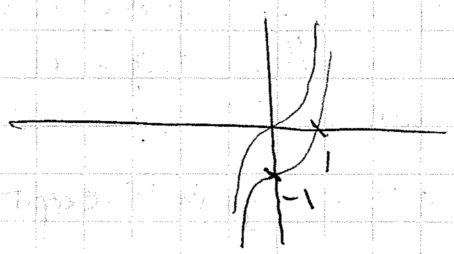
Oct 7

Multiplicity of roots of a polynomial

- A polynomial of degree n has at most n real roots. (It could have complex roots, but we will consider only real roots.)

ex $P(x) = x^3 - 1 = (x-1)(x^2 + x + 1) = 0$



$x = 1$ is its only real root

the roots of this factor are complex (imaginary)

ex $P(x) = (x^2 - 4)(x^2 - 7) = (x+2)(x-2)(x+\sqrt{7})(x-\sqrt{7}) = 0$

remember $x^2 - a$ factors as $(x - \sqrt{a})(x + \sqrt{a})$

$x = -2, 2$ are real

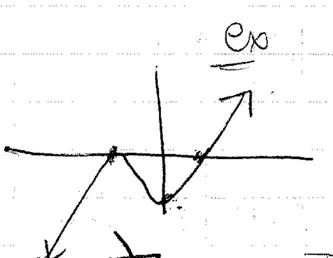
$x = -\sqrt{7}, \sqrt{7}$ are real, also, but not rational

- Once a polynomial is factored, we can see its roots and how many times the roots appear in the polynomial equation. This number is called the multiplicity of the root. We get this value by looking at the exponents of the factored eqn.

ex $P(x) = (x-1)^1(x+1)^2 = 0 \rightarrow x = 1, -1$ and the

$P(0) = (0-1)(0+1)^2 = -1$

multiplicity of 1 is one, while the multiplicity of -1 is 2.



ex $P(x) = (x^2 - 1)^2 (x^2 - 9)^3 = 0 \rightarrow x = \pm 1, \pm 3$

are the roots. What is the multiplicity of these roots?

$$P(x) = (x^2 - 1)^2 (x^2 - 9)^3 = [(x+1)(x-1)]^2 [(x+3)(x-3)]^3$$

$$= (x+1)^2 (x-1)^2 (x+3)^3 (x-3)^3$$

Thus, multiplicity of $x = \pm 1$ is two each and of $x = \pm 3$ is three each.

What is the degree of $P(x) = (x^2-1)^2(x^2-9)^3$? $10 \leftarrow$ Ans

(How did we get this? We're interested only in the highest power of x, which will be the leading term's exponent when this is expanded: $\underbrace{x^2 \cdot x^2}\cdot\underbrace{x^2 \cdot x^2 \cdot x^2} = x^{10}$)

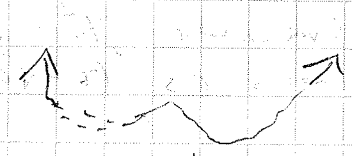
- The multiplicity of the roots is the key to sketching a graph of $P(x)$.

Knowing the roots & their multiplicity, along with the degree of the polynomial & the sign of the leading coeff (<0 or >0) we have a lot of clues.

Throw in $P(0)$ and we have the y-intercept, too!

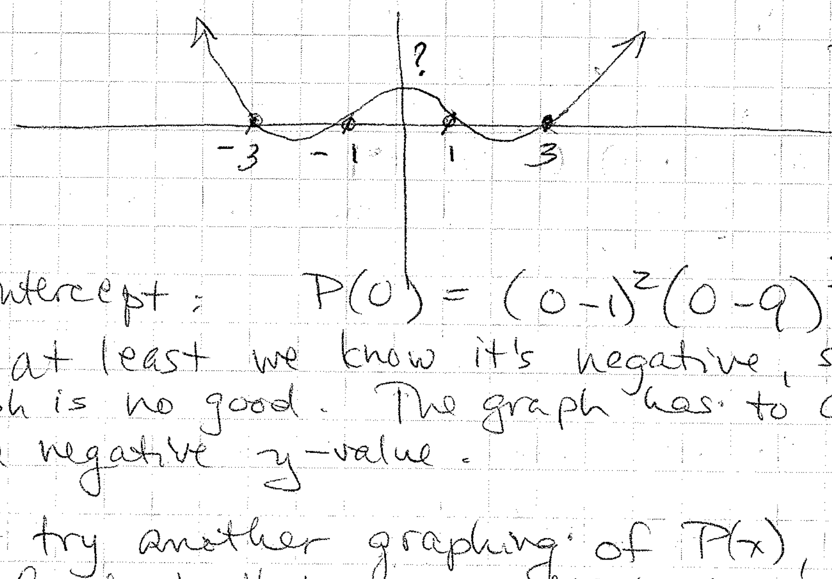
Graphing the same ex: ☆ $P(x) = (x^2-1)^2(x^2-9)^3$, we note the following facts:

1. deg = 10 (even)
2. l. coeff > 0



end behavior

3. roots, ± 1, ± 3
4. mult of $+1 = 2$ mult of $+3 = 3$
   mult of $-1 = 2$ mult of $-3 = 3$



Possible graph

5. y-intercept: $P(0) = (0-1)^2(0-9)^3 = (-9)^3 = $ 😞 but at least we know it's negative, so the possible graph is no good. The graph has to cross the y-axis at a negative y-value.

