MATH 220

Binghamton University
Department of Mathematical Sciences

J. Brewster and
R. Geoghegan

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CALCULUS FOR BUSINESS AND MANAGEMENT

A TEXT FOR

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Jean Brewster

Ross Geoghegan

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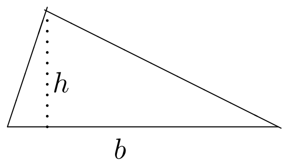
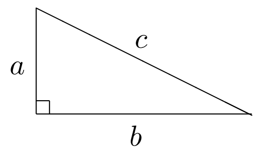
USEFUL FORMULAS FROM GEOMETRY AND ALGEBRA

Triangles



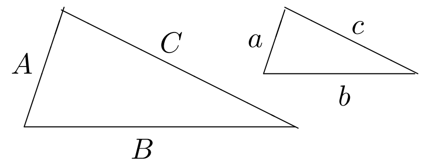
Area $= \frac{1}{2}bh$

Right Triangles



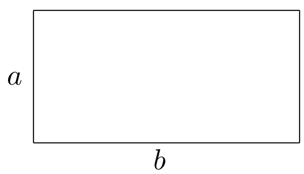
Pythagorean Theorem
$a^2 + b^2 = c^2$

Similar Triangles ($\cong$ angles)



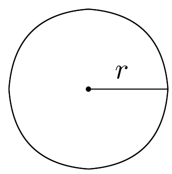
Sides Are Proportional
$\frac{a}{A} = \frac{b}{B} = \frac{c}{C}$; $\frac{A}{B} = \frac{a}{b}$, etc.

Rectangles



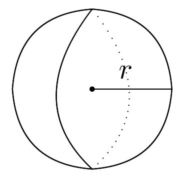
Area $= ab$
Perimeter $= 2a + 2b$

Circles



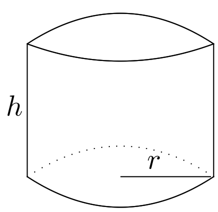
Area $= \pi r^2$
Circumference $= 2\pi r$

Spheres



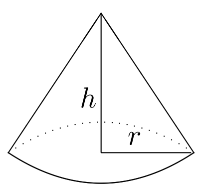
Volume $= \frac{4}{3}\pi r^3$
Surface Area $= 4\pi r^2$

Cylinders



Volume $= \pi r^2 h$
Surface Area $= 2\pi r^2 + 2\pi rh$

Cones



Volume $= \frac{1}{3}\pi r^2 h$
Surface Area $= \pi r\sqrt{r^2 + h^2}$

Distance and Midpoint Formulas for points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$:

Distance between P_1 and P_2 is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Midpoint of line segment $\overline{P_1P_2}$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Quadratic Formula The real solutions to $y = ax^2 + bx + c$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ when } (b^2 - 4ac) \geq 0.$$

Contents

Part I
SOME PRE-CALCULUS TOPICS

In the first four sections we review some topics from high school math. They are meant to be a review of things on which you may have gotten rusty. If you find that you are seriously challenged by these sections, you should talk to your instructor about taking Math 108 or Math 223 before this course. You need a good understanding of these topics for calculus.

1 Real Numbers, Calibrated Lines, Axes

Real Numbers

Examples of real numbers are: $1, 0, -3, \frac{4}{5}, \pi, \ .\overline{8}, \ -\sin 41°, \sqrt{17}$. A real number is any number that *can be* written as a decimal. Some of the numbers in the list of examples are not written as decimals, but they all *can* be written as decimals. We will drop the word "real" and just call them numbers[1] from now on.

The numbers $0, \pm 1, \pm 2, \pm 3, \cdots$ are called *integers*. The fractions, i.e. $\frac{\text{integer}}{\text{non-zero integer}}$ are also called *rational numbers* (so called because each is the ratio of two integers). The integer n is the same as the fraction $\frac{n}{1}$, so we often think of the set of all integers as a subset of the set of all fractions.

