Sec 7 + Sec 8

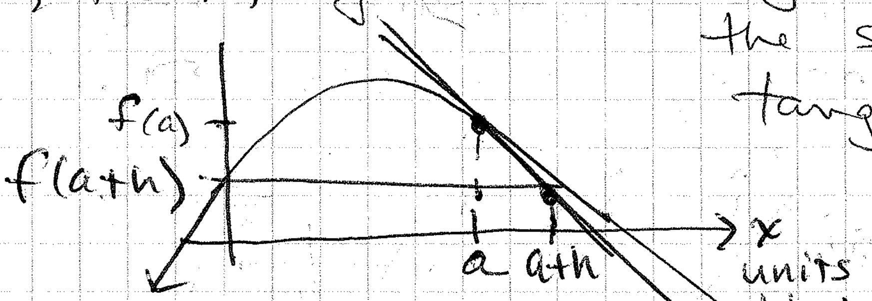
Instantaneous Rate of Change (IRC) of a Function

— That is, Derivative of a Function $f(x)$ at any given value of x where a function is "differentiable"

There are several ways to find the instaneous rate of change of a function at a given value of x, say, $x = a$.

1. You could graph the function (a polynomial for example) and draw with a straight edge the tangent line to the pt on the curve $(a, f(a))$ you're investigating. Then take the slope of this tangent line.

$$m = \frac{\Delta y}{\Delta x}$$



Do this for every point where you seek the IRC. This is the least efficient method and least accurate.

2. You ~~can~~ could write the difference quotient (DQ) for $x=a$. This represents the slope of a secant line from $(a, f(a))$ to a point h distance from a.

$$DQ = \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h) - f(a)}{h}$$

Then take the limit as $h \to 0$ of this quotient, thus giving you the value of the ~~instan~~ IRC at $x = a$.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \text{IRC at } x = a$$

Do this for each value a you seek the instaneous rate of change for.

This is also inefficient. Probably worse than method 1.

3. You can write the <u>general</u> <u>expression</u> of the limit of the difference quotient for the independent variable x -

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad \underset{\text{derivative}}{f'(x)}$$

Then, given a value of "a", plug it into this expression $f'(x)$.

This way you've taken the limit <u>only once</u>. Apply it to any a value.

(HW + possibly 1 test question)

4. For a given class of functions, say polynomials, you can memorize the formula for the derivative

(which came from someone once proving that this is the limit of the difference quotient — and even you will have done this for at least one polynomial)

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x^1 + a_0$$

$$f'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1$$

Formula

This is the preferred way and that you will ultimately use each time.

Ex Consider the polynomial

$$f(x) = -x^2 + 10x + 15$$

It's a parabola. If we complete the square we find it's this:

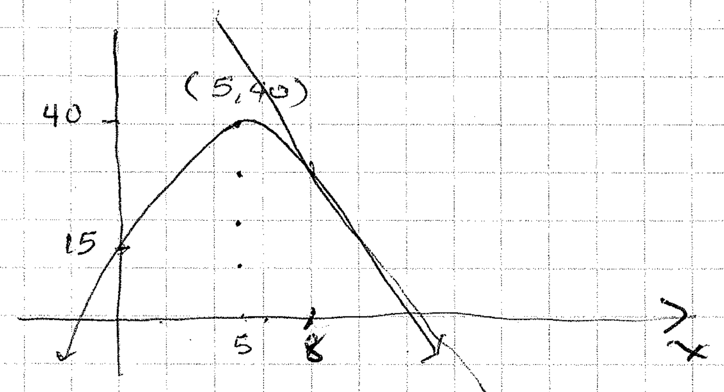
$$f(x) = -(x-5)^2 + 40$$

Graphing this is easy if you rewrite it as

$$y - 40 = -(x-5)^2 \qquad \begin{array}{c}\text{Template} \\ y-k=(x-h)^2\end{array}$$

The vertex is $(5,40)$ and it opens down. The y-intercept is $f(0)$:

$$f(0) = -(0-5)^2 + 40 = -25 + 40 = \underline{15}$$



By the way, what are the roots?

Look at $f(x) = -x^2 + 10x + 15$ and use the quad. formula.

