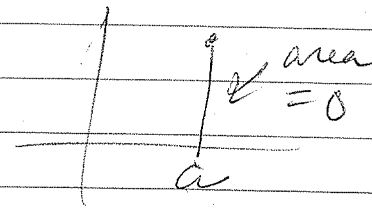
Sec 33 + 34 Facts and formulas to know

(1) $$\int_a^b f(x)\,dx = F(b) - F(a)$$

(2) $$\int_a^a f(x)\,\cancel{dx} = F(a) - F(a) = 0$$



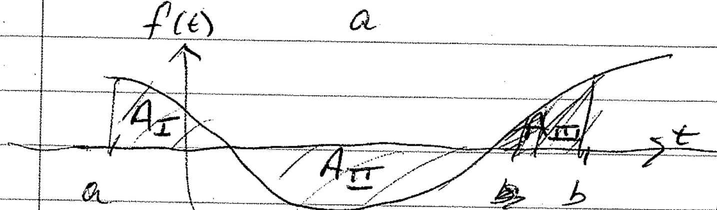
(3) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

(4) $\int_a^b f(x)\,dx$ makes no sense if a, b are not in dom f.

(5) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$ for any $c \in$ dom f

Fundamental Theorem of Calculus

(6) $$\int_a^b f(t)\,dt = \text{"signed area" of } f \text{ on } [a, b]$$



where, by signed area we mean the difference between areas of regions above t-axis & those below

$A_I + A_{III} - A_{II}$
= signed area

Alternately, $\int_1^x f'(t)\,dt = f(x)$ is another ~~statement of the FTC~~ form of the FTC

(7) Area enclosed by two function f & g:

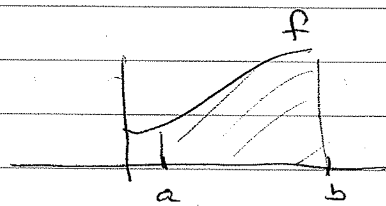
$$\int_a^b f(x)\,dx \quad - \quad \int_a^b g(x)\,dx$$

you won't always have a & b

where $f(a) = g(a)$ and $f(b) = g(b)$

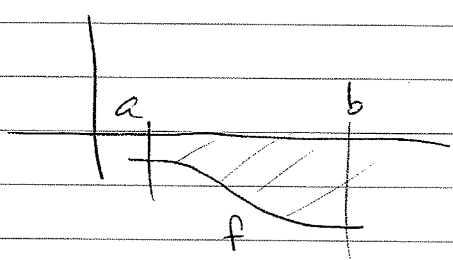
In other words, f & g intersect at $x = a$ & b.

More facts.

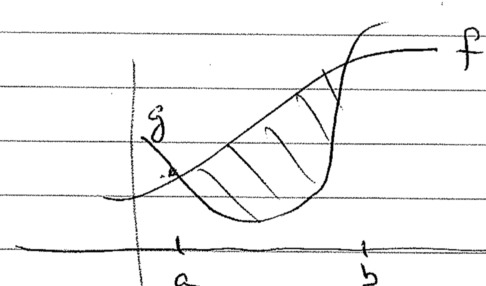


$\int_a^b f(x)\,dx$ = Area under f and above x-axis.

The integral = area when f lies above the x-axis on $[a, b]$



$\int_a^b f(x)\,dx$ = $-$ Area because f lies below the x-axis.



$\int_a^b f\,dx - \int_a^b g\,dx$ is the area enclosed by the curves.

Notice both f & g are above x-axis.
Also $f(a) = g(a)$ and $f(b) = g(b)$ as is needed.

When is area under f between a & b $= \int_a^b f\,dx$?

when $f > 0$ on $[a, b]$.

What do we call the definite integral $\int_a^b f\,dx$?

The "signed area"

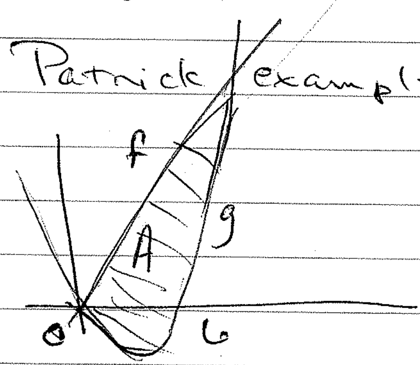
8 It turns out (a direct result of FTC is ~~this~~ corollary) that the area between two curves is simply

$$\int_a^b (f - g)\,dx$$

as long as $f \geq g$ on $[a, b]$.

