

Math 223

Midterm (8:00 AM version)

Sept 28, 2015

Name: Scott Key

 Section: ∞

Closed book and closed notes.

Answers must include supporting work.

Calculators and cell phones out of sight.

1. (10 pts) Solve the following inequalities:

(a) $|2+3x| \leq 3$

Method 1

$$\begin{aligned} -3 &\leq 2+3x \leq 3 \\ -5 &\leq 3x \leq 1 \\ -\frac{5}{3} &\leq x \leq \frac{1}{3} \\ \left[-\frac{5}{3}, \frac{1}{3}\right] & \end{aligned}$$

(b) $|x-1| > 3$

Method 2 (two cases)

$$\begin{aligned} 2+3x &\geq 0 \text{ and } 2+3x \leq 3 \\ 2+3x < 0 &\text{ or } -(2+3x) \leq 3 \\ 3x \geq -2 &\text{ and } 3x \leq 1 \\ x \geq -\frac{2}{3} &\text{ and } x \leq \frac{1}{3} \\ \left[-\frac{2}{3}, \frac{1}{3}\right] & \end{aligned}$$

(or)

$$\begin{aligned} 2+3x &< 0 \\ 3x &< -2 \\ x &< -\frac{2}{3} \\ 2+3x &\geq -3 \\ 3x &\geq -5 \\ x &\geq -\frac{5}{3} \\ \left[-\frac{5}{3}, -\frac{2}{3}\right) & \end{aligned}$$

Solution:

$$\left[-\frac{5}{3}, \frac{1}{3}\right] \cup \left[-\frac{2}{3}, \frac{1}{3}\right]$$

2. (15 pts)

Either

$$\begin{aligned} x-1 &\geq 0 \text{ and } x-1 > 3 \\ x &\geq 1 \text{ and } x > 4 \quad \text{For both,} \\ &\text{need } x > 4 \quad (4, \infty) \end{aligned}$$

or

$$\begin{aligned} x-1 &< 0 \text{ and } -(x-1) > 3 \\ x &< 1 \text{ and } x < -3 \quad \text{For both,} \\ &x < -2 \\ &\text{need } x < -2 \quad (-\infty, -2) \end{aligned}$$

 Solution: $(-\infty, -2) \cup (4, \infty)$

 (a) Determine f and g such that $h(x) = f(g(x))$ for $h(x) = \sqrt{4x-x^2}$.

 (b) Find the domain of $h(x)$.

(a) $f(x) = \sqrt{x}$

$$g(x) = 4x-x^2$$

$$h(x) = f(g(x)) = \sqrt{4x-x^2}$$

(b) $\text{dom}(h) = \{x \mid 4x-x^2 \geq 0\} = [0, 4]$

$$4x-x^2 = x(4-x)$$

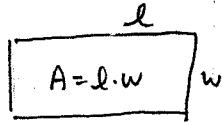
Case 1: $x \geq 0$ and $4-x \geq 0$
 $x \geq 0$ and $4 \geq x$

$$[0, 4]$$

Case 2: $x < 0$ and $4-x < 0$
 $4 < x$

No real x satisfies both $x < 0$ and $4 < x$.

3. (10 pts) Given a rectangle; its length is three times its width. Express its area A as a function of its perimeter P .



$$\begin{aligned} l &= 3w & P &= 2l + 2w = 2(3w) + 2w \\ A &= l \cdot w & &= 6w + 2w \\ & & &= 8w \end{aligned}$$

Let l be the length and w the width of the rectangle; A its area and P its perimeter.

$$\begin{aligned} &= 3w \cdot w \\ &= 3w^2 \\ &= 3\left(\frac{P}{8}\right)^2 \\ &= \frac{3P^2}{64} \end{aligned}$$

$$A(P) = \frac{3P^2}{64}.$$

4. (10 pts) The equation $2x^2 - 4x + 2y^2 + 8y + 1 = 0$ describes a circle. Determine the circles center and radius by completing the square.