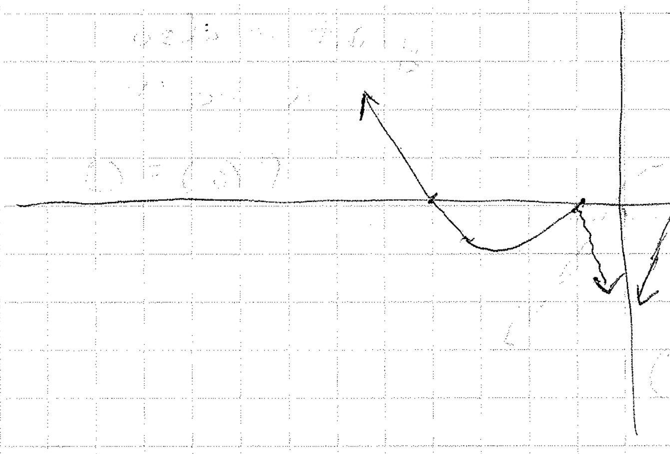
Before we try another graphing of $P(x)$, we need the following, final bullet on multiplicity:

- If the multiplicity of a root is odd, the graph crosses the x-axis at that root.
- If the multiplicity of a root is even, the graph touches there:

6. So, since $x=1$ has even mult, and so does $x=-1$, we know the graph of $P(x)$ touches the x-axis at these values. ↳ "bounces"

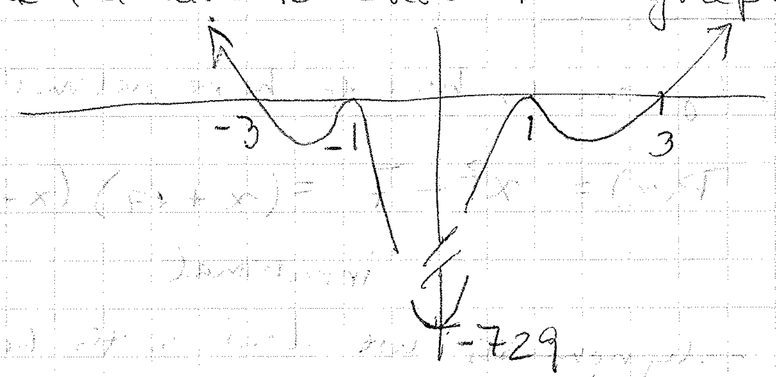
Whereas, since $x=3$ and $x=-3$ have odd multiplicity, the graph of $P(x)$ crosses the x-axis at these values.

(The reason for this goes to the behavior of $P(x)$ on either side of the root in a small distance from that root. It's part of mathematical analysis.)

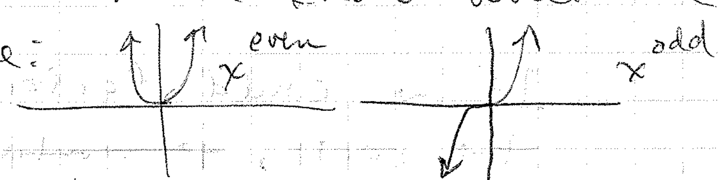


It's heading down to intersect the y-axis at $(-9)^3$ or -729.

I break the axis to show this graph:



Final, helpful idea: Think of a factor $(x-r)^n$ of a polynomial as just x^n for a moment. We know what the various $y=x^n$ look like:



The even-powered x touches the axis, while the odd-powered x crosses it. Likewise, the graph of $(x-r)^{\text{even}}$ touches at $x=r$ but $(x-r)^{\text{odd}}$ crosses at $x=r$.