In other words, the definite integral corresponds to the area btwn curves f, g when $f \geq g$ on $[a, b]$.

Patrick example

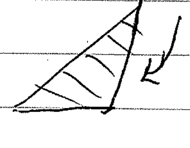


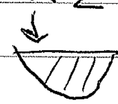
$$A = \int_0^6 2x - (x^2 - 4x)\,dx$$

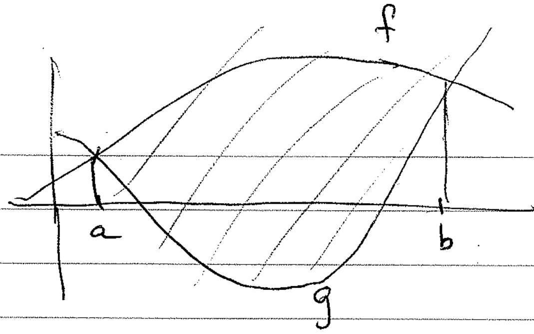
$$= \int_0^6 2x\,dx - \underbrace{\int_0^6 (x^2 - 4x)\,dx}_{\text{negative}}$$

negative b/c it's below the x-axis

The same as $A_1 + A_2$

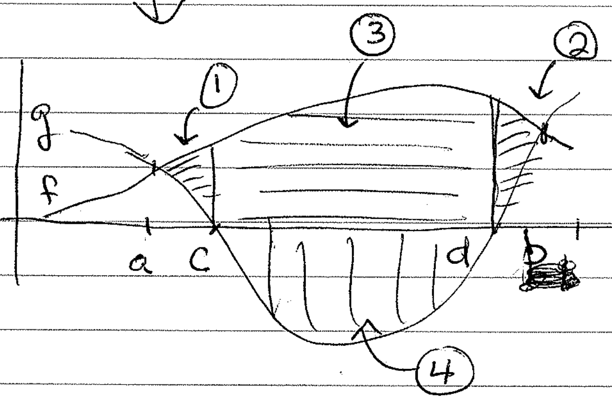






You can't find the area enclosed by f & g by merely subtracting the integrals because g is not above x-axis on all $[a, b]$

Solution



$A_{1} + A_{2} + A_{3} + A_{4}$ = area enclosed by f & g.

How do we represent each region with integrals?

$$A_{1} + A_{2} = \int_a^c (f-g)\,dx + \int_d^b (f-g)\,dx$$

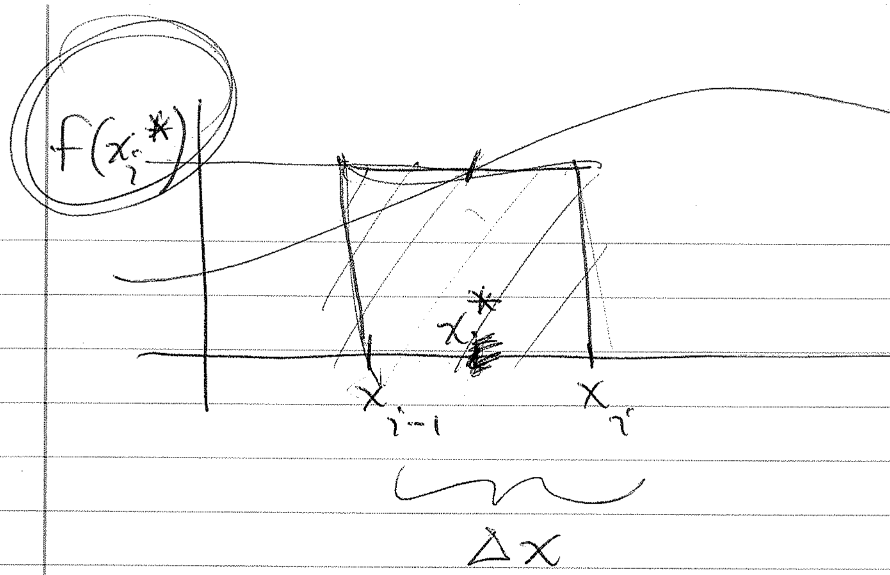
A_{3} is the area under f & above the x-axis.