Method 1

$$\begin{aligned} 2x^2 - 4x + 2y^2 + 8y + 1 &= 0 \\ 2(x^2 - 2x) + 2(y^2 + 4y) + 1 &= 0 \\ 2(x^2 - 2x + \underline{\quad} - \underline{\quad}) + 2(y^2 + 4y + \underline{\quad} - \underline{\quad}) + 1 &= 0 \\ 2(x^2 - 2x + 1 - 1) + 2(y^2 + 4y + 4 - 4) + 1 &= 0 \\ 2[(x-1)^2 - 1] + 2[(y+2)^2 - 4] + 1 &= 0 \end{aligned}$$

$$2(x-1)^2 - 2 + 2(y+2)^2 - 8 + 1 = 0$$

$$2(x-1)^2 + 2(y+2)^2 - 9 = 0$$

$$2(x-1)^2 + 2(y+2)^2 = 9$$

$$(x-1)^2 + (y+2)^2 = \frac{9}{2} = \left(\frac{3}{\sqrt{2}}\right)^2$$

Center: $(+1, -2)$

$$\text{Radius: } \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}}$$



Method 2

$$2x^2 - 4x + 2y^2 + 8y = -1$$

$$x^2 - 2x + y^2 + 4y = -\frac{1}{2}$$

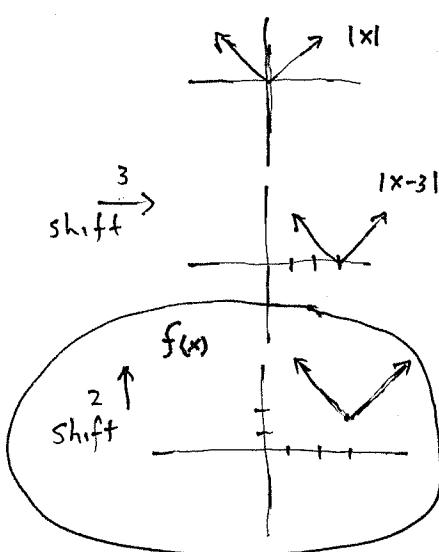
$$x^2 - 2x + \underline{1} + y^2 + 4y + \underline{4} = -\frac{1}{2} + \underline{1} + \underline{4}$$

$$\begin{aligned} (x-1)^2 + (y+2)^2 &= 5 - \frac{1}{2} \\ &= \frac{9}{2} \end{aligned}$$

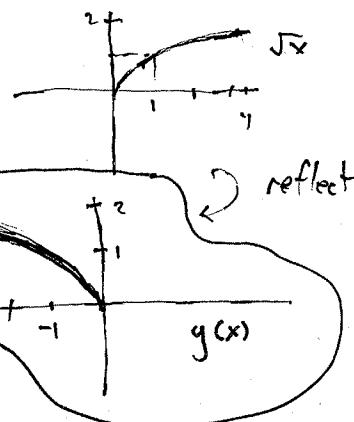
$$(x-1)^2 + (y+2)^2 = \left(\frac{3}{\sqrt{2}}\right)^2$$

5. (15 pts) Sketch the following graphs:

$$(a) y = |x - 3| + 2 = f(x)$$



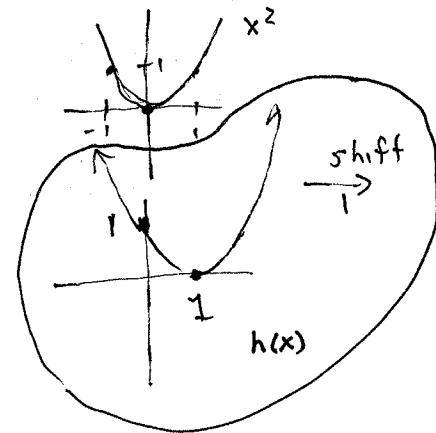
$$(b) y = \sqrt{-x} = g(x)$$



$$(c) y = x^2 - 2x + 1 = h(x)$$

$$= (x-1)(x-1)$$

$$= (x-1)^2$$



6. (10 pts)

(a) Find a polynomial of degree 3 with zeros $-3, 0$, and 3 .