Question

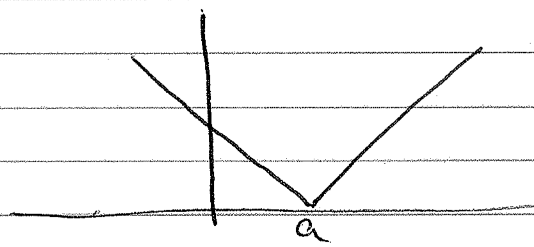
Find the IRC of $f(x)$ at $x = 8$. ← "a value"

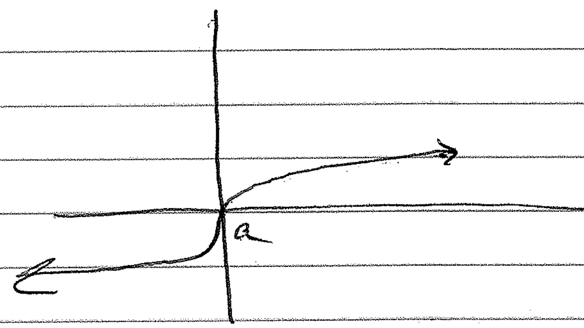
Method 1. Draw a tangent line at $(8, f(8))$ and take its slope. → OK, sure.

Method 2. Plot a point $(8+h, f(8+h))$, look at the secant line through this and $(8, f(8))$ and realize that if $h \to 0$, the slope of this secant goes to the slope of the tangent.

i.e. $$\lim_{h \to 0} \frac{f(8+h) - f(8)}{h} = \text{IRC at } x = 8$$

2. But there are two other places where $f'(x)$ cannot be taken, even though the fcn. is continuous there:





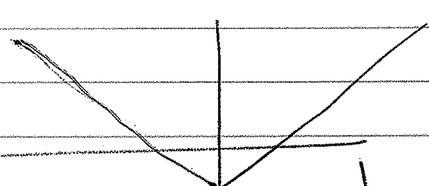
Cannot ~~take~~ draw a tangent at a corner

A vertical tangent (here, the y-axis is the tangent line) has an infinite slope.

So, derivative $f'(a)$ exists only if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists <u>and</u> is <u>finite</u>.

<u>Major example</u> $f(x) = |x|$ at $x = 0$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



$$\boxed{\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}} = \begin{cases} +1 & \text{if } h > 0 \\ -1 & \text{if } h < 0 \end{cases}$$

Method 3. Derive for any x the expression for the IRC, namely, the function's derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \quad f(x) = -x^2 + 10x + 15$$

$$= \lim_{h \to 0} \frac{[-(x+h)^2 + 10(x+h) + 15] - [-x^2 + 10x + 15]}{h}$$

$$= \lim_{h \to 0} \frac{-(x^2 + 2xh + h^2) + 10x + 10h + 15 + x^2 - 10x - 15}{h}$$

$$= \lim_{h \to 0} \frac{-x^2 - 2xh - h^2 + 10x + 10h + 15 + x^2 - 10x - 15}{h}$$

$$= \lim_{h \to 0} \frac{-2x\cancel{h} - h^{\cancel{2}} + 10\cancel{h}}{\cancel{h}} = \lim_{h \to 0} -2x - h + 10$$

$$\boxed{f'(x) = -2x + 10}$$

The derivative of $f(x)$ for any x where the fcn. is differentiable.

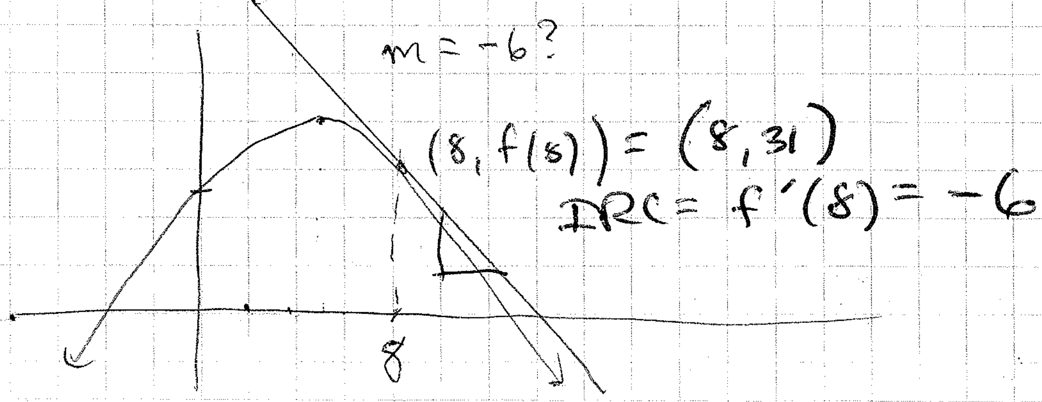
Method 4. Given polynomial $f(x) = -x^2 + 10x + 15$

