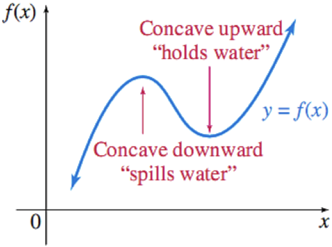
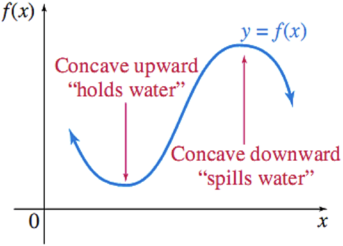
The second derivative gives us information about the shape of the graph of a function.

The Second Derivative Test for Concavity

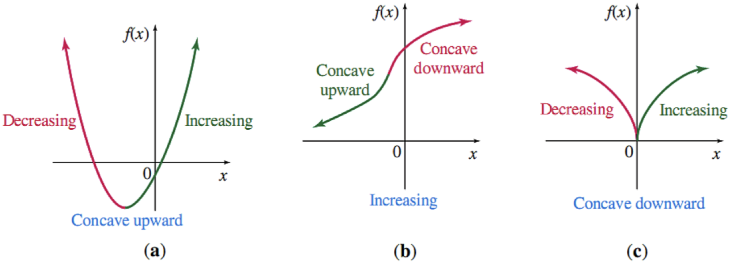
Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.





The concavity of a function is independent of whether the function is increasing or decreasing.



Notice that all 4 combinations occur.

Example

Find the intervals of concavity for $f(x) = x^4 - 8x^3 + 18x^2$.

Compute the first and second derivatives.

$$\begin{aligned} f'(x) &= 4x^3 - 24x^2 + 36x \\ f''(x) &= 12x^2 - 48x + 36 \end{aligned}$$

Set the second derivative equal to zero and solve the resulting equation by factoring.

$$\begin{aligned} 0 &= 12x^2 - 48x + 36 \\ &= 12(x^2 - 4x + 3) \\ &= 12(x-1)(x-3) \end{aligned}$$

So $x = 1$ or $x = 3$.

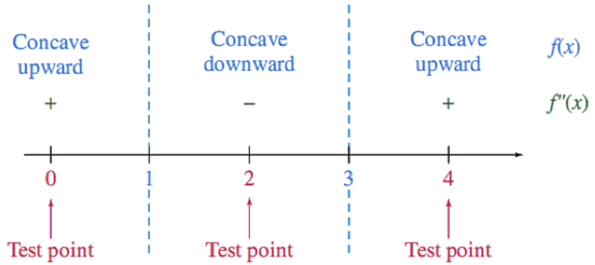
We will do a sign analysis for the second derivative.

Divide a number line into three intervals.
$(-\infty, 1), (1, 3), (3, \infty)$.

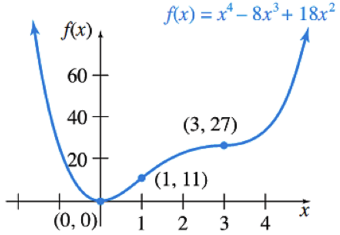
Any number from each of the three intervals can be used as a test point to find the sign of the second derivative.

We will use the test points $x = 0, 2, 4$.

$$\begin{aligned}
f''(0) &= +(-)(-) &= + \\
f''(2) &= +(+)(-) &= - \\
f''(4) &= +(+)(+) &= +
\end{aligned}$$



Since the second derivative is positive on $(-\infty, 1)$, negative on $(1, 3)$, and positive on $(3, \infty)$ we conclude that the graph is concave up on $(-\infty, 1)$ and $(3, \infty)$, and concave down on $(1, 3)$.



Example

Find the intervals of concavity for $f(x) = 6x - \frac{x^3}{2}$.