(b) Is $x - 1$ a factor of $P(x) = x^7 + 4x^6 - 2x^5 + x^4 - x^2 + 2x - 5$? Explain your answer. [Hint: use the Factor Theorem]

(a) Since $-3, 0$, and 3 are roots, $(x+3), (x-0)$, and $(x-3)$ must be factors. Take $p(x) = (x+3) \cdot x \cdot (x-3)$. \leftarrow Full credit

$$\begin{aligned} &= x(x+3)(x-3) \\ &= x(x^2 - 9) \\ &= x^3 - 9x \end{aligned} \quad \left. \right\}$$

Not needed; usually more complicated!

$$\begin{aligned} (b) \text{ Observe: } P(1) &= 1^7 + 4 \cdot 1^6 - 2 \cdot 1^5 + 1^4 - 1^2 + 2 \cdot 1 - 5 \\ &= 1 + 4 - 2 + 1 - 1 + 2 - 5 \\ &= 0. \end{aligned}$$

The Factor Theorem tells us that $x-1$ is a factor of $P(x)$. Yes.

7. (10 pts) Find the quotient and the remainder using long division.

$$\begin{array}{r} 3x^4 - 5x^3 - 20x - 5 \\ \hline x^2 + x + 3 \end{array}$$

$\frac{3x^2 - 8x - 1}{3x^4 - 5x^3 + 0x^2 - 20x - 5}$

$\frac{-8x^3 - 9x^2 - 20x - 5}{-8x^3 - 8x^2 - 24x}$

$\frac{-x^2 + 4x - 5}{-x^2 - x - 3}$

$5x - 2$

← Quotient: $3x^2 - 8x - 1$

← Remainder: $5x - 2$

8. (10 pts) Write the following polynomial in factored form.

$$P(x) = x^3 + 3x^2 - x - 3$$

Possible integer roots: $+1, -1, +3, -3$.

$$P(1) = 1 + 3 - 1 - 3 = 0, \text{ so } (x-1) \text{ is a factor.}$$

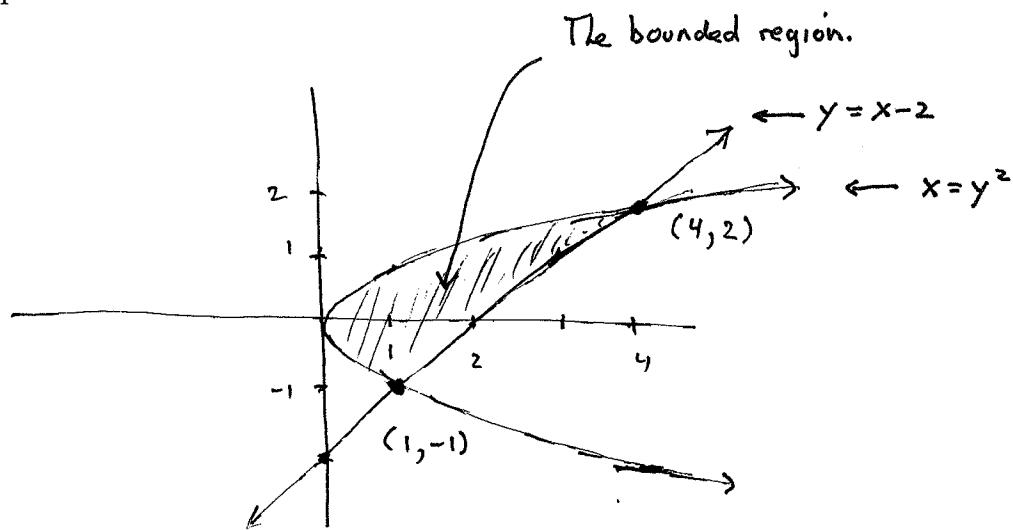
$$P(-1) = -1 + 3 + 1 - 3 = 0, \text{ so } (x+1) \text{ is a factor.}$$