~~Rational~~

HW examples

#3d $f(x) = (2x+1)(-x+4)x^2 = (-2x^2+7x+4)(x^2)$

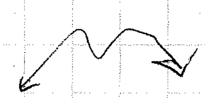
$= -2x^4 + \dots$

roots: $x = -\frac{1}{2}, 4, 0$

$m = 1, 1, 2$

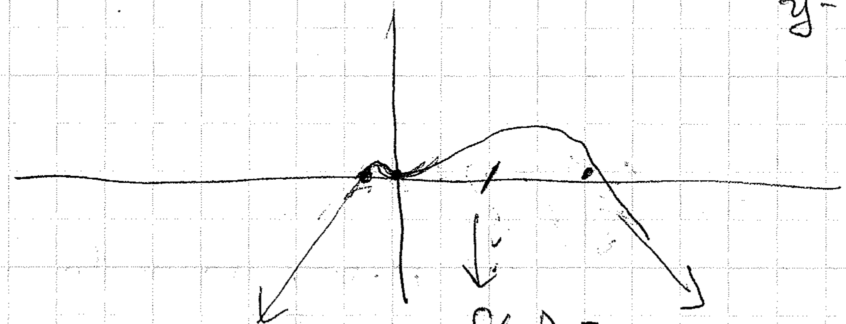
l.c. < 0 deg $= 4$

General shape



Notice that y-int is also a root

$f(0) = 0$



$f(2) =$

Rational Root Theorem

Do all polynomials have to have rational roots? NO

ex $P(x) = x^2 - 7 = (x + \sqrt{7})(x - \sqrt{7})$

irrational

But, most polynomials we deal with have had rational roots.

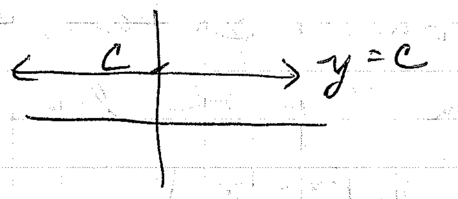
$$P(x) = x^4 - 3x^3 + 2x^2 - x + 1$$

If we could factor this to reveal the roots, ~~it would have~~ if it had rational roots, they would be of the form: $\frac{+1}{+1}, \frac{+1}{-1}, \frac{-1}{1}, \frac{-1}{-1} = 1$ or ± 1

<u>5.2</u> <u>Lines</u> - polynomials of degree zero or one

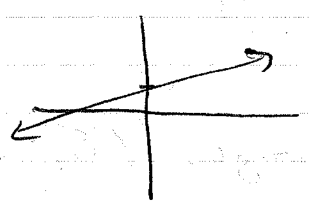
$P(x) = \text{constant}$ is a horizontal line,

that is, the line $y = \text{constant}$ for all x.



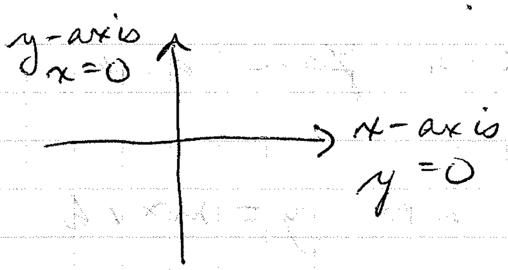
Since $P(x) = a_0 x^0 = a_0 = \text{const}$ is a zero degree polynomial we see the horizontal line is such a $P(x)$.

$P(x) = a_1 x^1 + a_0$ is a first degree polynomial which we recognize as the line $y = mx + b$, a non-horizontal line of slope m when m is not zero.



$P(0) = a_0$ (the y-intercept)

We are more familiar with lines written as $y = mx + b$ where b is the y-intercept.



Vertical lines (like the y-axis) are not functions because they fail ~~the~~ VLT!

Equations of vertical lines are like

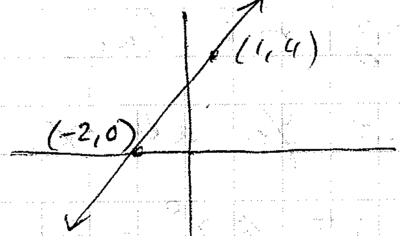
$x = \text{constant}$.

(A) We need only 2 pts to graph a line or write its equation.

(B) Alternately, we need only 1 pt and the slope to do so.

<u>Ex</u> (A) Given pts $(1,4)$, $(-2,0)$, find the equation of the line that passes through them.

Since slope m is defined as $\dfrac{\text{change in } y}{\text{change in } x}$, we find first the slope: $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4-0}{1-^-2} = \dfrac{4}{3}$



Choose either pt to write this line's eqn., along with slope found above. The formula for this, the "pt - slope form" of the line, is

This equation comes directly from def. of slope

$m = \dfrac{y - y_1}{x - x_1}$

$\rightarrow m(x - x_1) = y - y_1$

where we allow one of the pts (say, (x_2, y_2)) to remain variable

$$y - y_1 = m(x - x_1)$$

$$y - 0 = \frac{4}{3}(x - ^-2)$$

$$y = \frac{4}{3}(x + 2)$$

$$\boxed{y = \frac{4}{3}x + \frac{8}{3}}$$

Ex B Given that a line passes through $(5, ^-3)$ and has a slope of $-\frac{1}{2}$, what is the equation of this line?

To find this, choose the $y = mx + b$ form and perform 2 steps:

1. Substitute (x, y) + m into $y = mx + b$ and solve for b

$$y = mx + b \qquad b = -\frac{1}{2}$$

$$-3 = -\frac{1}{2}(5) + b$$

2. Substitute m + b into $y = mx + b$, this time leaving x + y as variables:

$$y = -\frac{1}{2}x - \frac{1}{2}$$