Compute the first and second derivatives.

$$\begin{aligned} f'(x) &= 6 - \frac{3x^2}{2} \\ f''(x) &= -3x \end{aligned}$$

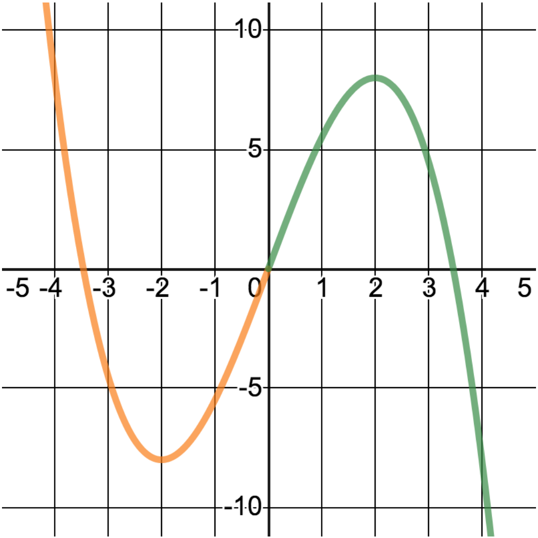
Set the second derivative equal to zero and solve.

$$\begin{aligned} 0 &= -3x \\ &\implies x = 0 \end{aligned}$$

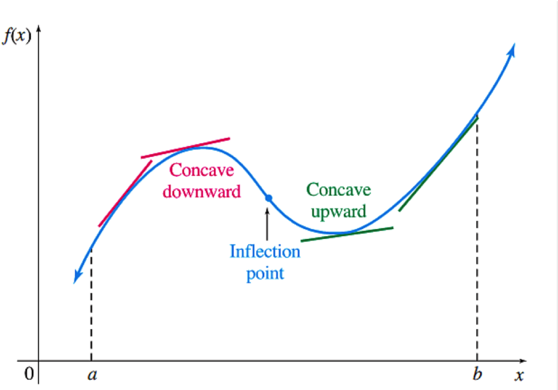
Divide a number line into two intervals. $(-\infty, 0) \cup (0, \infty)$.

$$\begin{aligned} f''(-\pi) &= -(-) = + \\ f''(3^{2.1}) &= -(+) = - \end{aligned}$$

The second derivative is positive on $(-\infty, 0)$ and negative on $(0, \infty)$.
The graph is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$.



DEFINITION A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.



At an inflection point for a function f, the second derivative is 0 or does not exist.

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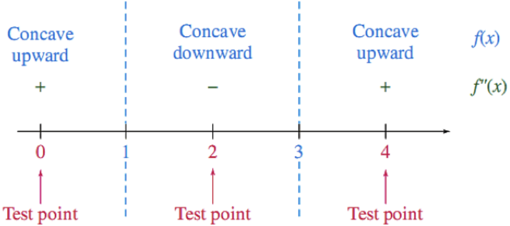
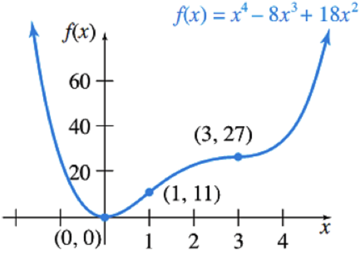
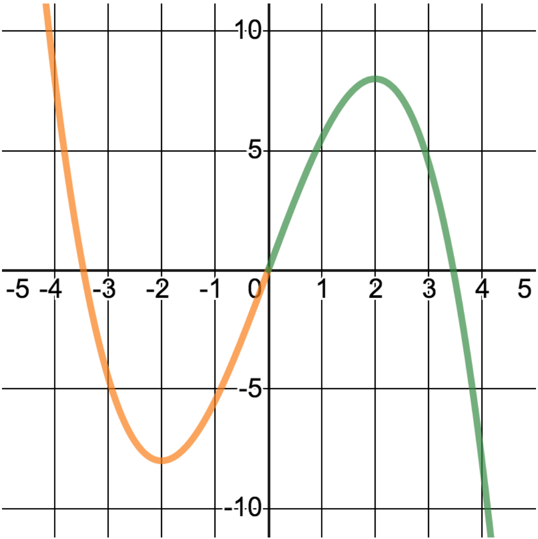


