2.9 Linear Approximations and Differentials

1. $f(x) = x^3 - x^2 + 3 \;\Rightarrow\; f'(x) = 3x^2 - 2x$, so $f(-2) = -9$ and $f'(-2) = 16$. Thus,

$L(x) = f(-2) + f'(-2)(x - (-2)) = -9 + 16(x + 2) = 16x + 23.$

2. $f(x) = \sin x \;\Rightarrow\; f'(x) = \cos x$, so $f\left(\frac{\pi}{6}\right) = \frac{1}{2}$ and $f'\left(\frac{\pi}{6}\right) = \frac{1}{2}\sqrt{3}$. Thus,

$L(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) = \frac{1}{2} + \frac{1}{2}\sqrt{3}\left(x - \frac{\pi}{6}\right) = \frac{1}{2}\sqrt{3}\,x + \frac{1}{2} - \frac{1}{12}\sqrt{3}\,\pi.$

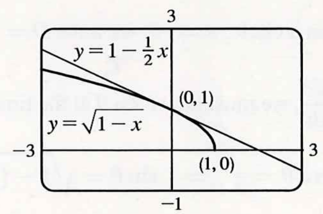
3. $f(x) = \sqrt{x} \;\Rightarrow\; f'(x) = \frac{1}{2}x^{-1/2} = 1/(2\sqrt{x})$, so $f(4) = 2$ and $f'(4) = \frac{1}{4}$. Thus,

$L(x) = f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4) = 2 + \frac{1}{4}x - 1 = \frac{1}{4}x + 1.$

4. $f(x) = 2/\sqrt{x^2 - 5} = 2(x^2 - 5)^{-1/2} \;\Rightarrow\; f'(x) = 2\left(-\frac{1}{2}\right)(x^2 - 5)^{-3/2}(2x) = -\dfrac{2x}{(x^2 - 5)^{3/2}}$, so $f(3) = 1$ and

$f'(3) = -\frac{3}{4}$. Thus, $L(x) = f(3) + f'(3)(x - 3) = 1 - \frac{3}{4}(x - 3) = -\frac{3}{4}x + \frac{13}{4}$.

5. $f(x) = \sqrt{1 - x} \;\Rightarrow\; f'(x) = \dfrac{-1}{2\sqrt{1 - x}}$, so $f(0) = 1$ and $f'(0) = -\frac{1}{2}$.



Therefore,

$\sqrt{1 - x} = f(x) \approx f(0) + f'(0)(x - 0) = 1 + \left(-\frac{1}{2}\right)(x - 0) = 1 - \frac{1}{2}x.$

So $\sqrt{0.9} = \sqrt{1 - 0.1} \approx 1 - \frac{1}{2}(0.1) = 0.95$

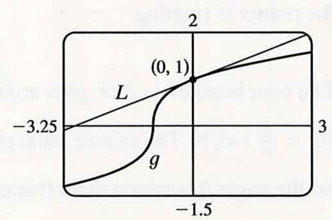
and $\sqrt{0.99} = \sqrt{1 - 0.01} \approx 1 - \frac{1}{2}(0.01) = 0.995$.

6. $g(x) = \sqrt[3]{1 + x} = (1 + x)^{1/3} \;\Rightarrow\; g'(x) = \frac{1}{3}(1 + x)^{-2/3}$, so $g(0) = 1$ and

$g'(0) = \frac{1}{3}$. Therefore, $\sqrt[3]{1 + x} = g(x) \approx g(0) + g'(0)(x - 0) = 1 + \frac{1}{3}x$.

So $\sqrt[3]{0.95} = \sqrt[3]{1 + (-0.05)} \approx 1 + \frac{1}{3}(-0.05) = 0.98\overline{3}$,

and $\sqrt[3]{1.1} = \sqrt[3]{1 + 0.1} \approx 1 + \frac{1}{3}(0.1) = 1.0\overline{3}$.

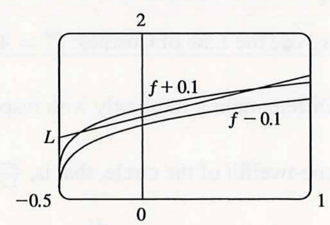


7. $f(x) = \sqrt[4]{1 + 2x} \;\Rightarrow\; f'(x) = \frac{1}{4}(1 + 2x)^{-3/4}(2) = \frac{1}{2}(1 + 2x)^{-3/4}$, so

$f(0) = 1$ and $f'(0) = \frac{1}{2}$. Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 1 + \frac{1}{2}x$.

