Sec. 30 → the end — Integration!!

Consider the following notation:

As Dictated

$$F(x) = \int f(x)\,dx$$

The fcns. $F(x)$ & $f(x)$ in this equation are related. F is a fcn whose derivative is f.

That is, $\boxed{F'(x) = f(x)}$

We say $F(x)$ is the antiderivative of $f(x)$.

This relationship is predicated on $f(x)$ being continuous on its domain (in this case) and later, on its being cts. on an interval.

Notice that f itself does not have to be differentiable. And the symbol $\int$ says "integrate" or "antidifferentiate".

The antiderivative is also called the indefinite integral.

Later, note that $\int_a^b f(x)\,dx$, is called the definite integral.

So, as the book shows, a fcn. $F(x)$ is differentiated wrt x to give $f(x)$:

$F(x) = \int f(x)\,dx$

e.g. $F(x) = \dfrac{x^4}{4} + \dfrac{7x^2}{2} + x$ $\qquad \dfrac{x^4}{4} + \dfrac{7x^2}{2} + x = \int x^3 + 7x + 1\,dx$

soon

$F'(x) = \dfrac{\cancel{4}x^{\overset{4-1}{3}}}{\cancel{4}} + \dfrac{7 \cdot \cancel{2}x^{2-1}}{\cancel{2}} + 1x^{1-1}$ now

$= x^3 + 7x + 1 = f(x)$

But any fcn. $\boxed{F(x) + C}$, where C is constant, also has derivative $f(x)$:

e.g. $F(x) + C = \dfrac{x^4}{4} + \dfrac{7x^2}{2} + x + \underline{6}$

Recall: $(f+g)' = f' + g'$

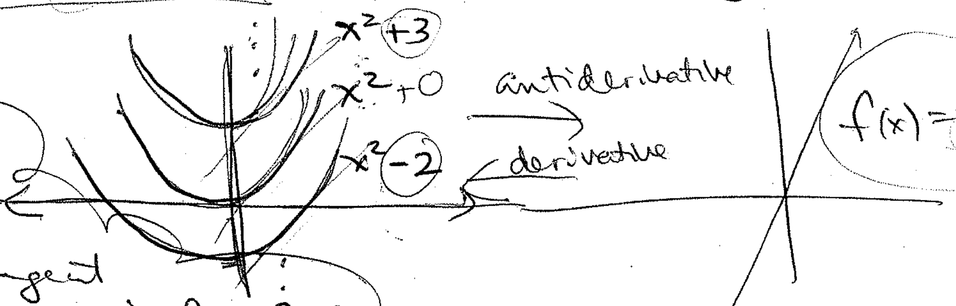
$(F(x) + C)' = F'(x) + C' = \underbrace{x^3 + 7x + 1}_{F'} + \underbrace{0}_{C'}$

$= f(x)$ still this, no difference

Thus, in $\underbrace{F(x) + C} = \int f(x)\,dx$, the left represents the <u>family</u> of antiderivatives of $f(x)$. "C" is called the <u>constant</u> of <u>integration</u>.

Ex $\boxed{F(x) = x^2 + C}$ is the family of

parabolas



The fcn that describes the slope of the tangent to F at $x \in$ dom is $f = 2x$

What is the rule that gets us from a polynomial to its antiderivative? It's this:

(for $f(x) = x^n$)

Power rule of integration

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

start with monomial

where $n \in \mathbb{R}$ not $= -1$

Ex $\int x^4 \, dx = \frac{x^{4+1}}{4+1} + C = \boxed{\frac{x^5}{5} + C}$ ← essential

The derivative of any fcn. in the family of fcns (or curves) $F(x) + C = \frac{x^5}{5} + C$

is x^4. C is the const. of $\int$.

In this rule, $n \neq -1$, which is reasonable, since $\frac{x^{n+1}}{n+1}$ would be $\frac{x^{-1+1}}{-1+1} = \frac{x^0}{0}$ undef

There are several rules of integration you need, one which handles $n = -1$.

Rules of Integration $\quad f, g$ continuous

1. $\int dx$, i.e., $\int 1\,dx$, i.e., $\int x^0\,dx$

$$\boxed{\int dx = x + C}$$

Makes sense, since $\int x^0\,dx = \dfrac{x^{0+1}}{0+1} + C = x + C$

k is const.

1a. $\int k\,dx = k\int dx = k(x + C) = kx + C$

Notice $kC = C$. We don't bother with distinguishing between $C + C$. We call all constants of integration C.

$$\boxed{\int k\,dx = kx + C}$$

k, C constants

C is a catchall of constants of $\int$

k const.