An *irrational number* is a number which is not rational. Examples of irrational numbers: $\sqrt{2}$ and π.

The numbers $0, 1, 2, 3, 4, 5, 6, 7, 8,$ and 9 are called *digits*. A *decimal* is an integer followed by a decimal point and then a string of digits. Example: -26.79. The string of digits might be infinitely long. If the infinitely long string consists of a finite string repeated indefinitely we call the whole thing a *repeating decimal*. Example: $65.91235235235235235235\ldots$ This is usually shortened to $65.91\overline{235}$, where the bar over 235 means those three digits are to be repeated forever. A decimal is *terminating* if the string to the right of the decimal point is finite in length[2].

Here is the relationship between fractions and decimals:

- A decimal is a rational number if it is terminating or repeating.
- A decimal is an irrational number if it is neither terminating nor repeating.

For example, nobody knows, or can possibly know, the full decimal version of π, because that would require an infinite amount of knowledge[3]!

Here is an example of changing a repeating decimal to a fraction:

Example 1.1. Change $.\overline{57}$ into its rational form.

Suppose we call the desired number x. That is, $x = .\overline{57}$. Then:

$$100x = 57.\overline{57}$$

$$100x - x = 57.\overline{57} - .\overline{57}$$

$$99x = 57$$

$$x = \frac{57}{99}$$

[1]There is a more general kind of number called a "complex number"; the word "real" is used to indicate that one is not talking about complex numbers.

[2]Actually, a terminating decimal can be thought of as repeating: for example, 3.151 is the same thing as $3.151\overline{0}$.

[3]There are things hidden here which you don't need for this course. It can be proved, using difficult mathematics, that π is irrational. The proof doesn't involve the decimal version; rather, one shows that if π were rational that would imply something which is known to be false, e.g. $1 = 2$.

The symbol for the set of all real numbers is $\mathbb{R}$.

Important note There is no number called ∞ and no number called $-\infty$. Yet in calculus you will see these symbols often. How to use them correctly is part of the calculus story which we will discuss as the course unfolds.

Numbers on a line

Draw a horizontal line. Pick a point on that line and label it 0. Pick a point to the right of 0 and label it 1. Now label the points twice as far to the right, three times as far to the right etc. by $2, 3, \cdots$. To the left of 0 label the mirror image points $-1, -2, -3, \cdots$. This associates a point on the line with every integer. Now proceed to associate a point on the line with every rational number (i.e. fraction) in the obvious way: for example $2\frac{1}{3}$ is the label of the point between 2 and 3 which is half as far from 2 as from 3.

What about the irrational numbers? For that we need to think in terms of decimals. So, once you have used every rational number (terminating or repeating decimal) to label a point on the line, you can (sort of) see how to squeeze in all the irrationals, since an irrational is closely approximated by chopping off the decimal digits from some point onward and replacing them with $\overline{0}$: the further out you chop, the better the approximation.

The process of associating each number with a point on the line is called *calibrating* the line. This is all done much more precisely in higher mathematics courses.

Putting 1 to the right of 0 rather than the left is a convention, but everyone follows this convention. If the line is vertical the convention is to put 1 above 0.

Axes

In analytic geometry and calculus, when you "draw an x-axis and a y-axis" in a plane you are really drawing horizontal and vertical calibrated lines. The point of intersection of the two lines is 0 on both. It is customary to use the same calibration on both lines: i.e., if 1 is one inch to the right of 0 on the x-axis, then 1 is one inch above 0 on the y-axis. Sometimes this is not practical, such as when the horizontal axis represents the number of airplanes sold and the vertical axis represents millions of dollars. In this course we will often (but not always) use the same calibration for the x-axis and the y-axis.

A point in the plane is named by two numbers, the *x-coordinate* and the *y-coordinate*, e.g. (x, y) could be $(-4, 1)$, $(0, 0)$, $(\pi, \frac{1}{\sqrt{2}})$.