$$\begin{aligned} P(-3) &= (-3)^3 + 3(-3)^2 - (-3) - 3 \\ &= -27 + 27 + 3 - 3 = 0, \text{ so } (x+3) \text{ is a factor.} \end{aligned}$$

$$P(x) = (x-1)(x+1)(x+3).$$

These were other ways to solve this problem.

9. (10 pts) Sketch the region bounded by the parabola $x = y^2$ and the line $y = x - 2$ and label their points of intersection.



$$x = (x-2)^2$$

$$= x^2 - 4x + 4$$

$$0 = x^2 - 5x + 4$$

$$= (x-4)(x-1)$$

$$x=4 \text{ or } x=1$$

$$y=4-2 \quad y=1-2$$

$$= 2 \quad = -1$$

(4, 2) (1, -1) ← Intersection points, i.e.,
points that lie on both equations' graphs.

1. $\frac{d}{dx} \left(\frac{1}{x} \right) = -\frac{1}{x^2}$
2. $\frac{d}{dx} \left(\frac{1}{x^2} \right) = -\frac{2}{x^3}$
3. $\frac{d}{dx} \left(\frac{1}{x^3} \right) = -\frac{3}{x^4}$
4. $\frac{d}{dx} \left(\frac{1}{x^4} \right) = -\frac{4}{x^5}$
5. $\frac{d}{dx} \left(\frac{1}{x^5} \right) = -\frac{5}{x^6}$

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Answers must include supporting work.

Calculators and cell phones out of sight.

1. (10 pts) Solve the following inequalities:

(a) $|1-2x| \leq 3$

(b) $|x-2| > 1$

Method 1

$$\begin{aligned} -3 &\leq 1-2x \leq 3 \\ -4 &\leq -2x \leq 2 \\ 2 &\geq x \geq -1 \\ [-1, 2] & \end{aligned}$$

Method 2

$$\begin{cases} \text{Case 1: } 1-2x \geq 0 \text{ and } 1-2x \leq 3 \\ \quad \uparrow \quad -2x \geq -1 \quad " \quad -2x \leq 2 \\ \quad \text{or} \quad x \leq \frac{1}{2} \quad " \quad x \geq -1 \\ \quad \downarrow \quad [-1, \frac{1}{2}] \\ \text{Case 2: } 1-2x < 0 \text{ and } -(1-2x) \leq 3 \\ \quad -2x < -1 \quad " \quad 1-2x \geq -3 \\ \quad x > \frac{1}{2} \quad " \quad -2x \geq -4 \\ \quad \quad \quad \quad x \leq 2 \\ \quad (\frac{1}{2}, 2] \end{cases}$$

$$\begin{aligned} \text{Solution: } & [-1, \frac{1}{2}] \cup (\frac{1}{2}, 2] \\ & = [-1, 2]. \end{aligned}$$

$$\begin{cases} \text{Case 1: } x-2 \geq 0 \text{ and } x-2 > 1 \\ \quad x \geq 2 \quad \text{and} \quad x > 3 \\ \quad \text{or} \\ \quad (3, \infty) \end{cases}$$

$$\begin{cases} \text{Case 2: } x-2 < 0 \text{ and } -(x-2) > 1 \\ \quad x < 2 \quad \text{and} \quad x-2 < -1 \\ \quad x < 1 \\ (-\infty, 1) \end{cases}$$

$$\text{Solution: } (-\infty, 1) \cup (3, \infty).$$

2. (15 pts)

 (a) Determine f and g such that $h(x) = f(g(x))$ for $h(x) = \sqrt{2x-x^2}$.