We need $\sqrt[4]{1 + 2x} - 0.1 < 1 + \frac{1}{2}x < \sqrt[4]{1 + 2x} + 0.1$, which is true when

$-0.368 < x < 0.677$.

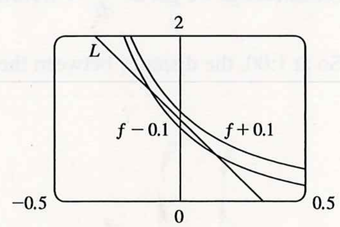


8. $f(x) = (1 + x)^{-3} \;\Rightarrow\; f'(x) = -3(1 + x)^{-4}$, so $f(0) = 1$ and

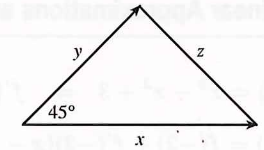
$f'(0) = -3$. Thus, $f(x) \approx f(0) + f'(0)(x - 0) = 1 - 3x$. We need

$(1 + x)^{-3} - 0.1 < 1 - 3x < (1 + x)^{-3} + 0.1$, which is true when

$-0.116 < x < 0.144$.



48. We are given that $\frac{dx}{dt} = 3$ mi/h and $\frac{dy}{dt} = 2$ mi/h. By the Law of Cosines,



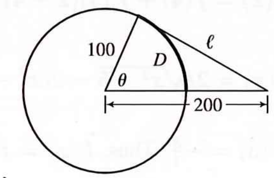
$z^2 = x^2 + y^2 - 2xy\cos 45° = x^2 + y^2 - \sqrt{2}\,xy \Rightarrow$

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} - \sqrt{2}\,x\frac{dy}{dt} - \sqrt{2}\,y\frac{dx}{dt}$. After 15 minutes $\left[= \frac{1}{4}\text{ h}\right]$,

we have $x = \frac{3}{4}$ and $y = \frac{2}{4} = \frac{1}{2} \Rightarrow z^2 = \left(\frac{3}{4}\right)^2 + \left(\frac{2}{4}\right)^2 - \sqrt{2}\left(\frac{3}{4}\right)\left(\frac{2}{4}\right) \Rightarrow z = \frac{\sqrt{13 - 6\sqrt{2}}}{4}$ and

$$\frac{dz}{dt} = \frac{2}{\sqrt{13 - 6\sqrt{2}}}\left[2\left(\tfrac{3}{4}\right)3 + 2\left(\tfrac{1}{2}\right)2 - \sqrt{2}\left(\tfrac{3}{4}\right)2 - \sqrt{2}\left(\tfrac{1}{2}\right)3\right] = \frac{2}{\sqrt{13 - 6\sqrt{2}}}\,\frac{13 - 6\sqrt{2}}{2} = \sqrt{13 - 6\sqrt{2}} \approx 2.125 \text{ mi/h}.$$

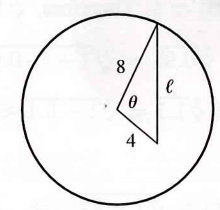
49. Let the distance between the runner and the friend be ℓ. Then by the Law of Cosines, $\ell^2 = 200^2 + 100^2 - 2 \cdot 200 \cdot 100 \cdot \cos\theta = 50{,}000 - 40{,}000\cos\theta$ $(\star)$. Differentiating implicitly with respect to t, we obtain $2\ell\frac{d\ell}{dt} = -40{,}000(-\sin\theta)\frac{d\theta}{dt}$. Now if D is the distance run when the angle is θ radians, then by the formula for the length of an arc on a circle, $s = r\theta$, we have $D = 100\theta$, so $\theta = \frac{1}{100}D \Rightarrow \frac{d\theta}{dt} = \frac{1}{100}\frac{dD}{dt} = \frac{7}{100}$. To substitute into the expression for $\frac{d\ell}{dt}$, we must know $\sin\theta$ at the time when $\ell = 200$, which we find from $(\star)$: $200^2 = 50{,}000 - 40{,}000\cos\theta \Leftrightarrow$



$\cos\theta = \frac{1}{4} \Rightarrow \sin\theta = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$. Substituting, we get $2(200)\frac{d\ell}{dt} = 40{,}000\frac{\sqrt{15}}{4}\left(\frac{7}{100}\right) \Rightarrow$

$d\ell/dt = \frac{7\sqrt{15}}{4} \approx 6.78$ m/s. Whether the distance between them is increasing or decreasing depends on the direction in which the runner is running.

