SECTION 1-8 Polynomial and Rational Inequalities

- Polynomial Inequalities
- Rational Inequalities

In this section we solve fairly simple polynomial and rational inequalities of the form

$$2x^2 - 3x - 4 < 0 \qquad \text{and} \qquad \frac{x - 2}{x^2 - x - 3} \geq 0$$

Even though we limit the discussion to quadratic inequalities and rational inequalities with numerators and denominators of degree 2 or less (you will see why below), the theory presented applies to polynomial and rational inequalities in general. In Chapter 3, with additional theory, we will be able to use the methods developed here to solve polynomial and rational inequalities of a more general nature. Also, the process—with only slight modification of key theorems—applies to other forms encountered in calculus.

Why so much interest in solving inequalities? Most significant applications of mathematics involve more use of inequalities than equalities. In the real world few things are exact.

• Polynomial Inequalities

We know how to solve linear inequalities such as

$$3x - 7 \geq 5(x - 2) + 3$$

But how do we solve quadratic (or higher-degree polynomial) inequalities such as the one given below?

$$x^2 + 2x < 8 \tag{1}$$

We first write the inequality in **standard form;** that is, we transfer all nonzero terms to the left side, leaving only 0 on the right side:

$$x^2 + 2x - 8 < 0 \qquad \text{Standard form} \tag{2}$$

In this example, we are looking for values of x that will make the quadratic on the left side less than 0—that is, negative.

The following theorem provides the basis for an effective way of solving this problem. Theorem 1 makes direct use of the *real zeros* of the polynomial on the left side of inequality (2). **Real zeros** of a polynomial are those real numbers that make the polynomial equal to 0—that is, the real roots of the corresponding polynomial equation. If a polynomial has one or more real zeros, then plotting these zeros on a real number line divides the line into two or more intervals.

Theorem 1

Sign of a Polynomial over a Real Number Line

A nonzero polynomial will have a constant sign (either always positive or always negative) within each interval determined by its real zeros plotted on a number line. If a polynomial has no real zeros, then the polynomial is either positive over the whole real number line or negative over the whole real number line.

We now complete the solution of inequality (1) using Theorem 1. After writing (1) in standard form, as we did in inequality (2), we find the real zeros of the polynomial on the left side by solving the corresponding polynomial equation:

$$x^2 + 2x - 8 = 0 \qquad \text{Can be solved by factoring}$$

$$(x - 2)(x + 4) = 0$$

$$x = -4, 2 \qquad \text{Real zeros of the polynomial } x^2 + 2x - 8$$

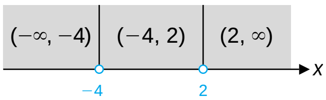


FIGURE 1 Real zeros of $x^2 + 2x - 8$.

Next, we plot the real zeros, -4 and 2, on a number line (Fig. 1) and note that three intervals are determined: $(-\infty, -4)$, $(-4, 2)$, and $(2, \infty)$.

From Theorem 1 we know that the polynomial has constant sign on each of these three intervals. If we select a **test number** in each interval and evaluate the polynomial at that number, then the sign of the polynomial at this test number must be the sign for the whole interval. Since any number within an interval can be used as a test number, we generally choose test numbers that result in easy computations. In this example, we choose -5, 0, and 3. Table 1 shows the computations.

TABLE 1 Polynomial: $x^2 + 2x - 8 = (x - 2)(x + 4)$

Test number	-5	0	3
Value of polynomial for test number	7	-8	7
Sign of polynomial in interval containing test number	+	−	+
Interval containing test number	$(-\infty, -4)$	$(-4, 2)$	$(2, \infty)$

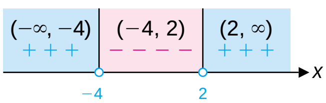


FIGURE 2 Sign chart for $x^2 + 2x - 8$.

Using the information in Table 1, we construct a **sign chart** for the polynomial, as shown in Figure 2.