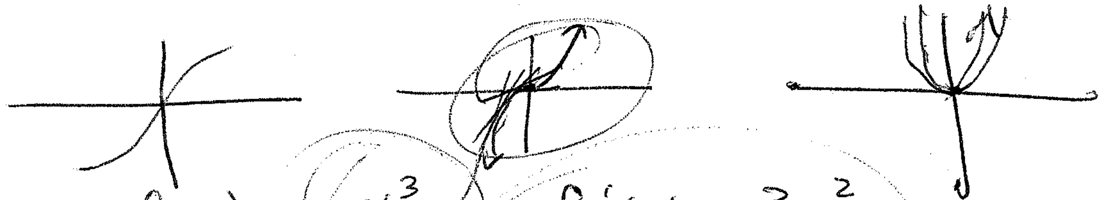
use the fact that $\boxed{f'(x) = -2x + 10}$

and evaluate $f'(x)$ at $x = 8$

$$f'(8) = -2(8) + 10 = -6$$

Look at the graph & verify the reasonableness of your answer.





Ex $f(x) = x^3$, $f'(x) = 3x^2$

Does $f'(a)$ exist for all $x = a$?
Yes, since the domain of the derivative is the same as that of the function.

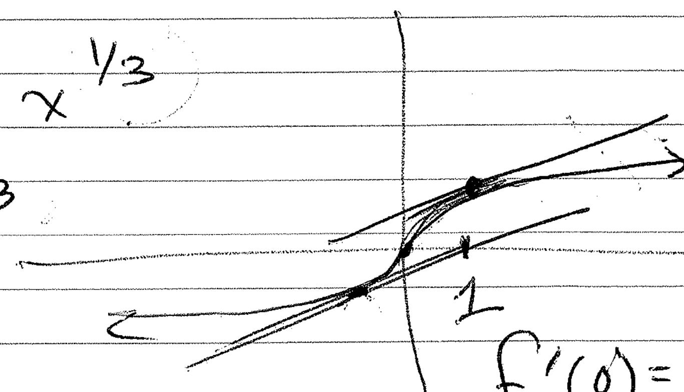
Namely, all $x \in \mathbb{R}$.

Ex $f(x) = x^{1/3}$ (the inverse of $y = x^3$)

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

$f'(0)$ DNE

$y = x^{1/3}$



$f'(0) = $ DNE

Derivative
$f'(x) = \frac{1}{3}x^{-2/3}$
$y' =$
$\frac{dy}{dx} =$

$\boxed{f'(x) = \frac{1}{3x^{2/3}}}$

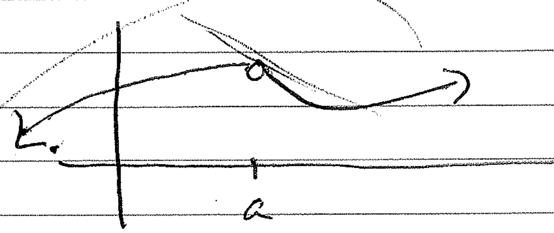
$x = 1$, $f'(1) = \frac{1}{3}$,

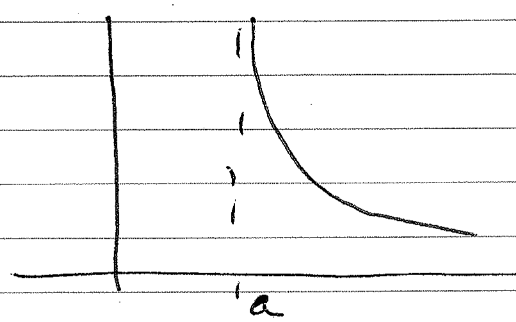
$x = -1$, $f'(-1) = \frac{1}{3(-1)^{2/3}} =$

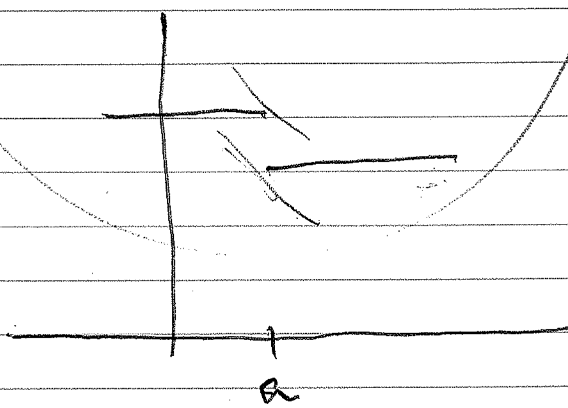
$\frac{1}{3((-1)^2)^{1/3}} = \frac{1}{3}$

What kinds of functions have places where an IRC cannot be found?

1. Any function that has a discontinuity. You cannot find IRC at such a point. That is, the function is <u>not</u> differentiable at $x = a$ if the function is discontinuous there.







3 kinds of <u>discontinuity</u>

- hole
- asymptote
- jump (step)

For all these situations, $f(x)$ is not differentiable at $x = a$