2. $\int k\,f(x)\,dx = k\int f(x)\,dx$,

3. $\int \left(f(x) \pm g(x)\right) dx = \int f(x)\,dx \pm \int g(x)\,dx$

4. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1$

Power rule for integration

4a. $\cancel{\int (x^m + x^n)\,dx}$

4a. $\int P(x)\,dx$, $P(x)$ is a polynomial

(from 3.)

$$= \int (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0)\,dx$$

$$= \int a_n x^n\,dx + \int a_{n-1}x^{n-1}\,dx + \cdots + \int a_0\,dx$$

Thus, the power rule $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

is applied ~~to~~ to each summand.

ex. $\int (x^4 - 2x^2 + 3x - 6)\,dx$

$$= \int x^4\,dx - \int 2x^2\,dx + \int 3x\,dx - \int 6\,dx$$

$$= \left(\frac{x^5}{5} + C_1\right) - \left(2\frac{x^3}{3} + C_2\right) + \left(\frac{3x^2}{2} + C_3\right) - (6x + C_4)$$

$$= \frac{x^5}{5} - \frac{2x^3}{3} + \frac{3x^2}{2} - 6x + C$$

the bunch of C_1, C_2, C_3, C_4 combined (summed)

5. Finally, what do we do with

$$\int x^{-1}\,dx \;?$$

This is given as (but not proven to be)

$$\boxed{f(x)=\frac{1}{x}} \qquad \int x^{-1}\,dx = \int \frac{1}{x}\,dx = \int \frac{dx}{x} \equiv \ln|x| + C$$

not part of antilog

The abs value on the log argument is needed to take in the entire domain of $f(x) = \frac{1}{x}$. Otherwise, only half the answer is shown $(\ln x + C)$.

Once we have more theory about integrals, in particular, the <u>definite integral</u> $\int_a^b f(x)\,dx$ as the area under the curve $f(x)$ between a & b, we can sketch out a motivation for this rule of integration.

$$\boxed{\int \frac{dx}{x} = \ln|x| + C}$$

6. One other integral (antiderivative)

$$\int e^x\,dx = e^x + C$$

This isn't surprising. $\frac{d}{dx}(e^x) = e^x$

and of course $\frac{d}{dx}(e^x + C) = e^x$

Thus, the family of antiderivatives

$$F(x) + C = e^x + C$$

result from taking the indefinite integral of the exponential fcn:

$$\int e^x\,dx = e^x + C$$

$\underline{Ex}$ $\int \frac{8dx}{x} = \int \frac{8}{x} dx = 8\ln|x| + C$

$\underline{Ex}$ $\int 7 dx = 7x + C$

$\underline{Ex}$ $\int \frac{dx}{4x} = \int \frac{1}{4} \cdot \frac{dx}{x} = \frac{1}{4} \int \frac{dx}{x}$ by 2.

$= \frac{1}{4} \ln|x| + C$

$\underline{Ex}$ $\int \left(\frac{1}{x} + \frac{1}{x^2} + x + 3 \right) dx$

by 3. $= \int \frac{1}{x} dx + \int \frac{1}{x^2} dx + \int x\, dx + \int 3 dx$

$= \int \frac{dx}{x} + \int x^{-2} dx + \int x^1 dx + \int 3 dx$

$= \ln|x| + \frac{x^{-2+1}}{-2+1} + \frac{x^{1+1}}{1+1} + 3x + C$

$= \ln|x| + \frac{x^{-1}}{-1} + \frac{x^2}{2} + 3x + C$

$= \boxed{\ln|x| - \frac{1}{x} + \frac{x^2}{2} + 3x + C}$

Ex $\int \frac{dx}{x^6} = \int x^{-6}\,dx$

$$= \frac{x^{-6+1}}{-6+1} + C$$

$$= \frac{x^{-5}}{-5} + C$$

$$= \boxed{-\frac{1}{5x^5} + C}$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{dx}{x^1} = \ln|x| + C$$

(from $\int x^{-1}\,dx$

$$\int k\,dx = kx + C$$

Ex

$$\int 4e^x \, dx = 4 \int e^x \, dx = 4e^x + 4C$$
$$= 4e^x + C$$

(you can skip the $4e^x + 4C$ step)

Ex

Finding a value for C depends on us having information on $F(x)$ at a given value of x. This information is called a boundary condition or initial condition

$$\begin{cases} F(x) + C = \int f(x)\,dx \\ F(x_0) = y_0 \end{cases}$$

Solving this system for C gives us the particular soln. of the family $F(x)+C$,