Thus, $x^2 + 2x - 8$ is negative within the interval $(-4, 2)$, and we have solved the inequality. The solution and graph are given below and in Figure 3:

$$-4 < x < 2 \qquad \text{Inequality notation}$$

$$(-4, 2) \qquad \text{Interval notation}$$

FIGURE 3 Solution of $x^2 + 2x < 8$.

Note: If $<$ in the original problem had been $\le$ instead, then we would have included the zeros of the polynomial in the solution set.

The steps in the above process are summarized in the following box:

Key Steps in Solving Polynomial Inequalities

Step 1. Write the polynomial inequality in standard form (a form where the right-hand side is 0).

Step 2. Find all real zeros of the polynomial (the left side of the standard form).

Step 3. Plot the real zeros on a number line, dividing the number line into intervals.

Step 4. Choose a test number (that is easy to compute with) in each interval, and evaluate the polynomial for each number (a small table is useful).

Step 5. Use the results of step 4 to construct a sign chart, showing the sign of the polynomial in each interval.

Step 6. From the sign chart, write down the solution of the original polynomial inequality (and draw the graph, if required).

With a little experience, many of the above steps can be combined and the process streamlined to two or three key operational steps. The critical part of the method is step 2, finding all real zeros of the polynomial. At this point we can find all real zeros of any quadratic polynomial (see Section 1-6). Finding real zeros of higher-degree polynomials is more difficult, and the process is considered in detail in Chapter 3.

EXPLORE-DISCUSS 1

We can solve a quadratic equation by factoring the quadratic polynomial and setting each factor equal to 0, as we did in the preceding example. Can we solve quadratic inequalities the same way? That is, can we solve

$$(x - 2)(x + 4) < 0$$

by considering linear inequalities involving the factors $x - 2$ and $x + 4$? Discuss how you could arrive at the correct solution, $-4 < x < 2$, by considering various combinations of

$$x - 2 < 0 \qquad x - 2 > 0 \qquad x + 4 < 0 \qquad x + 4 > 0$$

We now turn to a significant application that involves a polynomial inequality.

EXAMPLE 1 Profit and Loss Analysis

A company manufactures and sells flashlights. For a particular model, the marketing research and financial departments estimate that at a price of $\$p$ per unit, the weekly cost C and revenue R (in thousands of dollars) will be given by the equations

$$C = 7 - p \qquad \text{Cost equation}$$

$$R = 5p - p^2 \qquad \text{Revenue equation}$$

Find prices (including a graph) for which the company will realize:

(A) A profit (B) A loss

Solutions (A) A profit will result if cost is less than revenue, that is, if

$$C < R$$
$$7 - p < 5p - p^2$$

We solve this inequality following the steps outlined above.

Step 1. Write the polynomial inequality in standard form.

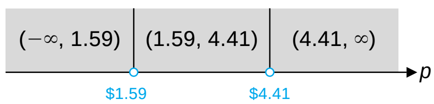
$$p^2 - 6p + 7 < 0 \quad \text{Standard form}$$

Step 2. Find all real zeros of the polynomial.

$$p^2 - 6p + 7 = 0$$
$$p = \frac{6 \pm \sqrt{36 - 28}}{2} \quad \text{Solve, using the quadratic formula.}$$
$$= 3 \pm \sqrt{2}$$
$$\approx \$1.59, \$4.41 \quad \text{Real zeros of the polynomial rounded to the nearest cent.}$$

Step 3. Plot the real zeros on a number line.

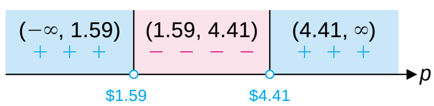
The two real zeros determine three intervals: $(-\infty, 1.59)$, $(1.59, 4.41)$, and $(4.41, \infty)$.



Step 4. Choose a test number in each interval, and construct a table.

Polynomial: $p^2 - 6p + 7$			
Test number	1	2	5
Value of polynomial for test number	2	−1	2
Sign of polynomial in interval containing test number	+	−	+
Interval containing test number	$(-\infty, 1.59)$	$(1.59, 4.41)$	$(4.41, \infty)$

Step 5. Construct a sign chart.