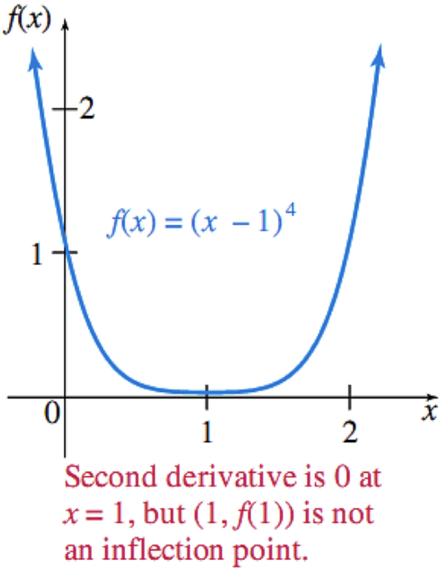


Example

Find any points of inflection for $f(x) = 6x - \dfrac{x^3}{2}$.



Warning Just because the second derivative is zero at a point, does not mean that there is a point of inflection there.



The second derivative can also be used to find extrema.

Second Derivative Test

Let f'' exist on some open interval containing c, (except possibly at c itself) and let $f'(c) = 0$.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum.
3. If $f''(c) = 0$ or $f''(c)$ does not exist, then the test gives no information about extrema, so use the first derivative test.

It is sometimes easier to apply this test...

And sometimes harder...

And at times the test is inconclusive.

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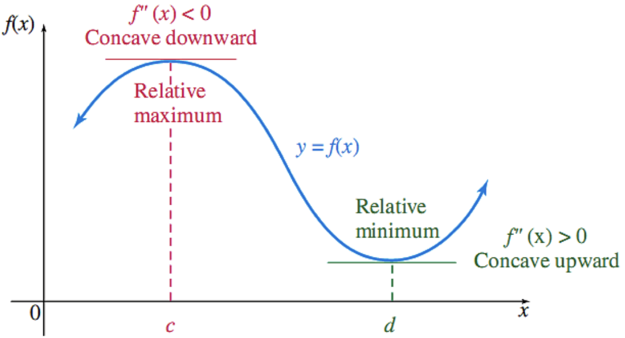
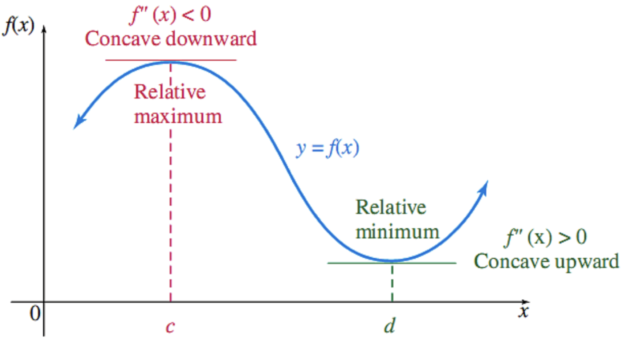


Remember that the derivative can be interpreted as a rate of change.



The second derivative is the rate of change of the slopes of the tangent lines to the curve.

Example

Find all relative extrema of $f(x) = -2x^3 + 3x^2 + 72x$ using the second derivative test.

We might as well compute both the first and second derivatives right away.

$$\begin{aligned} f'(x) &= -6x^2 + 6x + 72 \\ f''(x) &= -12x + 6 \end{aligned}$$

Find the critical numbers by setting the first derivative equal to zero and solving the resulting equation.

$$\begin{aligned} 0 &= -6x^2 + 6x + 72 \\ 0 &= -6(x-4)(x+3) \end{aligned}$$

The critical numbers for this function are $x = 4$ and $x = -3$.

Example

Find all relative extrema of $f(x) = -2x^3 + 3x^2 + 72x$ using the second derivative test.