50. The hour hand of a clock goes around once every 12 hours or, in radians per hour, $\frac{2\pi}{12} = \frac{\pi}{6}$ rad/h. The minute hand goes around once an hour, or at the rate of 2π rad/h. So the angle θ between them (measuring clockwise from the minute hand to the hour hand) is changing at the rate of $d\theta/dt = \frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$ rad/h. Now, to relate θ to ℓ, we use the Law of Cosines: $\ell^2 = 4^2 + 8^2 - 2 \cdot 4 \cdot 8 \cdot \cos\theta = 80 - 64\cos\theta$ $(\star)$.

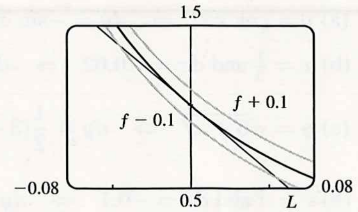


Differentiating implicitly with respect to t, we get $2\ell\frac{d\ell}{dt} = -64(-\sin\theta)\frac{d\theta}{dt}$. At 1:00, the angle between the two hands is one-twelfth of the circle, that is, $\frac{2\pi}{12} = \frac{\pi}{6}$ radians. We use $(\star)$ to find ℓ at 1:00: $\ell = \sqrt{80 - 64\cos\frac{\pi}{6}} = \sqrt{80 - 32\sqrt{3}}$.

Substituting, we get $2\ell\frac{d\ell}{dt} = 64\sin\frac{\pi}{6}\left(-\frac{11\pi}{6}\right) \Rightarrow \frac{d\ell}{dt} = \frac{64\left(\frac{1}{2}\right)\left(-\frac{11\pi}{6}\right)}{2\sqrt{80 - 32\sqrt{3}}} = -\frac{88\pi}{3\sqrt{80 - 32\sqrt{3}}} \approx -18.6$.

So at 1:00, the distance between the tips of the hands is decreasing at a rate of 18.6 mm/h ≈ 0.005 mm/s.

9. $f(x) = \dfrac{1}{(1+2x)^4} = (1+2x)^{-4} \quad \Rightarrow$

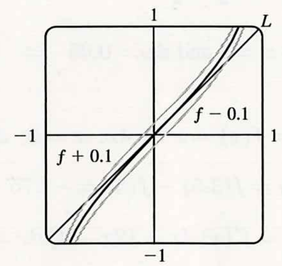


$f'(x) = -4(1+2x)^{-5}(2) = \dfrac{-8}{(1+2x)^5}$, so $f(0) = 1$ and $f'(0) = -8$.

Thus, $f(x) \approx f(0) + f'(0)(x-0) = 1 + (-8)(x-0) = 1 - 8x$.

We need $\dfrac{1}{(1+2x)^4} - 0.1 < 1 - 8x < \dfrac{1}{(1+2x)^4} + 0.1$, which is true when $-0.045 < x < 0.055$.

10. $f(x) = \tan x \quad \Rightarrow \quad f'(x) = \sec^2 x$, so $f(0) = 0$ and $f'(0) = 1$.



Thus, $f(x) \approx f(0) + f'(0)(x-0) = 0 + 1(x-0) = x$.

We need $\tan x - 0.1 < x < \tan x + 0.1$, which is true when $-0.63 < x < 0.63$.

11. (a) The differential dy is defined in terms of dx by the equation $dy = f'(x)\,dx$. For $y = f(x) = (x^2-3)^{-2}$,

$$f'(x) = -2(x^2-3)^{-3}(2x) = -\frac{4x}{(x^2-3)^3}, \text{ so } dy = -\frac{4x}{(x^2-3)^3}\,dx.$$

(b) For $y = f(t) = \sqrt{1-t^4}$, $f'(t) = \frac{1}{2}(1-t^4)^{-1/2}(-4t^3) = -\dfrac{2t^3}{\sqrt{1-t^4}}$, so $dy = -\dfrac{2t^3}{\sqrt{1-t^4}}\,dt$.

12. (a) For $y = f(u) = \dfrac{1+2u}{1+3u}$, $f'(u) = \dfrac{(1+3u)(2)-(1+2u)(3)}{(1+3u)^2} = \dfrac{-1}{(1+3u)^2}$, so $dy = \dfrac{-1}{(1+3u)^2}\,du$.

(b) For $y = f(\theta) = \theta^2 \sin 2\theta$, $f'(\theta) = \theta^2(\cos 2\theta)(2) + (\sin 2\theta)(2\theta)$, so $dy = 2\theta(\theta\cos 2\theta + \sin 2\theta)\,d\theta$.