Sign chart for $p^2 - 6p - 7$

Step 6. Write the solution and draw the graph.

Referring to the sign chart for the polynomial $p^2 - 6p + 7$ in step 5, we see that $p^2 - 6p + 7 < 0$, and a profit will occur ($C < R$), for

$$\$1.59 < p < \$4.41$$

Profit

(B) A loss will result if cost is greater than revenue; that is, if

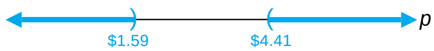
$$C > R$$
$$7 - p > 5p - p^2$$

Writing this polynomial inequality in standard form, we obtain the same inequality that was obtained in step 1 of part A, except the order of the inequality is reversed:

$$p^2 - 6p + 7 > 0 \qquad \text{Standard form}$$

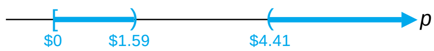
Referring to the sign chart for the polynomial $p^2 - 6p + 7$ in step 5 of part A, we see that $p^2 - 6p + 7 > 0$, and a loss will occur ($C > R$), for

$$p < \$1.59 \qquad \text{or} \qquad p > \$4.41$$



Since a negative price doesn't make sense, we must modify this result by deleting any number to the left of 0. Thus, a loss will occur for the following prices:

$$\$0 \le p < \$1.59 \qquad \text{or} \qquad p > \$4.41$$



The real zeros are not included, because they are the values for which $R = C$, the **break-even** values for the company.

Matched Problem 1 A company manufactures and sells computer printer ribbons. For a particular ribbon, the marketing research and financial departments estimate that at a price of $\$p$ per unit, the weekly cost C and revenue R (in thousands of dollars) will be given by the equations

$$C = 13 - p \qquad \text{Cost equation}$$
$$R = 7p - p^2 \qquad \text{Revenue equation}$$

Find prices (including a graph) for which the company will realize:

(A) A profit (B) A loss

• Rational Inequalities

The steps for solving polynomial inequalities can, with slight modification, be used to solve rational inequalities such as

$$\frac{x-3}{x+5} > 0 \qquad \text{and} \qquad \frac{x^2+5x-6}{5-x} \leq 3$$

If, after suitable operations on an inequality, the right side is 0 and the left side is of the form P/Q, where P and Q are nonzero polynomials, then the inequality is said to be a **rational inequality in standard form.** When the real zeros (if they exist) of the polynomials P and Q are plotted on a number line, they divide the line into two or more intervals. The following theorem, which includes Theorem 1 as a special case, provides a basis for solving rational inequalities in standard form.

Theorem 2

Sign of a Rational Expression over a Real Number Line

The rational expression P/Q, where P and Q are nonzero polynomials, will have a constant sign (either always positive or always negative) within each interval determined by the real zeros of P and Q plotted on a number line. If neither P nor Q have real zeros, then the rational expression P/Q is either positive over the whole real number line or negative over the whole real number line.

We will illustrate the use of Theorem 2 through an example.

EXAMPLE 2 Solving a Rational Inequality

Solve and graph: $\dfrac{x^2-3x-10}{1-x} \geq 2$

Solution

We might be tempted to start by multiplying both sides by $1-x$ (as we would do if the inequality were an equation). However, since we don't know whether $1-x$ is positive or negative, we don't know whether the order of the inequality is to be changed. We proceed instead as follows (modifying the steps for solving polynomial inequalities as needed):

Step 1. Write the inequality in standard form.

$$\frac{x^2-3x-10}{1-x} \geq 2$$

$$\frac{x^2-3x-10}{1-x} - 2 \geq 0 \qquad \text{Subtract 2 from both sides.}$$

$$\frac{x^2-3x-10-2(1-x)}{1-x} \geq 0 \qquad \text{Combine left side into a single fraction.}$$

$$\frac{x^2-x-12}{1-x} \geq 0 \qquad \text{Standard form: } \frac{P}{Q} \geq 0$$

The left side of the last inequality is a rational expression of the form P/Q, where $P = x^2 - x - 12$ and $Q = 1 - x$. Our problem now is to find all values of x so that $P/Q \geq 0$; that is, so that P/Q is positive or 0.

