Sec 8.7 Trig Identities - If possible, make a lot of sin, cos terms

#1 $\dfrac{\sec x \csc x}{\sec^2 x + \csc^2 x} =$ ~~sin x~~ ← Bad pen!

Rewrite in terms of sin x

$$= \left( \frac{\frac{1}{\cos x} \cdot \frac{1}{\sin x}}{\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}} \right) \cdot \left( \frac{\cos^2 x \sin^2 x}{\cos^2 x \sin^2 x} \right)$$ (LCD)

$$= \frac{\cos x \sin x}{\sin^2 x + \cos^2 x} = \frac{\cos x \sin x}{1}$$

$$= \boxed{\sqrt{1-\sin^2 x} \cdot \sin x}$$

#2 Rewrite in terms of cos x:

$$\frac{1}{1+\sin x} + \frac{1}{1-\sin x} \qquad LCD = 1-\sin^2 x$$

$$= \frac{1 - \cancel{\sin x} + 1 + \cancel{\sin x}}{1-\sin^2 x} = \boxed{\frac{2}{\cos^2 x}}$$

#3 Verify the identities - work on one side only since you cannot assume equality to prove equality

a) $\cos^2 x + \cos^2 x \tan^2 x \stackrel{?}{=} 1$

$\cos^2 x (1 + \tan^2 x) \stackrel{?}{=} 1$

$\cos^2 x \cdot \sec^2 x \stackrel{?}{=} 1$

$\cos^2 x \cdot \dfrac{1}{\cos^2 x} = 1$ ✓

b) $\cos x + \sin x \tan x \overset{?}{=} \sec x$

$$\frac{\cos x}{1} + \sin x \cdot \frac{\sin x}{\cos x} = \cos x + \frac{\sin^2 x}{\cos x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos x} = \frac{1}{\cos x} = \sec x$$

c) $(\sin x + \cos x)^2 \overset{?}{=} 1 + 2\sin x \cos x$

$$\underbrace{\sin^2 x + 2\sin x\cos x + \cos^2 x}_{=1} = 1 + 2\sin x \cos x$$

d) $\frac{1}{\sin x} - \sin x \overset{?}{=} \cot x \cos x$

$$\frac{1 - \sin^2 x}{\sin x} = \frac{\cos^2 x}{\sin x} = \frac{\cos x \cos x}{\sin x} = \cot x \cos x$$

e) $\frac{\sin x}{\csc x} + \frac{\cos x}{\sec x} \overset{?}{=} 1$

$$\sin x \cdot \sin x + \cos x \cdot \cos x$$

$$= \sin^2 x + \cos^2 x \overset{\checkmark}{=} 1$$

f) $\frac{\cos x - \sin x}{\cos x + \sin x} \overset{?}{=} \frac{\cot x - 1}{\cot x + 1}$

The $\cot x$ on right makes us think of $\cos x / \sin x$, which gives a motivation to divide each term on left by $\sin x$. if $\sin x \neq 0$.

$$\frac{\frac{\cos x}{\sin x} - \frac{\sin x}{\sin x}}{\frac{\cos x}{\sin x} + \frac{\sin x}{\sin x}} = \frac{\cot x - 1}{\cot x + 1} \Longleftarrow$$

g) $\frac{\cos^2 x}{\sin x} - 1 + \sin x \overset{?}{=} \frac{1 - \sin x}{\sin x}$

Get LCD on left ($\sin x$)

$$\frac{\cos^2 x - \sin x + \sin^2 x}{\sin x} = \frac{1 - \sin x}{\sin x}$$

h) $(\sec^2 x - 1)(\csc^2 x - 1) \stackrel{?}{=} 1$

$\tan^2 x \quad \cot^2 x = \tan^2 x \cdot \dfrac{1}{\tan^2 x} = 1$ ✓

i) $\dfrac{\sin x}{1+\cos x} + \cot x \stackrel{?}{=} \csc x$ LCD on right

$$\frac{\sin x + \cot x(1+\cos x)}{1+\cos x} = \frac{\sin x + \cot x + \cot x \cos x}{1+\cos x}$$

$$= \frac{\sin x + \frac{\cos x}{\sin x} + \frac{\cos x}{\sin x}\cdot \cos x}{1+\cos x}$$

Looking bad so far. Abandon ...

IDEA: Try the conjugate of $1+\cos x$!

$\otimes$ by $\dfrac{1-\cos x}{1-\cos x}$ if $\cos x \neq 1$

$$\frac{\sin x}{1+\cos x}\cdot\frac{1-\cos x}{1-\cos x} + \cot x = \frac{\sin x - \sin x\cos x}{1-\cos^2 x} + \frac{\cos x}{\sin x}$$

LCD $\sin^2 x$

$$= \frac{\sin x - \sin x\cos x}{\sin^2 x} + \frac{\cos x}{\sin x} = \frac{\sin x - \cancel{\sin x\cos x} + \cancel{\sin x\cos x}}{\sin^2 x}$$

$$= \frac{\sin x}{\sin^2 x} = \frac{1}{\sin x} = \csc x \checkmark$$

j) $\dfrac{\sin x + \tan x}{\sec x + 1} \stackrel{?}{=} \sin x$ where $\sec x \neq -1$

We started this in class by $\otimes$ by conjugate of $\sec x + 1$ top + bottom. This ends badly. Multiply top + bottom by $\cos x$ instead:

$$\frac{\sin x + \tan x}{\sec x + 1}\cdot\frac{\cos x}{\cos x} = \frac{\sin x\cos x + \sin x}{1+\cos x}$$