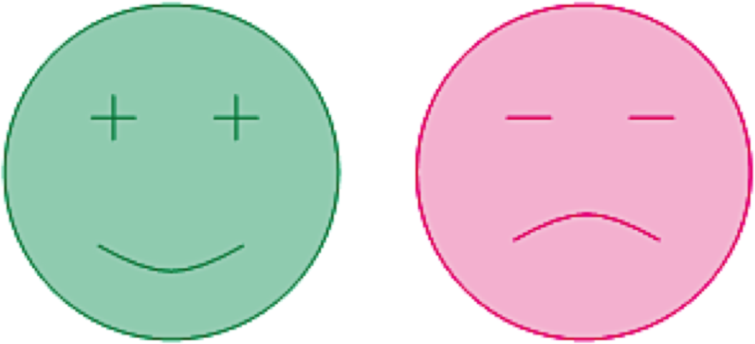
Evaluate the second derivative at the critical numbers, $x = 4$ and $x = -3$.

$$\begin{aligned} f''(x) &= -12x + 6 \\ f''(4) &= -42 \\ f''(-3) &= 42 \end{aligned}$$

Since $f''(4)$ is negative, $\boxed{\text{there is a local max at } x = 4}$.

Since $f''(-3)$ is positive, $\boxed{\text{there is a local min at } x = -3}$.

Minnie & Maxwell



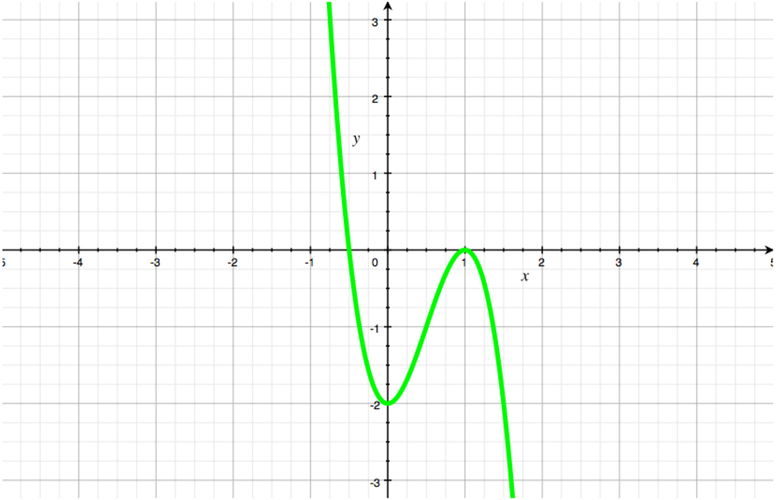
Example

Try to use the second derivative test for concavity, and the second derivative test for extrema to analyze the functions we worked on last time! It's OK to look back at your notes.

- $f(x) = x^3 + 3x^2 - 9x + 4$
- $f(x) = -x^3 - 2x^2 + 15x + 10$
- $f(x) = x^3$
- $f(x) = \dfrac{x-1}{x+1}$

- Can you find points of inflection?
- Can you find intervals of increase/decrease?
- Is the second derivative test inconclusive anywhere?

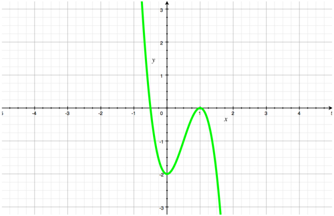
Here is the graph of the derivative of a function.



We can use calculus to interpret this graph, and infer certain properties about the shape of the original function.

First of all, recall that a function is increasing whenever its derivative is positive.

And that a function is decreasing whenever its derivative is negative.

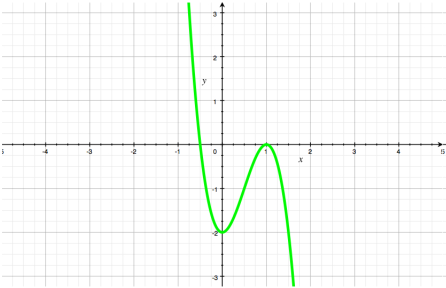


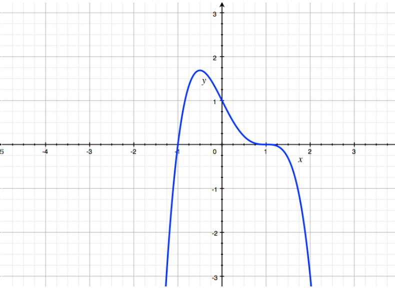
What we can infer here is that since the graph of the derivative is above the x-axis on the interval $(-\infty, \frac{-1}{2})$, then the original function must be increasing on that same interval.