 (b) Find the domain of $h(x)$.

(a) * $g(x) = 2x-x^2$

$$f(x) = \sqrt{x}$$

$$h(x) = f(g(x)) = \sqrt{2x-x^2}$$

The are, of course, other solutions, for example
 $g(x) = 2x$ but the first
 $f(x) = \sqrt{x-(\frac{x}{2})^2}$ solution $x=0$ is the most obvious and useful.

(b) $\text{dom}(h) = \{x \mid 2x-x^2 \geq 0\}$

$$2x-x^2 \geq 0$$

$$x(2-x) \geq 0$$

$$\text{Case 1: } x \geq 0 \text{ and } 2-x \geq 0$$

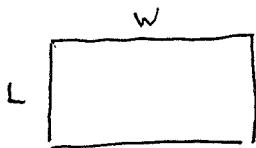
$$x \geq 0 \quad " \quad 2 \geq x$$

$$0 \leq x \leq 2$$

$$[0, 2]$$

$$\begin{cases} \text{Case 2: } x < 0 \text{ and } 2-x < 0 \\ \quad x < 0 \quad \text{and} \quad 2 < x \\ \quad \text{No } x \text{ satisfies both.} \end{cases}$$

3. (10 pts) A rectangle has perimeter 12 m. Express the area A of the rectangle as a function of the length, L , of one of its sides.



$$P = 2L + 2W = 12.$$

$$A = L \cdot W.$$

Let L be the length of one side, W the length of the other; A the area and P the perimeter.

$$A = L \cdot W$$

$$A(L) = L(6-L)$$

$$= 6L - L^2 \quad \leftarrow \text{Expanding}$$

not required and often discouraged.

$$2W = 12 - 2L$$

$$W = \frac{1}{2}(12 - 2L)$$

$$= 6 - L$$

4. (10 pts) The equation $x^2 + y^2 - 4x + 10y + 13 = 0$ describes a circle. Determine the circles center and radius by completing the square.