$(\cos x \neq 0)$

$\tan x \cos x = \sin x$

$\sec x \cos x = 1$

$$= \frac{\sin x\,\cancel{(\cos x + 1)}}{\cancel{(1+\cos x)}} = \sin x$$

k) $\dfrac{\sec x}{1-\tan x} + \dfrac{1+\tan x}{\sec x} \stackrel{?}{=} \dfrac{2\cos x}{1-\tan x}$

Try LCD $\sec x(1-\tan x)$

$$\frac{\sec^2 x + 1-\tan^2 x}{(1-\tan x)\sec x} = \frac{\tan^2 x + 1 + 1 - \tan^2 x}{(1-\tan x)\sec x}$$

$$= \frac{2}{\cancel{\sec x - \tan x \sec x}} = \frac{2}{\sec x(1-\tan x)} \stackrel{\checkmark}{=} \frac{2\cos x}{1-\tan x}$$

l) $\dfrac{\sin x}{\cot x+1} + \dfrac{\cos x}{\tan x+1} \stackrel{?}{=} \dfrac{1}{\sin x+\cos x}$

$$\frac{\sin x}{\frac{\cos x}{\sin x}+1} + \frac{\cos x}{\frac{\sin x}{\cos x}+1} = \frac{\sin^2 x}{\cos x+\sin x} + \frac{\cos^2 x}{\sin x+\cos x}$$

$\times$ by $\dfrac{\sin x}{\sin x}$ $\quad$ $\times$ by $\dfrac{\cos x}{\cos x}$

$$= \frac{\sin^2 x+\cos^2 x}{\cos x+\sin x} = \frac{1}{\cos x+\sin x} \checkmark$$

m) $\dfrac{1-\cos x}{\cos} \stackrel{?}{=} \dfrac{\tan^2 x}{1+\sec x}$

Let's work on the right, since we see an identity here.

$$\frac{\tan^2 x}{1+\sec x} = \frac{\sec^2 x - 1}{1+\sec x}$$

$$= \frac{(\sec x-1)\cancel{(\sec x+1)}}{\cancel{(\sec x+1)}}$$

$$= \frac{1}{\cos x} - 1 = \boxed{\frac{1-\cos x}{\cos x}} \checkmark$$

n) $\dfrac{\sin^3 x - 1}{\sin x - 1} \stackrel{?}{=} 2 + \sin x - \cos^2 x$

Difference of cubes: $\dfrac{\cancel{(\sin x - 1)}(\sin^2 x + \sin x + 1)}{\cancel{(\sin x - 1)}}$

$= (1 - \cos^2 x) + \sin x + 1$

$= 2 - \cos^2 x + \sin x$ ✓

o) $\underbrace{\sec^4 x - \tan^4 x}_{\text{Diff of squares}} \stackrel{?}{=} 2\tan^2 x + 1$

$(\sec^2 x + \tan^2 x)(\sec^2 x - \tan^2 x)$

$= (\sec^2 x + \tan^2 x)(1) = (\tan^2 x + 1) + \tan^2 x$

$= 2\tan^2 x + 1$ ✓

#4) Skipping for now

#5) Leave it !

#3 Going back to #3. We have to deal with any domain restrictions we introduced that the original didn't have.

For example, in #3i, the original:

$\dfrac{\sin x}{1+\cos x} + \cot x$ has a domain wherein $\boxed{\cos x \neq -1}$

+ $\cot x = \dfrac{\cos x}{\sin x}$, so $\boxed{\sin x \neq 0}$ →

When we then multiply by conjugate of $1+\cos x$, that is, $1-\cos x$ in the denominator, we introduce the restriction that $\cos x \neq 1$. To finish the proof of the identity

$$\frac{\sin x}{1+\cos x} + \cot x = \csc x$$

we have to consider the case where $\cos x = 1$, since we took this value out of the picture in performing the proof.

Go back to the original, letting $\cos x = 1$:

$\cot x = \frac{\cos x}{\sin x}$

$$\frac{\sin x}{1+1} + \frac{1}{\sin x}$$

$$= \frac{\sin x}{2} + \frac{1}{\sin x} \overset{?}{=} \csc x$$

However, if $\cos x = 1$, $\sin x = 0$, and this was a restriction on the original already, so it looks like we didn't introduce a new restriction after all!

#3 j is a better example. When we multiplied by $\sec x - 1$, we introduced a restriction that $\sec x \neq 1$, i.e., $\cos x \neq 1$ (why?) This means $\sin x \neq 0$ again, not a restriction on the original.

But it doesn't invalidate the identity, so the ~~proof~~ holds. Here are the details:-

**Orig** $\dfrac{\sin x + \tan x}{\sec x + 1}$ where $\sec x \neq -1 = \dfrac{1}{\cos x}$

i.e., $\cos x \neq -1$

During the proof, we ⊗ the numerator + denominator by $\sec x - 1$,

hence $\sec x \neq 1$, or $\dfrac{1}{\cos x} \neq 1$, i.e. $\begin{cases} \cos x \neq 1 \\ \sin x \neq 0 \end{cases}$

Substituting $\sec x = 1$, $\sin x = 0$. $\cos x = 1$ into original:

$$\frac{\sin x + \tan x}{\sec x + 1} = \frac{0 + 0/1}{1+1} = 0 \stackrel{?}{=} \sin x = 0 \checkmark$$

p. 268, Ex 8.7.6 explains further.