Step 2. Find all real zeros for polynomials P and Q.

$$x^2 - x - 12 = 0$$

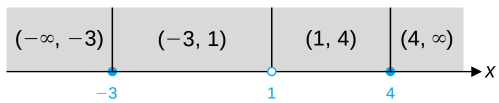
$$(x + 3)(x - 4) = 0$$

$$x = -3, 4 \quad \text{Real zeros for } P$$

$$1 - x = 0$$

$$x = 1 \quad \text{Real zero for } Q$$

Note: The real zeros for P make P/Q equal to 0; thus, the equality part of the original inequality is satisfied for these zeros and they must be included in the final solution set. On the other hand, since division by 0 is never allowed, P/Q is not defined at the zeros of Q. Thus, the real zeros of Q must *not* be included in the solution set.



Step 3. Plot the real zeros for P and Q on a number line.

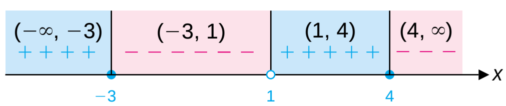
The three zeros of P and Q determine four intervals: $(-\infty, -3)$, $(-3, 1)$, $(1, 4)$, and $(4, \infty)$. Note that we use solid dots at -3 and 4 to indicate that these zeros of P are part of the solution set. However, we use an open dot at 1 to indicate that this zero of Q is not part of the solution set. Remember, P/Q is not defined at the zeros of Q.

Step 4. Choose a test number in each interval, and construct a table.

Rational expression: $\dfrac{x^2 - x - 12}{1 - x} = \dfrac{(x + 3)(x - 4)}{1 - x}$

Test number	-4	0	2	5
Value of P/Q	$\frac{8}{5}$	-12	10	-2
Sign of P/Q	+	−	+	−
Interval	$(-\infty, -3)$	$(-3, 1)$	$(1, 4)$	$(4, \infty)$

Step 5. Construct a sign chart.



Sign chart for $\dfrac{x^2 - x - 12}{1 - x}$

Step 6. Write the solution and draw the graph.

From the sign chart, we see that

$$\frac{x^2 - x - 12}{1 - x} \geq 0 \quad \text{and} \quad \frac{x^2 - 3x - 10}{1 - x} \geq 2$$

for

$x \leq -3$ or $1 < x \leq 4$ Inequality notation

$(-\infty, -3] \cup (1, 4]$ Interval notation

Matched Problem 2 Solve and graph: $\frac{3}{2 - x} \leq \frac{1}{x + 4}$

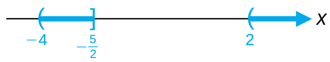
Answers to Matched Problems

1. (A) Profit: $\$2.27 < p < \$5.73$

(B) Loss: $\$0 \leq p < \$2.27$ or $p > \$5.73$

2. $-4 < x \leq -\frac{5}{2}$ or $x > 2$

$(-4, -\frac{5}{2}] \cup (2, \infty)$



EXERCISE 1-8

A

In Problems 1–14, solve and graph. Express answers in both inequality and interval notation.

1. $x^2 < 10 - 3x$

2. $x^2 + x < 12$

3. $x^2 + 21 > 10x$

4. $x^2 + 7x + 10 > 0$

5. $x^2 \leq 8x$

6. $x^2 + 6x \geq 0$

7. $x^2 + 5x \leq 0$

8. $x^2 \leq 4x$

9. $x^2 > 4$

10. $x^2 \leq 9$

11. $\frac{x - 2}{x + 4} \leq 0$

12. $\frac{x + 3}{x - 1} \geq 0$

13. $\frac{x + 4}{1 - x} \leq 0$

14. $\frac{3 - x}{x + 5} \leq 0$

B

In Problems 15–26, solve and graph. Express answers in both inequality and interval notation.