Similarly, notice that the graph of the derivative is below the x-axis on the union of the intervals $(\frac{-1}{2}, 1) \cup (1, \infty)$. So the original function must be decreasing there.

What we can infer here is that since the graph of the derivative is above the x-axis on the interval $(-\infty, \frac{-1}{2})$, then the original function must be increasing on that same interval.

Similarly, notice that the graph of the derivative is below the x-axis on the union of the intervals $(\frac{-1}{2}, 1) \cup (1, \infty)$. So the the original function must be decreasing there.

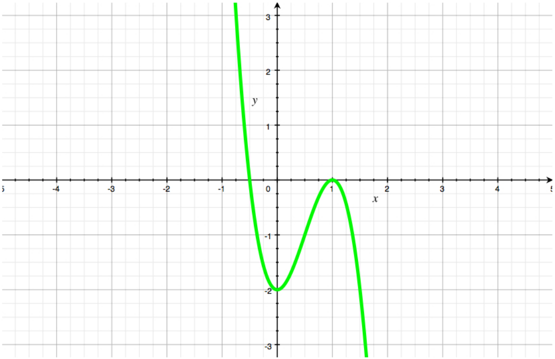




Also notice that the original function flattens out at $x = 1$. It stops decreasing ever so briefly, while the derivative equals zero at $x = 1$.

Next, recall that a function is concave upwards whenever its second derivative is positive.

And that a function is concave downwards whenever its second derivative is negative.



Even though we were only given the graph of the derivative, we can use it to infer certain properties about the second derivative.

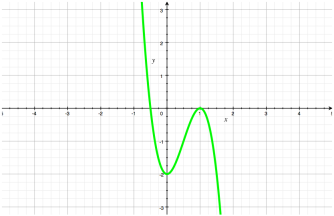
We know that a function is increasing whenever its derivative is positive.

And that a function is decreasing whenever its derivative is negative.

Use this to come up with a logically equivalent statement about a derivative and the second derivative:

The derivative is increasing when the second derivative is positive.

And the derivative is decreasing when second derivative is negative.

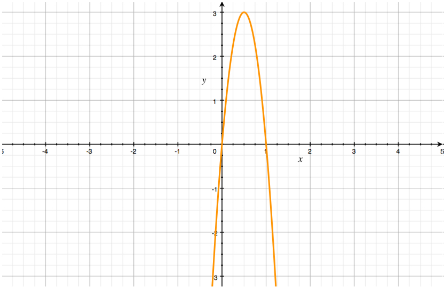


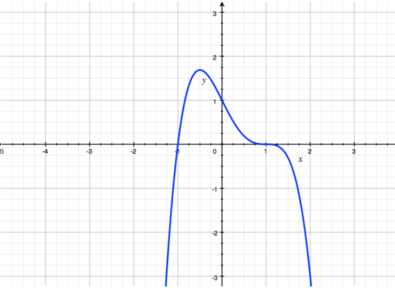
What we can infer here is that since the graph of the derivative is decreasing on the intervals $(-\infty, 0) \cup (1, \infty)$, then the second derivative must be negative there.

Similarly, the graph of the derivative is increasing on the interval $(0, 1)$. So the second derivative must be positive there.

The original function is concave down on $(-\infty, 0) \cup (1, \infty)$ because the second derivative is negative there.

And the original function is concave up on $(0, 1)$ since the second derivative is positive.