Numbers versus points

There is one number for each point on a calibrated line, and one point on the line for each number. In view of the discussion here, you can also think of $\mathbb{R}$ as a calibrated line, perhaps an x-axis or a y-axis. This connection gives us insight into two important properties of $\mathbb{R}$.

1. *The set of real numbers is ordered.* Each number has its unique place on the line. For any two *different* numbers a and b either $a < b$ or $b < a$.

2. *The set of real numbers is dense.* Between any two distinct points on the line, there is another point. For any two numbers a and b, where $a < b$, there is some number c such that [4] $a < c < b$.

Other useful vocabulary, symbols, and reminders

A fraction, $\frac{a}{b}$, is a ratio of numbers, $a : b$. It is also an expression of division, a divided by b. The number written on the top (in this case a) is the *numerator*. The number written on the bottom (b) is the *denominator*.

No fraction can have a 0 as its denominator. Division by 0 is *always* meaningless.

Positive numbers are those strictly greater than zero. *Negative* numbers are those strictly less than zero. Zero is neither positive nor negative. So, the expression "x is a positive number" is not the same as "x is a non-negative number."

Given two numbers $a < b$, the *open interval* (a, b) is the set of all numbers x such that $a < x < b$. Note that a and b are not members of this open interval, but all numbers between them are members. For example, $(-4, 1)$ denotes the set of numbers between [5] -4 and 1. The sets $(-\infty, 3)$ and $(-7, \infty)$ are also considered to be open intervals. The set $\mathbb{R}$ of all numbers is also sometimes written as $(-\infty, \infty)$.

A set of numbers is *bounded above* if there is some number, not necessarily in the set, greater than or equal to every number in the set. A set of numbers is *bounded below* if there is some number, not necessarily in the set, less than or equal to every number in the set. A set of numbers is *bounded* if it is both bounded above and bounded below. For example, the interval $(-3, 7)$ is bounded. The number -4 is less than every number in the set and the number 10 is greater than every number in the set. The set of all integers between 4 and 400 is a bounded set. The set of integers ≥ 15 is bounded below but it is not bounded above.

Unbounded means "not bounded" Certainly $(-\infty, \infty)$ is an unbounded set. Also, the set $(-\infty, 5)$ is an unbounded set because there is no number that is less than every number in the set.

Given two numbers $a < b$, the *closed* interval $[a, b]$ is the set of all numbers x such that $a \leq x \leq b$. For example, the set [5, 13] is a closed interval; it is the set of numbers x such that $5 \leq x \leq 13$. The smallest number in the interval is 5 and the largest in the interval is 13.

An interval such as $(1, 4]$ or $[-7, 0)$ is said to be *half-open* (or *half-closed*). The intervals $[-\infty, 3)$, $(-7, \infty]$ and $[-\infty, \infty]$ are not allowed because $\pm\infty$ are not numbers.

[4] Intuitively, you can keep squeezing in more and more points (or numbers) and you will never finish.

[5] Be careful here: the symbol $(-4, 1)$ can also mean the point in the plane with x-coordinate -4 and y-coordinate 1. You have to figure out from the context whether it means the interval $(-4, 1)$ in $\mathbb{R}$ or the point $(-4, 1)$ in the plane.

Section 1 - Exercises (answers follow)