15. $\frac{x^2 + 5x}{x - 3} \geq 0$

16. $\frac{x - 4}{x^2 + 2x} \leq 0$

17. $\frac{(x + 1)^2}{x^2 + 2x - 3} \leq 0$

18. $\frac{x^2 - x - 12}{x^2 + 4} \leq 0$

19. $\frac{1}{x} < 4$

20. $\frac{5}{x} > 3$

21. $\frac{3x + 1}{x + 4} \leq 1$

22. $\frac{5x - 8}{x - 5} \geq 2$

23. $\frac{2}{x + 1} \geq \frac{1}{x - 2}$

24. $\frac{3}{x - 3} \leq \frac{2}{x + 2}$

25. $x^3 + 2x^2 \le 8x$

26. $2x^3 + x^2 > 6x$

For what real values of x will each expression in Problems 27–32 represent a real number? Write answers using inequality notation.

27. $\sqrt{x^2 - 9}$

28. $\sqrt{4 - x^2}$

29. $\sqrt{2x^2 + x - 6}$

30. $\sqrt{3x^2 - 7x - 6}$

31. $\sqrt{\dfrac{x + 7}{3 - x}}$

32. $\sqrt{\dfrac{x - 1}{x + 3}}$

If a, b, and c are real numbers, the quadratic equation $ax^2 + bx + c = 0$ must have either two distinct real roots, one double real root, or two conjugate imaginary roots. In Problems 33–36, use the given information concerning the roots to describe the possible solution sets for the indicated inequality. Illustrate your conclusions with specific examples.

33. $ax^2 + bx + c > 0$, given distinct real roots r_1 and r_2 with $r_1 < r_2$.

34. $ax^2 + bx + c \le 0$, given distinct real roots r_1 and r_2 with $r_1 < r_2$.

35. $ax^2 + bx + c \ge 0$, given one (double) real root r.

36. $ax^2 + bx + c < 0$, given one (double) real root r.

37. Give an example of a quadratic inequality whose solution set is the entire real line.

38. Give an example of a quadratic inequality whose solution set is the empty set.

C

In Problems 39–50, solve and graph. Express answers in both inequality and interval notation.

39. $x^2 + 1 < 2x$

40. $x^2 + 25 < 10x$

41. $x^2 < 3x - 3$

42. $x^2 + 3 > 2x$

43. $x^2 - 1 \ge 4x$

44. $2x + 2 > x^2$

45. $x^3 > 2x^2 + x$

46. $x^3 \le 4x^2 + 3x$

47. $4x^4 + 4 \le 17x^2$

48. $x^4 + 36 \ge 13x^2$

49. $|x^2 - 1| \le 3$

50. $\left|\dfrac{x + 1}{x}\right| > 2$

APPLICATIONS

51. Profit and Loss Analysis. At a price of $\$p$ per unit, the marketing department in a company estimates that the weekly cost C and revenue R (in thousands of dollars) will be given by the equations

$$C = 28 - 2p \quad \text{Cost equation}$$

$$R = 9p - p^2 \quad \text{Revenue equation}$$

Find the prices for which the company has
(A) A profit (B) A loss

52. Profit and Loss Analysis. At a price of $\$p$ per unit, the marketing department in a company estimates that the weekly cost C and revenue R (in thousands of dollars) will be given by the equations

$$C = 27 - 2p \quad \text{Cost equation}$$

$$R = 10p - p^2 \quad \text{Revenue equation}$$

Find the prices for which the company has
(A) A profit (B) A loss

53. Physics. If an object is shot straight up from the ground with an initial velocity of 112 feet per second, its distance d (in feet) above the ground at the end of t seconds (neglecting air resistance) is given by $d = 112t - 16t^2$. Find the interval of time for which the object is 160 feet above the ground or higher.