Method 1

$$x^2 - 4x + y^2 + 10y = -13$$

$$x^2 - 4x + 4 + y^2 + 10y + 25 = -13 + 4 + 25$$

$$(x-2)^2 + (y+5)^2 = 16 = 4^2$$

Center: $(2, -5)$

Radius: 4

Method 2

$$x^2 + y^2 - 4x + 10y + 13 = 0$$

$$x^2 - 4x + 4 - 4 + y^2 + 10y + 25 - 25 + 13 = 0$$

$$(x-2)^2 + (y+5)^2 - 4 - 25 + 13 = 0$$

$$(x-2)^2 + (y+5)^2 - 16 = 0$$

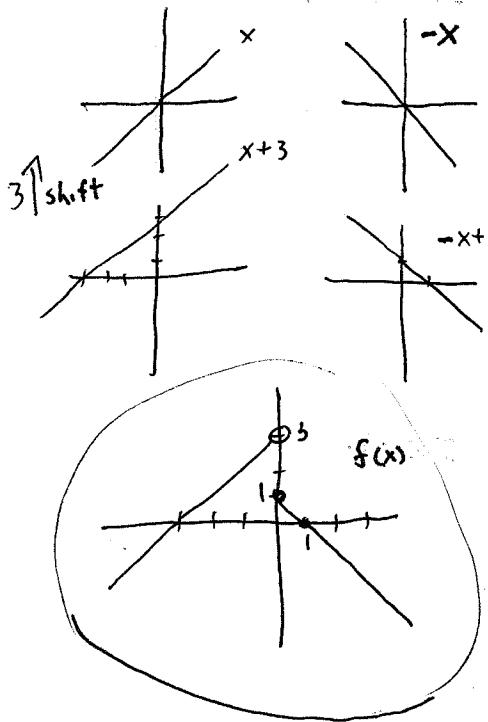
$$(x-2)^2 + (y+5)^2 = 16$$

Center: $(2, -5)$

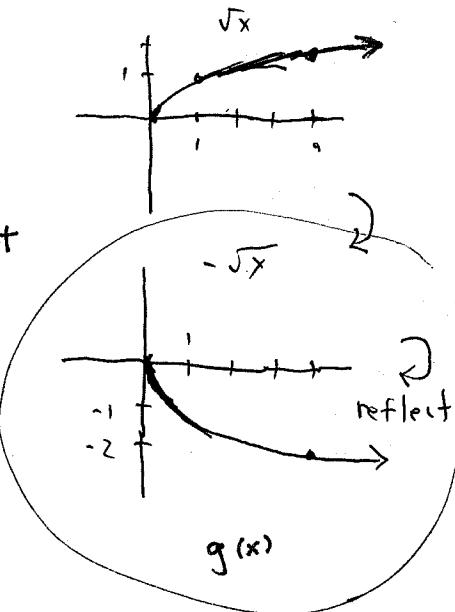
Radius: 4

5. (15 pts) Sketch the following functions:

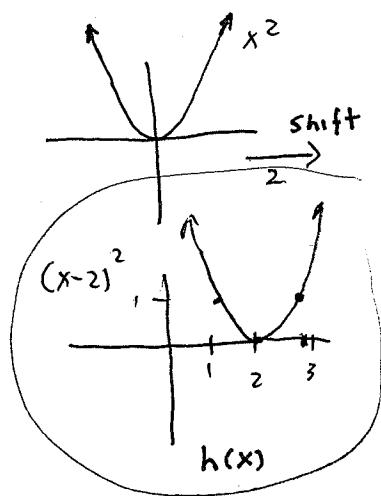
$$(a) f(x) = \begin{cases} x+3 & \text{if } x < 0 \\ 1-x & \text{if } x \geq 0 \end{cases}$$



$$(b) y = -\sqrt{x} = g(x) \quad (c) y = x^2 - 4x + 4 = h(x)$$



$$x^2 - 4x + 4 = (x-2)^2$$



6. (10 pts)

(a) Find a polynomial of degree 3 with zeros -2, 0, and 2.

(b) Is $x-1$ a factor of $P(x) = x^7 + 4x^6 - 2x^5 + x^4 - x^2 + 2x - 5$? Explain your answer. [Hint: use the Factor Theorem]

By the Factor Theorem, $(x-(-2))$, $(x-0)$, and $(x-2)$ must be factors.

$$\begin{aligned} (a) \quad P(x) &= (x-(-2))(x-0)(x-2) \\ &= (x+2) \cdot x \cdot (x-2) \quad \text{This clearly has degree 3.} \\ &= x(x-2)(x+2) \\ &= x(x^2-4) \\ &= x^3-4x \end{aligned}$$

$\left. \begin{array}{l} \text{Not needed; usually more} \\ \text{complicated!} \end{array} \right\}$

$$(b) \quad \text{Since } P(1) = 1^7 + 4 \cdot 1^6 - 2 \cdot 1^5 + 1^4 - 1^2 + 2 \cdot 1 - 5$$

$$= 1 + 4 - 2 + 1 - 1 + 2 - 5$$

$= 0$, by the Factor Theorem $(x-1)$ is a factor of $p(x)$. Yes.

7. (10 pts) Find the quotient and the remainder using long division.

$$\begin{array}{c} 3x^4 - 5x^3 - 20x - 5 \\ \hline x^2 + x + 3 \end{array}$$

$$3x^2 - 8x - 1 \quad \leftarrow \text{Quotient: } 3x^2 - 8x - 1$$

$$\begin{array}{r} 3x^4 - 5x^3 + 0x^2 - 20x - 5 \\ 3x^4 + 3x^3 + 9x^2 \\ \hline -8x^3 - 9x^2 - 20x - 5 \\ -8x^3 - 8x^2 - 24x \\ \hline -x^2 + 4x - 5 \\ -x^2 - x - 3 \\ \hline 5x - 2 \end{array}$$

$$\leftarrow \text{Remainder: } 5x - 2$$

8. (10 pts) Factor completely the following polynomial.

$$P(x) = x^3 + 3x^2 - x - 3$$

Possible integer roots: 1, -1, 3, -3.