1. Without using a calculator, write $\frac{2}{3}$, $\frac{6}{11}$ and $\frac{11}{6}$ as repeating decimals.
2. Again without a calculator, write $\frac{1}{11}$, $\frac{2}{11}$, $\frac{3}{11}$, and $\frac{4}{11}$ as repeating decimals. Look at the pattern of your answers. What do you expect $\frac{5}{11}$ to be? What about $\frac{6}{11}$, $\frac{9}{11}$, $\frac{10}{11}$?
3. Write $\frac{1}{9}$, $\frac{2}{9}$, $\frac{3}{9}$, and $\frac{4}{9}$ as repeating decimals. What do you expect $\frac{7}{9}$ to be? The pattern suggests an answer for $\frac{9}{9}$. Of course we know that $\frac{9}{9} = 1 = 1.0000\cdots$, so the number 1 has at least two decimal representations, one of which is eventually all 0's and the other of which is eventually all 9's. It is a fact that no fraction has more than two decimal representations and that the only way a fraction can have two decimal representations (rather than one) is if one of them is eventually all 0's and the other is eventually all 9's. It's a doable but challenging problem to figure out why this is true.
4. Using a computer or calculator, try to write $\frac{23}{17}$ as a repeating decimal. Unless you use a computer or a calculator that gives many decimal places you won't see the answer. But if you do it by hand using long division you'll get the answer in a short time. Have a race with a friend to see who gets it first.
5. On the basis of these division exercises can you figure out a general rule which will tell you, for a given fraction, the maximum number of decimal places that could be needed to get the repeater in the repeating decimal?
6. Using your answer to problem 4, write $\frac{23}{1700}$ as a repeating decimal.
7. Change the following decimals into fractions: $.75$, 45.024, $.\overline{85}$, $3.2\overline{85}$, $.3\overline{857}$.
8. Decide whether each statement below is TRUE or FALSE.
   (a) An irrational number is a real number.
   (b) ∞ is an irrational number.
   (c) Between any two rational numbers is another rational number.
   (d) Between any two rational numbers is an irrational number.
   (e) Between any two rational numbers is an integer.
   (f) Between any two irrational numbers is a rational number.
   (g) The number $34.\overline{9}$ sits right next to the number 35 on a number line.
   (h) $\frac{0}{0} = 1$
   (i) $\frac{0}{0} = 0$
   (j) $\frac{1}{0} = \infty$
   (k) $\infty - \infty = 0$
   (l) All unbounded sets are open intervals.
   (m) All closed intervals are bounded sets.

(n) All irrational numbers have exactly one decimal representation.

(o) Integers are the only numbers that have two decimal representations.

(p) $\frac{\sqrt{5}}{3}$ is a rational number.

(q) If a number is not irrational, then it must be a rational number.

Section 1 - Answers

1. $.\overline{6}$, $.\overline{54}$, $1.8\overline{3}$

2. $.\overline{09}$, $.\overline{18}$, $.\overline{27}$, $.\overline{36}$, $.\overline{45}$, $.\overline{54}$, $.\overline{81}$, $.\overline{90}$

3. $.\overline{1}$, $.\overline{2}$, $.\overline{3}$, $.\overline{4}$, $.\overline{7}$, $.\overline{9}$

4. $1.\overline{3529411764705882}$

5. Hint: When you do the long division, how many *possible* remainders are there for each subtraction step?

6. $0.01\overline{3529411764705882}$

7. $\frac{3}{4}$, $\frac{45024}{1000} = \frac{5628}{125}$, $\frac{85}{99}$, $\frac{3253}{990}$, $\frac{3854}{9990} = \frac{1927}{4995}$

8. (a) True

   (b) False. Irrational numbers are real. ∞ is not a real number.

   (c) True

   (d) True

   (e) False. Example: There is no integer between $\frac{1}{3}$ and $\frac{1}{2}$.

   (f) True

   (g) False. These two numbers are equal. They are the same point on the line.

   (h) False. Division by zero is never allowed.

   (i) False. Division by zero is never, never, allowed.

   (j) False. Division by zero is never, never, never, allowed.

   (k) False. You cannot use the arithmetic operations for real numbers on numbers that are not real.

   (l) False. Example: The set of integers is an unbounded set, but isn't an interval at all.

   (m) True

   (n) True

   (o) False. Any terminating decimal can be written in two ways. For example: $2.75 = 2.74\overline{9}$

   (p) False. A rational number is a ratio of *integers*. $\sqrt{5}$ is not an integer.

   (q) True

2 Overview of Functions

In this section we discuss functions and some points of algebra. These are topics that were covered in high school classes but perhaps need to be reviewed.