54. Physics. In Problem 53, find the interval of time for which the object is above the ground.

★ **55. Safety Research.** It is of considerable importance to know the shortest distance d (in feet) in which a car can be stopped, including reaction time of the driver, at various speeds v (in miles per hour). Safety research has produced the formula $d = 0.044v^2 + 1.1v$ for a given car. At what speeds will it take the car more than 330 feet to stop?

★ **56. Safety Research.** Using the information in Problem 55, at what speeds will it take a car less than 220 feet to stop?

★★ **57. Marketing.** When successful new software is first introduced, the weekly sales generally increase rapidly for a period of time and then begin to decrease. Suppose that the weekly sales S (in thousands of units) t weeks after the software is introduced are given by

$$S = \frac{200t}{t^2 + 100}$$

When will sales be 8 thousand units per week or more?

★★ **58. Medicine.** A drug is injected into the bloodstream of a patient through her right arm. The concentration (in milligrams per milliliter) of the drug in the bloodstream of the left arm t hours after the injection is given approximately by

$$C = \frac{0.12t}{t^2 + 2}$$

When will the concentration of the drug in the left arm be 0.04 milligram per milliliter or greater?

CHAPTER 1 GROUP ACTIVITY Rates of Change

1. Average Rate. If you score 90 on your first math exam and 100 on the second exam, then your average exam score for the two exams is $\frac{1}{2}(90 + 100) = 95$. The number 95 is called the **arithmetic average** of 90 and 100. Now suppose you walk uphill at a rate of 3 mph for 5 hours and then turn around and return to your starting point by walking downhill at 6 mph for 2.5 hours. The arithmetic average of the rates for each leg of the trip is $\frac{1}{2}(3 + 6) = 4.5$ mph. On the other hand, you walked a total distance of 30 miles in 7.5 hours so that the rate for the round-trip is $30/7.5 = 4$ mph. Which is your *average rate?* The basic formula $D = RT$ is valid whenever an object travels a distance D at a *constant* rate R for a fixed time T. If the rate is not constant, then this formula can still be used but must be interpreted differently. To be precise, for objects moving at nonconstant rates, **average rate is total distance divided by total time.** Thus, your average rate for the total trip up and down the hill is 4 mph, not 4.5 mph. The formula $R = D/T$ now has two interpretations: $R = D/T$ is the *rate* for an object moving at a constant rate and the *average rate* for an object whose rate is not always the same.

(A) If r is the rate for one leg of a round-trip and s is the rate for the return trip, express the average rate for the round-trip in terms of r and s.

(B) A boat can travel 10 mph in still water. The boat travels 60 miles up a river with a 5 mph current and then returns to its starting point. Find the average rate for the round-trip using the definition of average rate and then check with the formula you found in part A.

(C) Referring to the hill-climbing example discussed earlier, if you walk up the hill at 3 mph, how fast must you walk downhill to average 6 mph for the round-trip? (This is similar to a famous problem communicated to Albert Einstein by Max Wertheimer. See Abraham S. Luchins and Edith H. Luchins, The Einstein–Wertheimer Correspondence on Geometric Proofs and Mathematical Puzzles, *Mathematical Intelligencer 2*, Spring 1990, pp. 40–41. For a discussion of this and other interesting rate–time problems, see Lawrence S. Braden, My Favorite Rate–Time Problems, *Mathematics Teacher,* November 1991, pp. 635–638.)