Observe: $P(1) = 1^3 + 3 \cdot 1^2 - 1 - 3 = 1 + 3 - 1 - 3 = 0$ so $(x-1)$ is a factor.

$$P(-1) = (-1)^3 + 3 \cdot (-1)^2 - (-1) - 3 = -1 + 3 + 1 - 3 = 0 \text{ so }$$

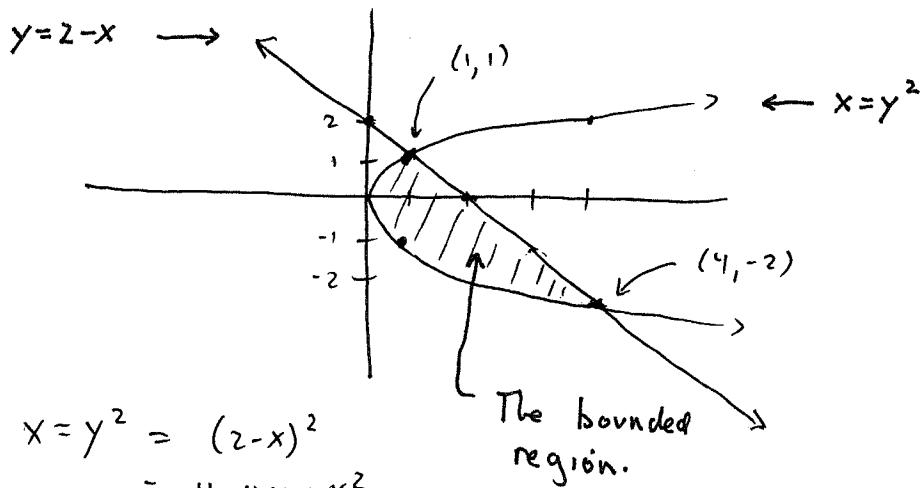
$$P(-3) = (-3)^3 + 3(-3)^2 - (-3) - 3 \quad (x+1) \text{ is a factor.}$$

$$= -27 + 27 + 3 - 3 = 0 \quad \text{so } (x-(-3)) = (x+3) \text{ is a factor.}$$

$$P(x) = (x-1)(x+1)(x+3).$$

There were other ways to solve this problem.

9. (10 pts) Sketch the region bounded by the parabola $x = y^2$ and the line $y = 2 - x$ and label their points of intersection.



$$\begin{aligned}x = y^2 &= (2-x)^2 \\&= 4-4x+x^2\end{aligned}$$

$$x^2 - 4x + 4 = x$$

$$x^2 - 5x + 4 = 0$$

"

$$(x-4)(x-1)$$

$$x=4 \text{ or } x=1$$

Curves

~~the~~ meet when $x=4$ and when $x=1$

$$\begin{aligned}y &= 2-4 \\&= -2\end{aligned}$$

$$\begin{aligned}y &= 2-1 \\&= 1\end{aligned}$$

(4, -2)

(1, 1)

Intersection
points — points
that lie on the
graphs of both
equations.

the first time in the history of the world, the
whole of the human race has been gathered
together in one place, and that is the
present meeting of the World's Fair.
The great number of people here,
from all parts of the globe, are
representing their countries, and
showing the progress they have made
in science, art, and industry.
The exhibits are indeed wonderful,
and it is a great privilege to see them.
The buildings are also very beautiful,
and the grounds are well kept.
The atmosphere is friendly and
welcoming, and everyone seems to be
enjoying themselves.
It is a truly remarkable gathering,
and it is a pleasure to be a part of it.