Here are some equations in the two variables x and y:

- $y = \sqrt{x^2+7} - 4$
- $y = -6x$
- $y = \pi$
- $x^2 + y^4 = y + 3$
- $x^2 + y^2 = 9$
- $x = 5x^2 - 4$

The symbol "=" is a verb meaning "is the same as" or "equals." An equation always involves at least one variable, some numbers, and an = symbol. In the first and second equations, above, y is on the left of the = symbol, and only x's and constants (i.e. numbers) are on the right. In the third equation the same is true (even though there are no x's on the right). But the other three equations are not of this form. To put this in different words, the first two equations look like $y =$ formula in x, or more briefly $y = f(x)$, while the last three are not of that form.

Whenever an equation has the form $y = f(x)$ we say that "*y is a function of x*". $f(x)$ might be a constant function, for example, $f(x) = -1$ or $y = 8$.

More abstractly, if $f(x)$ is a function and a specific value, say a, is used for x, then $f(a)$ is the called the *value* of f at $x = a$. To *evaluate* $f(x) = 2x + 3$ at $x = -4$ write $f(-4) = 2(-4) + 3 = -8 + 3 = -5$.

Domain and Definition of Function

When studying a function $f(x)$ in a particular problem, one often needs to be clear on which numbers x are to be permitted as "plug-ins" in the formula $f(x)$. This set of numbers x is called the *domain* of the function $f(x)$ and is denoted by D_f. The domain of $f(x)$ will always be found in one of three ways:

1. D_f may be the *natural domain* of the formula $f(x)$; this means: D_f is the set of all numbers x for which the formula $f(x)$ makes sense. For example the natural domain of $f(x) = \sqrt{x^2-7}$ is the union of two intervals: $(-\infty, -\sqrt{7}] \cup [\sqrt{7}, \infty)$, because if x is not in one of those intervals the formula would involve the square root of a negative number[6]. The natural domain of the function $f(x) = x^2 + 3x - 7$ is the set of all numbers, since the formula makes sense for any $x \in \mathbb{R}$.

[6]If $y < 0$ there is no number x such that $x^2 = y$. Remember, we only deal with real numbers.

2. Sometimes, the natural domain makes sense mathematically, but in the physical or real-life problem under consideration some of those allowable values of x do not make sense. For example, you know that the formula for the area of a circle of radius r is $A(r) = \pi r^2$. The natural domain for this function $A(r)$ is the set $\mathbb{R}$ of all numbers, since any number can be squared and then multiplied by π. But who ever heard of a circle of negative radius? So if the problem is about areas of circles it would be understood, even if not explicitly stated, that the domain D_A is the set of non-negative numbers (numbers ≥ 0) rather than the set of all numbers. More generally, if common sense tells you that the natural domain is too big to be useful in your problem, go with common sense in identifying the domain for your problem.

3. The person setting the problem may specify the domain explicitly for you. For example, in the problem *find the maximum value of the function* $f(x) = x^2$ *when* $-1 \leq x \leq 6$ the restricted domain D_f is specified as the closed interval $[-1, 6]$. (By the way, what is the answer?)

In summary: Always (*yes, always*) write down the domain of the function you are considering before you tackle a problem. The domain D_f is the appropriate set of numbers x to be considered in connection with the function $f(x)$ in your problem. Remember: D_f cannot be bigger than the natural domain, but it may be smaller.

Example 2.1. Find the domain for each of the following functions.

1. $f(x) = \frac{3+x}{x-2}$
   Answer: $D_f = (-\infty, 2) \cup (2, \infty)$ because the x cannot take on any value that would make the denominator equal to zero.

2. $g(x) = \sqrt{4-x}$
   Answer: $D_g = (-\infty, 4]$ because the square root of a negative number does not exist (remember that we are only working in $\mathbb{R}$). We can include $x = 4$ because $\sqrt{4-4}$ does exist. $\sqrt{0} = 0$.

