{C}_6\text{H}_5\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_3$

6. $\text{C}_6\text{H}_5\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_3$

7. $\text{C}_6\text{H}_5\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_2\text{CH}_3$

Power rule - very powerful -

$$f(x) = x^n \rightarrow f'(x) = nx^{n-1}, n \in \mathbb{R}$$

All your funcs are either polynomials, rational funcs, root funcs, exponential funcs, or some combination of these.

So, finding marginal funcs is often just a matter of applying power rule to each term.

$$\begin{aligned} C(x) &= x^2 - \sqrt{x} + 750 \\ &= x^2 - x^{1/2} + 750 \end{aligned}$$

$$C'(x) = 2x - \frac{1}{2}x^{-1/2} = \boxed{2x - \frac{1}{2\sqrt{x}}}$$

→ Domain $C(x)$: $x \geq 0$

$$\text{Fixed cost: } C(0) = 0^2 - \sqrt{0} + 750 = 750$$

$$R(x) = \$400x \quad R'(x) = 400x^0$$

$$P(x) = R(x) - C(x)$$

$$= 400x - x^2 + \sqrt{x} - 750$$

Break even: $R(x) = C(x)$, that is,

$$P(x) = 0$$

$$P(x) = 400x - x^2 + \sqrt{x} - 750 = 0$$

$$P'(x) = 400 - 2x + \frac{1}{\sqrt{x}}$$

See 10 #3

$$C(x) = 6x^2, R(x) = 8x, P(x) = \underline{8x - 6x^2}$$

$$C'(x) = 12x, R'(x) = 8, \underline{P'(x) = 8 - 12x}$$

Break even

$$P'(x) = 0$$

$$P(x) = 8x - 6x^2 = 0$$

$$P'(x) = 8 - 12x = 0$$

$$AB=0$$

$$(2x)(4 - 3x) = 0$$

$$A=0 \text{ or}$$

$$2x = 0, \text{ or } 4 - 3x = 0$$

$$B=0$$

$$\cancel{x=0}$$

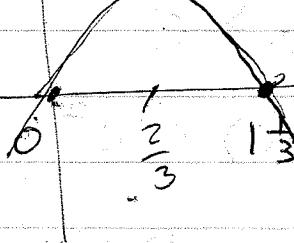
$$x = \frac{4}{3}$$

unit

$$P(x) = \$$$

say $\boxed{x=2}$ rounded up
to next integer

$2\frac{1}{3} \approx ?$



$$8 = 12x$$

$$\underline{8} = x$$

12

$$x = \frac{2}{3}$$

↓

After the $\frac{2}{3}$ item

(i.e., the first item)
 $x=1$
profit begins to fall

- You're still making a profit, but the return on your investment is falling!

Clean up points

#4 on Quiz $R(x) = \underline{15x}$ $R'(x) = 15$

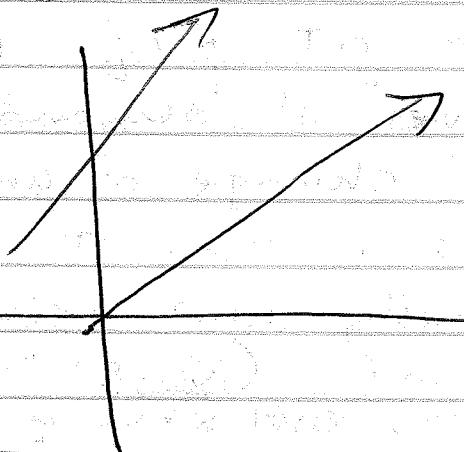
$$C(x) = \underline{20x + 320} \quad C'(x) = 20$$

$C(x)$ = cost to manufacturer

$+ C'(x)$ is cost of each new unit produced = \$20

$R(x)$ = ~~the~~ money coming in

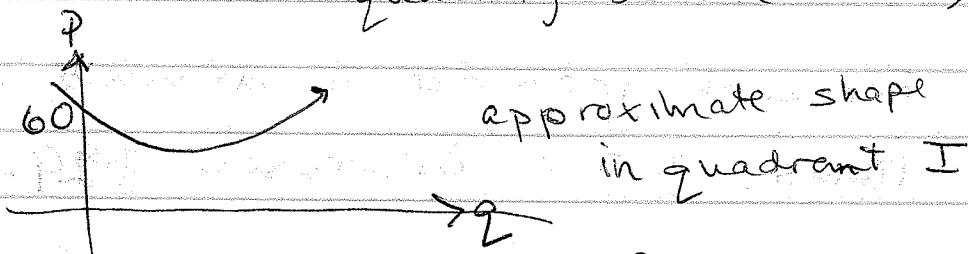
* $R'(x)$ is rev incr for ~~each~~ \rightarrow next unit sold = \$15



Increase the price to \$30

#4 in Sec 10

$p(q)$ is a price fn., q is quantity sold (demand)



$$P(q) = .1q^3 - .4q + 60$$

$p(0) = 60$ means if price were \$0 we'd "sell" 60 units

This fn. is a little odd because it treats price as the dependent variable; and we're more likely to see a demand fn. as $q(p)$, not $p(q)$.

Nevertheless since we begin pricing at ~~existing stock level~~ the scenario is not feasible.