13. (a) For $y = f(t) = \tan\sqrt{t}$, $f'(t) = \sec^2\sqrt{t}\cdot\frac{1}{2}t^{-1/2} = \dfrac{\sec^2\sqrt{t}}{2\sqrt{t}}$, so $dy = \dfrac{\sec^2\sqrt{t}}{2\sqrt{t}}\,dt$.

(b) For $y = f(v) = \dfrac{1-v^2}{1+v^2}$,

$$f'(v) = \frac{(1+v^2)(-2v)-(1-v^2)(2v)}{(1+v^2)^2} = \frac{-2v[(1+v^2)+(1-v^2)]}{(1+v^2)^2} = \frac{-2v(2)}{(1+v^2)^2} = \frac{-4v}{(1+v^2)^2},$$

so $dy = \dfrac{-4v}{(1+v^2)^2}\,dv$.

14. (a) For $y = f(t) = \sqrt{t-\cos t}$, $f'(t) = \frac{1}{2}(t-\cos t)^{-1/2}(1+\sin t) = \dfrac{1+\sin t}{2\sqrt{t-\cos t}}$, so $dy = \dfrac{1+\sin t}{2\sqrt{t-\cos t}}\,dt$.

(b) For $y = f(x) = \dfrac{1}{x}\sin x$, $f'(x) = \dfrac{1}{x}\cos x - \dfrac{1}{x^2}\sin x = \dfrac{x\cos x - \sin x}{x^2}$, so $dy = \dfrac{x\cos x - \sin x}{x^2}\,dx$.

15. (a) $y = \tan x \quad \Rightarrow \quad dy = \sec^2 x\,dx$

(b) When $x = \pi/4$ and $dx = -0.1$, $dy = [\sec(\pi/4)]^2(-0.1) = \left(\sqrt{2}\right)^2(-0.1) = -0.2$.

16. (a) $y = \cos \pi x \;\Rightarrow\; dy = -\sin \pi x \cdot \pi\, dx = -\pi \sin \pi x\, dx$

(b) $x = \frac{1}{3}$ and $dx = -0.02 \;\Rightarrow\; dy = -\pi \sin \frac{\pi}{3}(-0.02) = \pi\left(\sqrt{3}/2\right)(0.02) = 0.01\pi\sqrt{3} \approx 0.054.$

17. (a) $y = \sqrt{3 + x^2} \;\Rightarrow\; dy = \dfrac{1}{2}(3 + x^2)^{-1/2}(2x)\, dx = \dfrac{x}{\sqrt{3 + x^2}}\, dx$

(b) $x = 1$ and $dx = -0.1 \;\Rightarrow\; dy = \dfrac{1}{\sqrt{3 + 1^2}}(-0.1) = \dfrac{1}{2}(-0.1) = -0.05.$

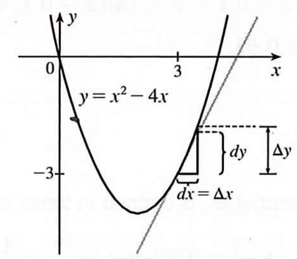
18. (a) $y = \dfrac{x + 1}{x - 1} \;\Rightarrow\; dy = \dfrac{(x - 1)(1) - (x + 1)(1)}{(x - 1)^2}\, dx = \dfrac{-2}{(x - 1)^2}\, dx$

(b) $x = 2$ and $dx = 0.05 \;\Rightarrow\; dy = \dfrac{-2}{(2 - 1)^2}(0.05) = -2(0.05) = -0.1.$

19. $y = f(x) = x^2 - 4x,\; x = 3,\; \Delta x = 0.5 \;\Rightarrow$

$\Delta y = f(3.5) - f(3) = -1.75 - (-3) = 1.25$

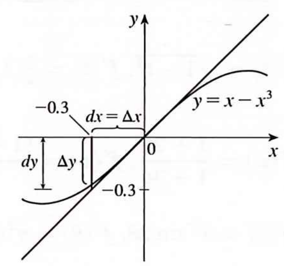
$dy = f'(x)\, dx = (2x - 4)\, dx = (6 - 4)(0.5) = 1$



20. $y = f(x) = x - x^3,\; x = 0,\; \Delta x = -0.3 \;\Rightarrow$

$\Delta y = f(-0.3) - f(0) = -0.273 - 0 = -0.273$

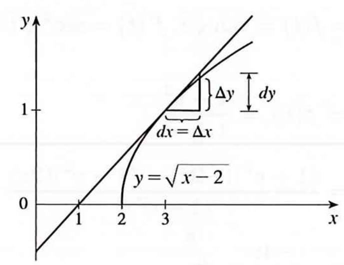
$dy = f'(x)\, dx = (1 - 3x^2)\, dx = (1 - 0)(-0.3) = -0.3$