3. $C(p) = 1.25p$ where C represents the cost to buy p slices of pizza.
   Answer: D_C is the set of non-negative integers. We assume here that in real life the vendor doesn't sell fractions of slices.

4. $K(s) = 125s$ where K represents the number of calories in s 8 oz. cans of soda.
   Answer: $D_K = [0, \infty)$. Here, fractional values of s do make sense.

5. $h(x) = x^2 + 3x - 7$ where $-1 < x \leq 8$
   Answer: $D_h = (-1, 8]$. Here the domain was explictly given.

To summarize: A *function* consists of two sets of numbers X (the *domain*) and Y (the *range*) and a *rule* that assigns to each number x in the domain exactly one number y in the range; x is the *input* or *independent variable*; y is the *output* or *dependent variable.*

Graphs of Equations and Functions

Here are some of the equations we began with:

(i) $y = \sqrt{x^2 + 7} - 4$

(ii) $y = -6x$

(iii) $x^2 + y^4 = y + 3$

(iv) $x^2 + y^2 = 9$

These are examples of equations that involve two variables x and y. The *graph* of such an equation is a picture in the xy-plane that shows information about the equation in visual form. As an example, let's think about the graph of the equation $x^2 + y^4 = y + 3$. The point $(\sqrt{3}, 1)$ is on the graph because when you plug in those values for x and y you get a true statement: both sides equal 4. The point $(\sqrt{3}, 0)$ is also on the graph because when you plug in those values for x and y you also get a true statement: this time both sides equal 3. The point $(2, \frac{1}{2})$ is not on the graph because $4 + \frac{1}{16} \neq \frac{1}{2} + 3$. In general the *graph* of an equation is the set of points in the plane such that when the first coordinate of the point is plugged in for x and the second coordinate of the point is plugged in for y you get a true statement.

Now consider a particular function, say, $y = 8x^3 - 6$. We can consider this as an equation in two variables, so it has a graph. The graph of this function is the set of all points $(x, 8x^3 - 6)$. For example, $(0, -6)$ is on the graph. The natural domain of this function is the set of all numbers, so there will be a point on the graph for each value of x. The equation actually says that there is only one point on the graph for each x (why?). To find other points simply substitute any number in $\mathbb{R}$ for x in $8x^3 - 6$.

In the case of $x^2 + y^4 = y + 3$ we saw a choice of x, namely $x = \sqrt{3}$, that gave rise to two different points on the graph of the equation. In this example, there is *more than one value of y for a given value of x*. So, according to the definition of "function" this equation does not define a function (because there is not a unique y for a given x). This illustrates a general fact: *a vertical line in the plane will intersect the graph of a function at most once.*

Piecewise Defined Functions

One type of function that occurs frequently in real life, but which you may not have studied previously, is the piecewise-defined function. This is a function whose domain can be thought of as broken into pieces. Each piece of the domain has its own "rule" for finding the function values (y values). Some examples ofpiecewise-defined functions are below.

Example 2.2.

1. A car rental company charges \$270 per week to rent a compact car. The first 300 miles driven are "free." If more than 300 miles are driven, the company charges an additional 55 cents per mile. A function that describes the cost, C, in dollars, to rent a car driven m miles in one week is given by:
$$C(m) = \begin{cases} 270 & 0 \leq m \leq 300 \\ 270 + .55(m - 300) & 300 < m \end{cases}$$

2. In 2011, the federal income tax owed by a single person with a taxable income of \$100,000 or more is figured by the following function. T represents the tax due. I represents the taxable

income.[7]

$$T(I) = \begin{cases} .28I - 6,383 & 100,000 \le I \le 174,400 \\ .33I - 15,103 & 174,400 < I \le 379,150 \\ .35I - 22,686 & I > 379,150 \end{cases}$$

3.

$$f(x) = \begin{cases} 1 & x \text{ is rational} \\ -1 & x \text{ is irrational} \end{cases}$$

4.

$$g(x) = \begin{cases} -x & x < 0 \\ x & x \ge 0 \end{cases}$$

In the first example, the domain of the function is $[0, \infty)$. The domain is split into two pieces: $[0, 300]$ and $(300, \infty)$ For each piece, the function value (y value, or $C(m)$ value) is calculated by a different rule. For values of m in $[0, 300]$ the rule simply assigns the function value 270 (dollars). For m in $(300, \infty)$, the rule calculates the function value using the mathematical formula $270 + .55(m - 300)$.