$$A_{3} = \int_c^d f\,dx$$

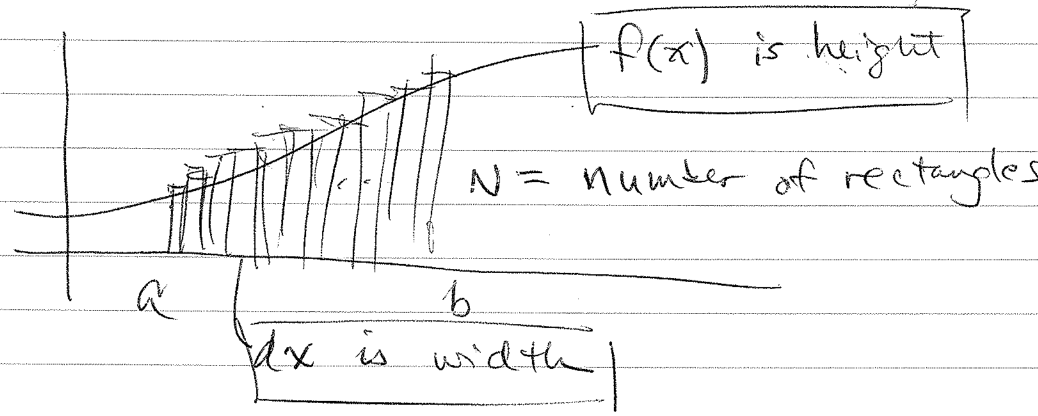
A_{4} is the ~~area~~ area above g and below x-axis.

As such $\quad A_{4} = -\int_c^d g\,dx \quad$ or $\quad \int_d^c g\,dx$

Notice the order of the limits of integration are switched.



Now we simplify this x_i^* notation in the process of making those rectangles thinner + thinner, until we have



Thus (9) $$\int_a^b f(x)\,dx = \lim_{N \to \infty} \Big(\underbrace{\sum_{i=1}^{N} f(x_i)\,\Delta x}_{\text{areas of rectangles}} \Big)$$

(10) The useful aspect of Riemann sum integration is in "average value of a fcn $f(x)$ on $[a, b]$"

Which is this:

$$\underset{\text{on } [a,b]}{\text{Avg value } f(x)} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

3 "Definitions" of the Integral ⓐⓑⓒ below

Idea

9 Fundamental Thm of Calculus & f cts. on [a, b]

as seen ⓐ $$\int_a^b f(x)\,dx = \text{signed area between } f(x) \text{ \& x-axis on } [a,b]$$

where "signed area" is difference of areas above & below x-axis.

another statement of FTC ⓑ $$\boxed{g(x)} = \int_a^x \boxed{f(t)\,dt}, \quad a \le x \le b$$

where $g'(x) = f(x)$ for $a < x < b$

We won't explore this statement of FTC further, but you need to see this truth.

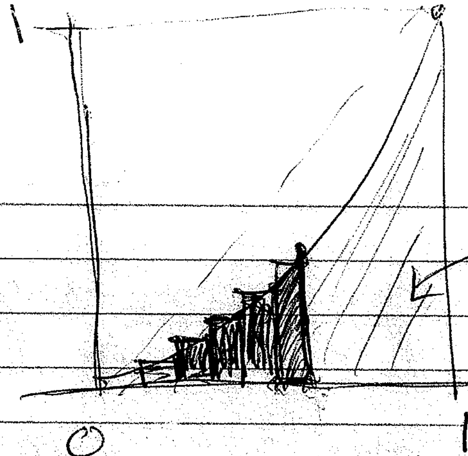
HARDEST CONCEPT TO GRASP

ⓒ $$\int_a^b f(x)\,dx = \lim_{N \to \infty} \left(\sum_{i=1}^{N} f(x_i)\Delta x \right)$$

The above requires f be continuous on [a, b]. What this statement says is that the signed are between f(x) & the x-axis on [a, b] is the "Riemann sum" of the rectangles of height "$y = f(x_i^*)$" and width [illegible]