21. $y = f(x) = \sqrt{x - 2},\; x = 3,\; \Delta x = 0.8 \;\Rightarrow$

$\Delta y = f(3.8) - f(3) = \sqrt{1.8} - 1 \approx 0.34$

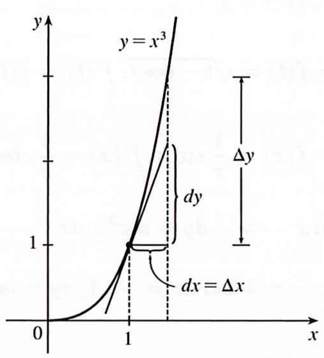
$dy = f'(x)\, dx = \dfrac{1}{2\sqrt{x - 2}}\, dx = \dfrac{1}{2(1)}(0.8) = 0.4$



22. $y = x^3,\; x = 1,\; \Delta x = 0.5 \;\Rightarrow$

$\Delta y = (1.5)^3 - 1^3 = 3.375 - 1 = 2.375.$

$dy = 3x^2\, dx = 3(1)^2(0.5) = 1.5$



23. To estimate $(1.999)^4$, we'll find the linearization of $f(x) = x^4$ at $a = 2$. Since $f'(x) = 4x^3$, $f(2) = 16$, and $f'(2) = 32$, we have $L(x) = 16 + 32(x - 2)$. Thus, $x^4 \approx 16 + 32(x - 2)$ when x is near 2, so $(1.999)^4 \approx 16 + 32(1.999 - 2) = 16 - 0.032 = 15.968$.

24. $y = f(x) = 1/x \quad \Rightarrow \quad dy = -1/x^2\,dx$. When $x = 4$ and $dx = 0.002$, $dy = -\frac{1}{16}(0.002) = -\frac{1}{8000}$, so $\frac{1}{4.002} \approx f(4) + dy = \frac{1}{4} - \frac{1}{8000} = \frac{1999}{8000} = 0.249875$.

25. $y = f(x) = \sqrt[3]{x} \quad \Rightarrow \quad dy = \frac{1}{3}x^{-2/3}\,dx$. When $x = 1000$ and $dx = 1$, $dy = \frac{1}{3}(1000)^{-2/3}(1) = \frac{1}{300}$, so $\sqrt[3]{1001} = f(1001) \approx f(1000) + dy = 10 + \frac{1}{300} = 10.00\overline{3} \approx 10.003$.

26. $y = f(x) = \sqrt{x} \quad \Rightarrow \quad dy = \frac{1}{2}x^{-1/2}\,dx$. When $x = 100$ and $dx = 0.5$, $dy = \frac{1}{2}(100)^{-1/2}\left(\frac{1}{2}\right) = \frac{1}{40}$, so $\sqrt{100.5} = f(100.5) \approx f(100) + dy = 10 + \frac{1}{40} = 10.025$.

27. $y = f(x) = \tan x \quad \Rightarrow \quad dy = \sec^2 x\,dx$. When $x = 0°$ [i.e., 0 radians] and $dx = 2°$ [i.e., $\frac{\pi}{90}$ radians], $dy = (\sec^2 0)\left(\frac{\pi}{90}\right) = 1^2\left(\frac{\pi}{90}\right) = \frac{\pi}{90}$, so $\tan 2° = f(2°) \approx f(0°) + dy = 0 + \frac{\pi}{90} = \frac{\pi}{90} \approx 0.0349$.

28. $y = f(x) = \cos x \quad \Rightarrow \quad dy = -\sin x\,dx$. When $x = 30°$ $[\pi/6]$ and $dx = -1°$ $[-\pi/180]$, $dy = \left(-\sin\frac{\pi}{6}\right)\left(-\frac{\pi}{180}\right) = -\frac{1}{2}\left(-\frac{\pi}{180}\right) = \frac{\pi}{360}$, so $\cos 29° = f(29°) \approx f(30°) + dy = \frac{1}{2}\sqrt{3} + \frac{\pi}{360} \approx 0.875$.

29. $y = f(x) = \sec x \quad \Rightarrow \quad f'(x) = \sec x\,\tan x$, so $f(0) = 1$ and $f'(0) = 1 \cdot 0 = 0$. The linear approximation of f at 0 is $f(0) + f'(0)(x - 0) = 1 + 0(x) = 1$. Since 0.08 is close to 0, approximating $\sec 0.08$ with 1 is reasonable.