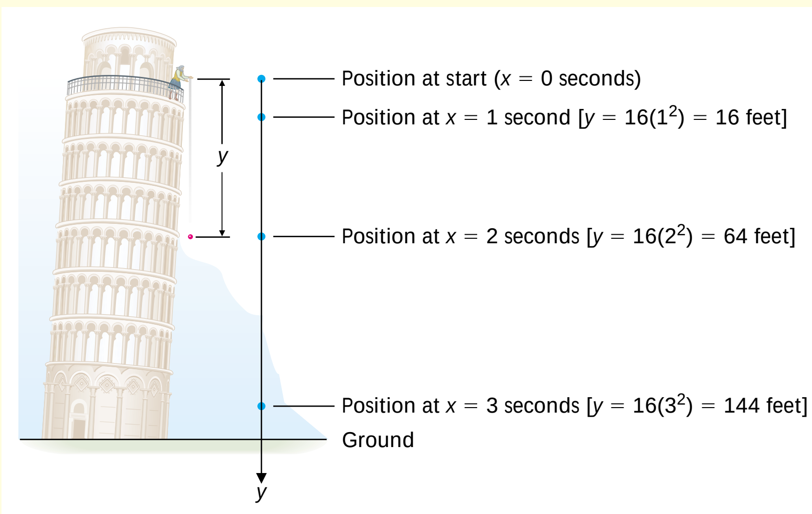
2. Instantaneous Rate. One of the fundamental concepts of calculus is the *instantaneous rate* of a moving object, which is closely related to the average rate discussed above. To introduce this concept, consider the following problem.

A small steel ball dropped from a tower will fall a distance of y feet in x seconds, as given approximately by the formula (from physics)

$$y = 16x^2$$

Figure 1 shows the position of the ball on a number line (positive direction down) at the end of 0, 1, 2, and 3 seconds. Clearly, the ball is not falling at a constant rate.

FIGURE 1 Position of a falling object [*Note:* Positive direction is down.]



(A) What is the average rate that the ball falls during the first second (from $x = 0$ to $x = 1$ second)? During the second second? During the third second?

By definition, average rate involves the distance an object travels over an *interval* of time, as in part A. How can we determine the rate of an object at a *given instant* of time? For example, how fast is the ball falling at exactly 2 seconds after it was released? We will approach this problem from two directions, numerically and algebraically.

(B) Complete the following table of average rates. What number do these average rates appear to approach?

Time interval	[1.9, 2]	[1.99, 2]	[1.999, 2]	[1.9999, 2]
Distance fallen				
Average rate				

(C) Show that the average rate over the time interval $[t, 2]$ is $\dfrac{64 - 16t^2}{2 - t}$. Simplify this algebraic expression and discuss its values for t very close to 2.

(D) Based on the results of parts B and C, how fast do you think the ball is falling at 2 seconds?

Chapter 1 Review

1-1 LINEAR EQUATIONS AND APPLICATIONS

A **solution** or **root** of an equation is a number in the **domain** or **replacement set** of the variable that when substituted for the variable makes the equation a true statement. An equation is an **identity** if it is true for all values from the domain of the variable and a **conditional equation** if it is true for some domain values and false for others. Two equations are **equivalent** if they have the same **solution set.** The **properties of equality** are used to solve equations:

1. If $a = b$, then $a + c = b + c$. — Addition Property

2. If $a = b$, then $a - c = b - c$. — Subtraction Property

3. If $a = b$, then $ca = cb$, $c \neq 0$. — Multiplication Property

4. If $a = b$, then $\dfrac{a}{c} = \dfrac{b}{c}$, $c \neq 0$. — Division Property

5. If $a = b$, then either may replace the other in any statement without changing the truth or falsity of statement. — Substitution Property

An equation that can be written in the **standard form** $ax + b = 0$, $a \neq 0$, is a **linear** or **first-degree equation.**

Strategy for Solving Word Problems

1. Read the problem carefully—several times if necessary—that is, until you understand the problem, know what is to be found, and know what is given.

2. Let one of the unknown quantities be represented by a variable, say x, and try to represent all other unknown quantities in terms of x. This is an important step and must be done carefully.

3. If appropriate, draw figures or diagrams and label known and unknown parts.

4. Look for formulas connecting the known quantities to the unknown quantities.

5. Form an equation relating the unknown quantities to the known quantities.

6. Solve the equation and write answers to *all* questions asked in the problem.

7. Check and interpret all solutions in terms of the original problem—not just the equation found in step 5—since a mistake may have been made in setting up the equation in step 5.