"$x_i - x_{i-1}$" [illegible]

x_i^* is a value of x in $[x_{i-1}, x_i]$

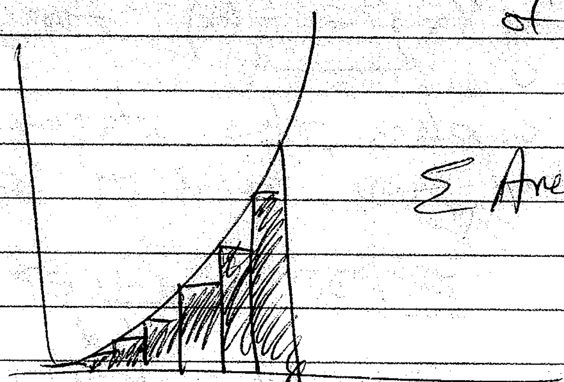


$y = x^2$

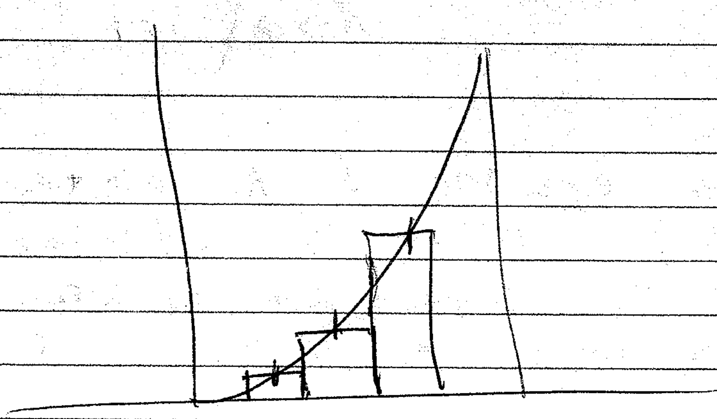
Area $< \frac{1}{2}$

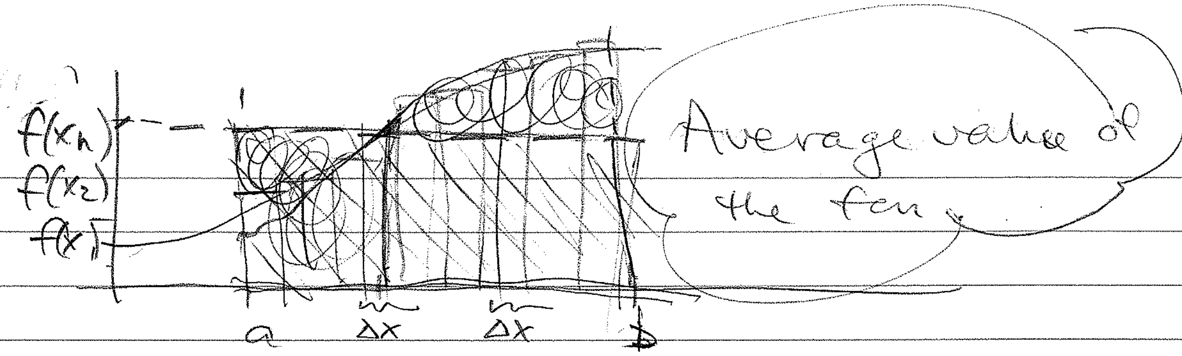
Σ Area rect $> \int$

$\int_0^1 x^2\,dx$ = Sum of many rectangle areas under curve, and the more rectangles you slice it up into, the closer your area gets the the sum of rectangles' areas.



Σ Area rect $< \int$





$$f_{avg} = \frac{f(x_1) + f(x_2) + \cdots + f(x_N)}{N}$$

$$= \frac{1}{N\Delta x}\left(f(x_1)\Delta x + \cdots + f(x_N)\Delta x\right)$$

But N of these Δx is $b-a$

and the more rectangles you use, the closer the sum gets to the integral.

Thus:

(10)

$$\lim_{N\to\infty} \frac{1}{N\Delta x}\sum_{i=1}^{N} f(x_i)\Delta x = \boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}$$

The big idea here is that somewhere on $[a,b]$ the fcn. takes on the average value of the ~~area of the rectangles~~ heights of all the rectangles, and the more rectangles, the better the approximation to this avg value via the integral $\frac{1}{b-a}\int_a^b f(x)\,dx$