30. $y = f(x) = \sqrt{x} \quad \Rightarrow \quad f'(x) = 1/(2\sqrt{x})$, so $f(4) = 2$ and $f'(4) = \frac{1}{4}$. The linear approximation of f at 4 is $f(4) + f'(4)(x - 4) = 2 + \frac{1}{4}(x - 4)$. Now $f(4.02) = \sqrt{4.02} \approx 2 + \frac{1}{4}(0.02) = 2 + 0.005 = 2.005$, so the approximation is reasonable.

31. (a) If x is the edge length, then $V = x^3 \quad \Rightarrow \quad dV = 3x^2\,dx$. When $x = 30$ and $dx = 0.1$, $dV = 3(30)^2(0.1) = 270$, so the maximum possible error in computing the volume of the cube is about 270 cm^3. The relative error is calculated by dividing the change in V, ΔV, by V. We approximate ΔV with dV.

Relative error $= \dfrac{\Delta V}{V} \approx \dfrac{dV}{V} = \dfrac{3x^2\,dx}{x^3} = 3\,\dfrac{dx}{x} = 3\left(\dfrac{0.1}{30}\right) = 0.01$.

Percentage error = relative error × 100% = 0.01 × 100% = 1%.

(b) $S = 6x^2 \quad \Rightarrow \quad dS = 12x\,dx$. When $x = 30$ and $dx = 0.1$, $dS = 12(30)(0.1) = 36$, so the maximum possible error in computing the surface area of the cube is about 36 cm^2.

Relative error $= \dfrac{\Delta S}{S} \approx \dfrac{dS}{S} = \dfrac{12x\,dx}{6x^2} = 2\,\dfrac{dx}{x} = 2\left(\dfrac{0.1}{30}\right) = 0.00\overline{6}$.

Percentage error = relative error × 100% = $0.00\overline{6}$ × 100% = $0.\overline{6}$%.

(b) From the graph, we see that $f'(x)$ is positive and decreasing. This means that the slopes of the tangent lines are positive, but the tangents are becoming less steep. So the tangent lines lie *above* the curve. Thus, the estimates in part (a) are too large.

42. (a) $g'(x) = \sqrt{x^2+5} \;\Rightarrow\; g'(2) = \sqrt{9} = 3$. $g(1.95) \approx g(2) + g'(2)(1.95-2) = -4 + 3(-0.05) = -4.15$.

$g(2.05) \approx g(2) + g'(2)(2.05-2) = -4 + 3(0.05) = -3.85$.

(b) The formula $g'(x) = \sqrt{x^2+5}$ shows that $g'(x)$ is positive and increasing. This means that the slopes of the tangent lines are positive and the tangents are getting steeper. So the tangent lines lie *below* the graph of g. Hence, the estimates in part (a) are too small.

LABORATORY PROJECT Taylor Polynomials

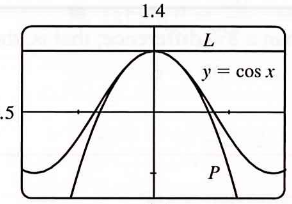
1. We first write the functions described in conditions (i), (ii), and (iii):

$$P(x) = A + Bx + Cx^2 \qquad f(x) = \cos x$$
$$P'(x) = B + 2Cx \qquad f'(x) = -\sin x$$
$$P''(x) = 2C \qquad f''(x) = -\cos x$$

So, taking $a = 0$, our three conditions become

$$P(0) = f(0): \quad A = \cos 0 = 1$$
$$P'(0) = f'(0): \quad B = -\sin 0 = 0$$
$$P''(0) = f''(0): \quad 2C = -\cos 0 = -1 \;\Rightarrow\; C = -\tfrac{1}{2}$$

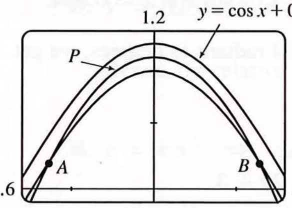
The desired quadratic function is $P(x) = 1 - \frac{1}{2}x^2$, so the quadratic approximation is $\cos x \approx 1 - \frac{1}{2}x^2$.



The figure shows a graph of the cosine function together with its linear approximation $L(x) = 1$ and quadratic approximation $P(x) = 1 - \frac{1}{2}x^2$ near 0. You can see that the quadratic approximation is much better than the linear one.