Rather than analyze the usefulness of this model, we just follow the steps to get $p'(q)$ (marginal price at changing ~~some~~ level of demand), then answer rate of change of unit price (marginal price as a fn. of quantity sold, demand), and finally unit price at that level of demand. (Note: "unit price" is price per 1000 ~~and~~ since q is measured in thousands)

$$a) p'(q) = .3q^2 - .4$$

$$b) p'(10) = .3(10^2) - .4 = 300 - .40 = 29.60$$

For each increase in demand of 1000 items, price increases \$29.60

At $q=10$, the unit price $p(q)$ is $p(10) = .1(10^3) - .4(10) + 60 = 100 - 4 + 60 = \$156 / \text{thousand items}$

Review fcn. composition

Given fns. f, g , we find fns.

$f \circ g$ by:

$$f(g(x))$$

↓
first

$g \circ f$ by:
 $g(f(x))$

$$g \circ f \quad x \xrightarrow{f} f(x) \xrightarrow{g} g(f(x))$$

$$f \circ g \quad x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x))$$

$$\text{For ex } f(x) = x^8, \quad g(x) = x+1$$

$$f \circ g \equiv f(g(x)) = f(x+1) = (x+1)^8$$

$$g \circ f \equiv g(f(x)) = g(x^8) = x^8 + 1$$

Chain rule of differentiation:

$$(f \circ g)'(x) = [f(g(x))]' = f'(g(x))g'(x)$$

$$f(x) = x^{20} \quad g(x) = 2x - 9$$

$$(f \circ g)(x) = f(g(x)) = f(2x-9) = (2x-9)^{20}$$

So far, we can differentiate by product rule:

$$[f(g(x))]' \stackrel{?}{=} \text{Useful, Expand } (2x-9)^{20}$$

$$= (2x-9)(2x-9)(2x-9)\dots$$

= 3 days later ...

$$f(g(x)) = (2x+9)^{20} \quad \text{where } f(x) = x^{20}$$

$$g(x) = 2x+9$$

$$[f(g(x))]' = f'(g(x)) \cdot g'(x) = 20(2x+9)^{19} \cdot 2 \quad \text{power rule}$$

So $f(g(x))' = 40(2x+9)^{19}$

Ex $f(x) = \sqrt{3x^2 - 2} = (3x^2 - 2)^{\frac{1}{2}}$

$$[f(g(x))]' \quad f'(x) = ? = \frac{1}{2}(3x^2 - 2)^{-\frac{1}{2}} \cdot 6x$$

$$= f'(g(x))g'(x) \quad \text{same}$$

Final $\frac{3x}{(3x^2 - 2)^{\frac{1}{2}}} = \frac{3x}{\sqrt{3x^2 - 2}}$

Both forms
are fine

Ex $F(x) = e^{4x^2 - 2x + 6} \rightarrow \text{not an } n$

$$F'(x) = \cancel{4x^2 - 2x + 6}$$

Notice $e^{\cancel{4x^2 - 2x + 6}} = e^{\underline{g(x)}} = \underline{e^{f(g(x))}}$

where $f(x) = e^x$

Use chain rule on $e^{g(x)} = F(x)$

$$F'(x) = [e^{g(x)}]' \cdot g'(x) = (e^{4x^2 - 2x + 6})(8x - 2)$$

Easier ex $F(x) = (e^{3x})^7$

$$F'(x) = e^{3x} \cdot 3 = [3e^{3x}]$$

Your task is to recognize which diff'g rule is most efficient.

$$F(x) = \frac{(x+1)^7}{\sqrt{x}} \quad \text{no chain rule}$$

(1)

rewrite

$$(x+1)^7$$

~~$x^{1/2}$~~

quotient rule

$$\frac{7(x+1)^6 \cdot x^{1/2} - (x+1)^7 \left(\frac{1}{2}x^{-1/2}\right)}{(x^{1/2})^2}$$

$$\frac{(x+1)^6 7x^{1/2}}{2x} - \frac{(x+1)^7}{2x^{3/2}}$$

Horrible form
of an answer
using quotient
rule.

(2)

rewrite

$$\frac{(x+1)^7}{x^{1/2}} \rightarrow 7(x+1)^6$$

$$x^{1/2} \rightarrow \frac{1}{2}x^{-1/2}$$

$$(x+1)^7 (x^{-1/2})$$

prod. rule

$$7(x+1)^6 (x^{-1/2})$$

$$+ (x+1)^7 \left(-\frac{1}{2}x^{-3/2}\right)$$

$$\frac{7(x+1)^6}{x^{1/2}} - \frac{(x+1)^7}{2x^{3/2}}$$

This is
nicer.

∴ (2) rewrite for product rule
is our choice

Wisdom for today: A polynomial is often composed with a function resulting in the need to use the chain rule.

There are 2 scenarios: $[f(x)]^n \neq e^{f(x)}$

$$F(x) = (f(x))^n$$

$$F'(x) = n(f(x))^{n-1} f'(x)$$

$$F(x) = e^{f(x)}$$

$$F'(x) = e^{f(x)} \cdot f'(x)$$

Ex $F(x) = (2x^3 - 3x^2 + 1)^5$

~~$F'(x) = 5(2x^3 - 3x^2 + 1)^4 (6x^2 - 6x)$~~

Ex $F(x) = e^{12x-5}$

~~$F'(x) = e^{12x-5} (12)$~~

$$F'(x) = 5(2x^3 - 3x^2 + 1)^4 (6x^2 - 6x)$$

$$= [30x \cdot (x-1)(2x^3 - 3x^2 + 1)^4]$$

$$F'(x) = e^{12x-5} (12)$$

$$= [12e^{12x-5}]$$

Usually only the minimal, obvious simplification is sufficient.