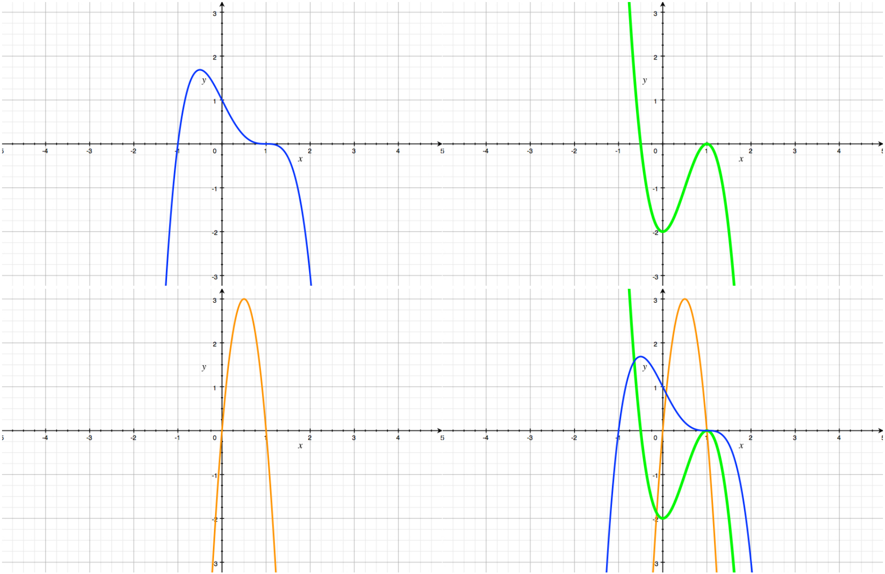


$$***STARING***$$

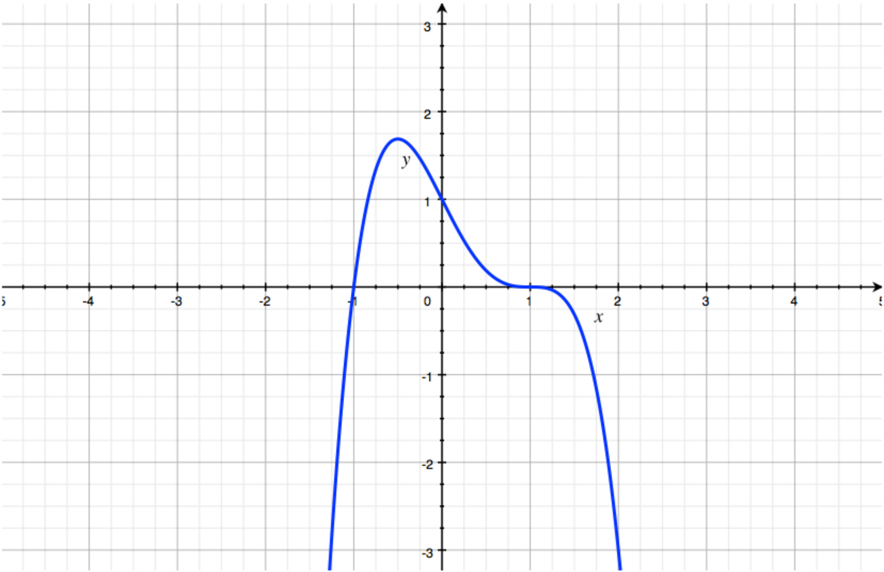
$$f(x) = -x^4 + 2x^3 - 2x + 1$$

$$f'(x) = -4x^3 + 6x^2 - 2$$

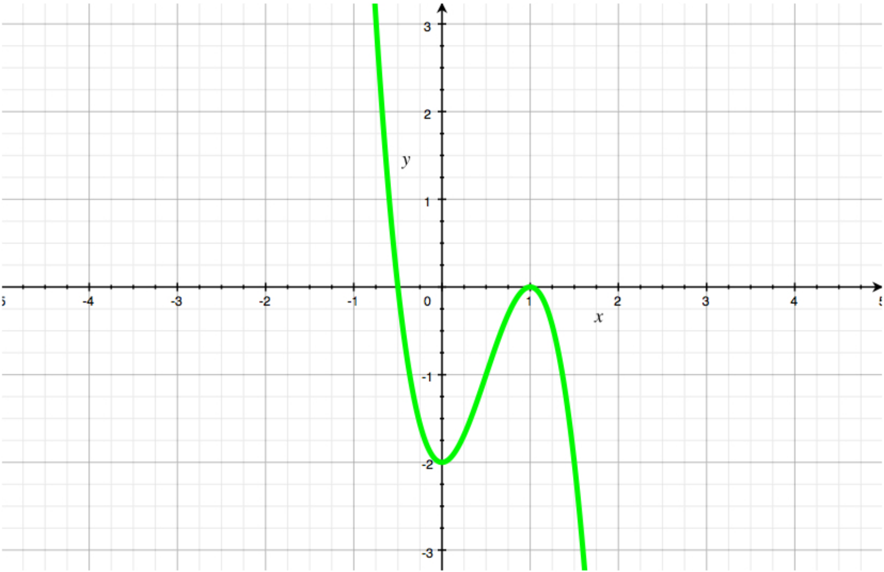
$$f''(x) = -12x^2 + 12x$$



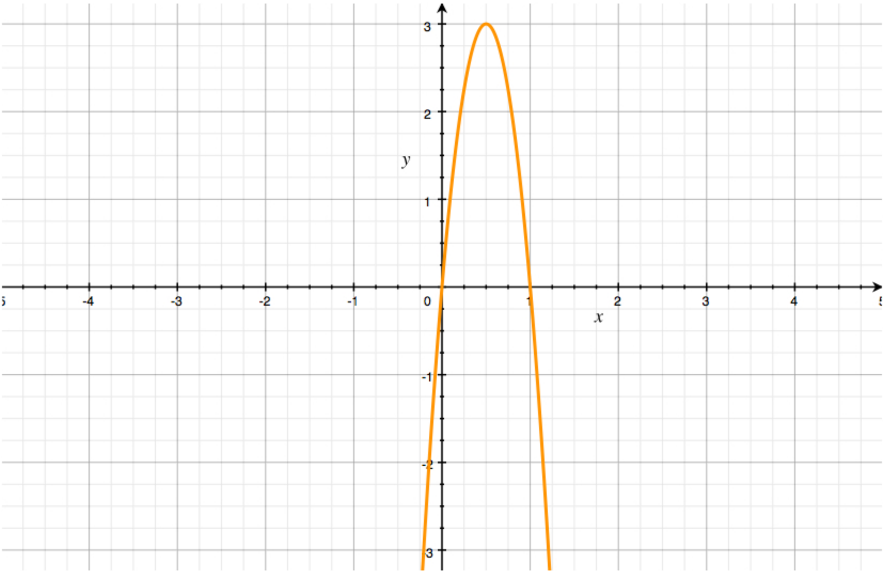
y
x
-5 -4 -3 -2 -1 0 1 2 3 4 5
3 2 1 -1 -2 -3



3
2
1
0
-1
-2
-3
-4
-3
-2
-1
1
2
3
4
y
x



y
x
3
2
1
0
-1
-2
-3
-4
1
2
3
4



y
x
3
2
1
-1
-2
-3
-4
-3
-2
-1
0
1
2
3
4

Which should we sketch next?

1. $f(x) = x^2 - 10x - 8$
2. $g(x) = 0.25x^4 - 8x + 12$
3. $h(x) = x^3 - x^2 - 10$
4. $q(t) = (t^2 - 1)^2$
5. $r(t) = \sqrt{t}$
6. $s(t) = t^2\sqrt{t-4}$
7. $A(x) = x^{1/3} - x^{-1/3}$
8. $B(x) = 15x^{2/3} + 6x^{5/3}$
9. $C(x) = x \ln x$
10. $D(x) = \dfrac{\ln x}{x}$
11. $E(x) = xe^x$
12. $F(x) = x^2e^x$

$$D(x) = \frac{\ln x}{x}$$

$$D'(x) = \frac{1 - \ln x}{x^2}$$

$$D''(x) = \frac{-3 + 2\ln x}{x^3}$$

crit numbers: $\left\{ \begin{array}{l} \cancel{0} \\ e \end{array} \right.$

$D(e) = \frac{1}{e}$

possible poi: $\left\{ \begin{array}{l} \cancel{0} \\ e^{3/2} \end{array} \right.$

$D(e^{3/2}) = \frac{3}{2}e^{-3/2}$

$D(x)$	concave down increasing	concave down decreasing	concave up decreasing
$D'(x)$	+	−	−
$D''(x)$	−	−	+

0   e $e^{3/2}$ e^2

Test Value (e)   Test Value (e^2)