2. Accuracy to within 0.1 means that $\left|\cos x - \left(1 - \frac{1}{2}x^2\right)\right| < 0.1 \;\Leftrightarrow\; -0.1 < \cos x - \left(1 - \frac{1}{2}x^2\right) < 0.1 \;\Leftrightarrow$

$0.1 > \left(1 - \frac{1}{2}x^2\right) - \cos x > -0.1 \;\Leftrightarrow\; \cos x + 0.1 > 1 - \frac{1}{2}x^2 > \cos x - 0.1 \;\Leftrightarrow\; \cos x - 0.1 < 1 - \frac{1}{2}x^2 < \cos x + 0.1$.



From the figure we see that this is true between A and B. Zooming in or using an intersect feature, we find that the x-coordinates of B and A are about ± 1.26. Thus, the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ is accurate to within 0.1 when $-1.26 < x < 1.26$.

3. If $P(x) = A + B(x-a) + C(x-a)^2$, then $P'(x) = B + 2C(x-a)$ and $P''(x) = 2C$. Applying the conditions (i), (ii), and (iii), we get

$$P(a) = f(a): \quad A = f(a)$$
$$P'(a) = f'(a): \quad B = f'(a)$$
$$P''(a) = f''(a): \quad 2C = f''(a) \quad \Rightarrow \quad C = \tfrac{1}{2}f''(a)$$

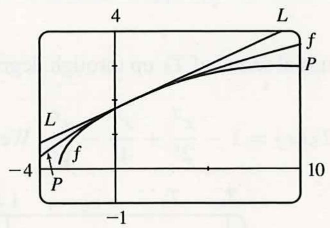
Thus, $P(x) = A + B(x-a) + C(x-a)^2$ can be written in the form $P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$.

4. From Example 2.9.1, we have $f(1) = 2$, $f'(1) = \frac{1}{4}$, and $f'(x) = \frac{1}{2}(x+3)^{-1/2}$.

So $f''(x) = -\frac{1}{4}(x+3)^{-3/2} \quad \Rightarrow \quad f''(1) = -\frac{1}{32}$.

From Problem 3, the quadratic approximation $P(x)$ is

$\sqrt{x+3} \approx f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 = 2 + \frac{1}{4}(x-1) - \frac{1}{64}(x-1)^2$.



The figure shows the function $f(x) = \sqrt{x+3}$ together with its linear approximation $L(x) = \frac{1}{4}x + \frac{7}{4}$ and its quadratic approximation $P(x)$. You can see that $P(x)$ is a better approximation than $L(x)$ and this is borne out by the numerical values in the following chart.

	from $L(x)$	actual value	from $P(x)$
$\sqrt{3.98}$	1.9950	1.99499373...	1.99499375
$\sqrt{4.05}$	2.0125	2.01246118...	2.01246094
$\sqrt{4.2}$	2.0500	2.04939015...	2.04937500

5. $T_n(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$. If we put $x = a$ in this equation, then all terms after the first are 0 and we get $T_n(a) = c_0$. Now we differentiate $T_n(x)$ and obtain $T_n'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + nc_n(x-a)^{n-1}$. Substituting $x = a$ gives $T_n'(a) = c_1$. Differentiating again, we have $T_n''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a^2) + \cdots + (n-1)nc_n(x-a)^{n-2}$ and so $T_n''(a) = 2c_2$. Continuing in this manner, we get $T_n'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + \cdots + (n-2)(n-1)nc_n(x-a)^{n-3}$ and $T_n'''(a) = 2\cdot 3c_3$. By now we see the pattern. If we continue to differentiate and substitute $x = a$, we obtain $T_n^{(4)}(a) = 2\cdot 3\cdot 4c_4$ and in general, for any integer k between 1 and n, $T_n^{(k)}(a) = 2\cdot 3\cdot 4\cdot 5\cdot \cdots \cdot kc_k = k!\,c_k \quad \Rightarrow$

$c_k = \dfrac{T_n^{(k)}(a)}{k!}$. Because we want T_n and f to have the same derivatives at a, we require that $c_k = \dfrac{f^{(k)}(a)}{k!}$ for $k = 1, 2, \ldots, n$.

6. $T_n(x) = f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$. To compute the coefficients in this equation we need to calculate the derivatives of f at 0:

$$f(x) = \cos x \qquad f(0) = \cos 0 = 1$$
$$f'(x) = -\sin x \qquad f'(0) = -\sin 0 = 0$$
$$f''(x) = -\cos x \qquad f''(0) = -1$$
$$f'''(x) = \sin x \qquad f'''(0) = 0$$
$$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = 1$$

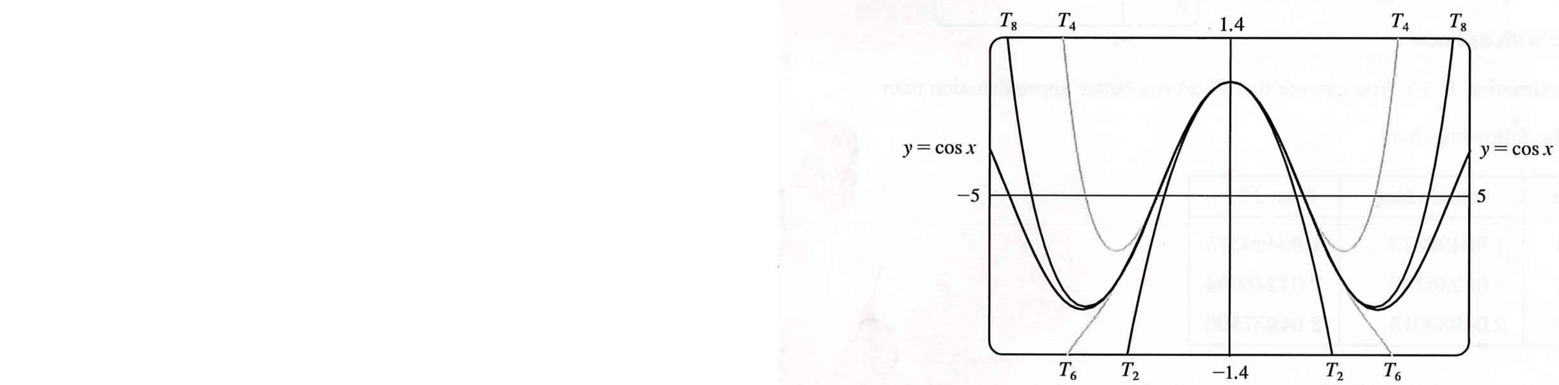
We see that the derivatives repeat in a cycle of length 4, so $f^{(5)}(0) = 0$, $f^{(6)}(0) = -1$, $f^{(7)}(0) = 0$, and $f^{(8)}(0) = 1$.

From the original expression for $T_n(x)$, with $n = 8$ and $a = 0$, we have

$$T_8(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3 + \cdots + \frac{f^{(8)}(0)}{8!}(x-0)^8$$

$$= 1 + 0\cdot x + \frac{-1}{2!}x^2 + 0\cdot x^3 + \frac{1}{4!}x^4 + 0\cdot x^5 + \frac{-1}{6!}x^6 + 0\cdot x^7 + \frac{1}{8!}x^8 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

and the desired approximation is $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$. The Taylor polynomials T_2, T_4, and T_6 consist of the initial terms of T_8 up through degree 2, 4, and 6, respectively. Therefore, $T_2(x) = 1 - \frac{x^2}{2!}$, $T_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$, and $T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$. We graph T_2, T_4, T_6, T_8, and f:



Notice that $T_2(x)$ is a good approximation to $\cos x$ near 0, $T_4(x)$ is a good approximation on a larger interval, $T_6(x)$ is a better approximation, and $T_8(x)$ is better still. Each successive Taylor polynomial is a good approximation on a larger interval than the previous one.

2 Review

TRUE-FALSE QUIZ

1. False. See the note after Theorem 2.2.4.

2. True. This is the Sum Rule.

3. False. See the warning before the Product Rule.

4. True. This is the Chain Rule.

5. True. $\frac{d}{dx}\sqrt{f(x)} = \frac{d}{dx}[f(x)]^{1/2} = \frac{1}{2}[f(x)]^{-1/2}\,f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$

6. False. $\frac{d}{dx}f(\sqrt{x}) = f'(\sqrt{x})\cdot\frac{1}{2}x^{-1/2} = \frac{f'(\sqrt{x})}{2\sqrt{x}}$, which is not $\frac{f'(x)}{2\sqrt{x}}$.

7. False. $f(x) = |x^2 + x| = x^2 + x$ for $x \geq 0$ or $x \leq -1$ and $|x^2 + x| = -(x^2 + x)$ for $-1 < x < 0$. So $f'(x) = 2x + 1$ for $x > 0$ or $x < -1$ and $f'(x) = -(2x + 1)$ for $-1 < x < 0$. But $|2x + 1| = 2x + 1$ for $x \geq -\frac{1}{2}$ and $|2x + 1| = -2x - 1$ for $x < -\frac{1}{2}$.