Does this make sense? What should it cost if the renter only drives 70 miles? $C(70) = 270$ dollars. What should it cost if the renter drives 400 miles? $C(400) = 270 + .55(400 - 300) = 270 + 55 = 325$ dollars. What is $C(450)$? $C(45)$? Answers: \$352.50, \$270.

In the second example, how much tax is owed if the taxable income is \$200,000?
$T(200,000) = 200,000 \times .33 - 15,103 = 50,897$ What is T(1,000,000)? What does it mean? Answer: \$327,314 is the tax owed for a taxable income of \$1,000,000

In the third example, what is $f(4)$? Since 4 is a rational number, $f(4) = 1$. What is $f(\pi)$? Since π is irrational, $f(\pi) = -1$. What is $f(\frac{1}{2})$? $f(-10)$? Answers: 1, 1

Try some values in the last function. You should recognize $g(x)$ as one way to express the absolute value function.

The domain for a piecewise defined function is very easy to determine because it is given directly. You only need to look in the right column of the function and see the possible values for the independent variable. $D_C = [0, \infty)$, $D_T = [100,000, \infty)$, and $D_f = \mathbb{R}$.

It is emphasized here that a piecewise defined function *is* a function. For each independent variable there is only one function value. The function may use multiple ways to find values, but only one way is appropriate for any given domain element.

Function Operations

Functions can be added, subtracted, multiplied, divided (being careful not to divide by zero) and composed. We review those operations and notations with the following example:

[7] From www.irs.gov

Example 2.3. Suppose $f(x) = \frac{x}{x^2+3}$ and $g(x) = \sqrt{x}$. Then:

$(f+g)(x) = f(x) + g(x) = \frac{x}{x^2+3} + \sqrt{x}$
$(f-g)(x) = f(x) - g(x) = \frac{x}{x^2+3} - \sqrt{x}$
$(f \cdot g)(x) = f(x) \cdot g(x) = \frac{x}{x^2+3} \cdot \sqrt{x} = \frac{x\sqrt{x}}{x^2+3}$
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \dfrac{\frac{x}{x^2+3}}{\sqrt{x}} = \dfrac{x}{(x^2+3)\sqrt{x}}$
$(f \circ g)(x) = f\left(g(x)\right) = f(\sqrt{x}) = \frac{\sqrt{x}}{(\sqrt{x})^2+3}$

Certainly the subtraction and division operations are not, in general, commutative[8]. $(f-g)(x) \neq (g-f)(x)$ and $\left(\frac{f}{g}\right)(x) \neq \left(\frac{g}{f}\right)(x)$. The composition of functions is not commutative either. $(f \circ g)(x) \neq (g \circ f)(x)$. Using the functions from Example 2.3 we get: $(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x^2+3}\right) = \sqrt{\frac{1}{x^2+3}}$. This clearly not the same as $(f \circ g)(x)$.

When algebraically combining functions, you must be careful about the domain of the newly created function. When you look at $\left(\frac{f}{g}\right)(x)$ and $(f \circ g)(x)$ in Example 2.3 you might be tempted to automatically simplify the expressions on the far right. You may do so, but you must do so correctly.

The expression $\dfrac{x}{(x^2+3)\sqrt{x}}$ is not defined at $x = 0$. So the domain for $\left(\frac{f}{g}\right)(x)$ cannot include zero. The expression $\dfrac{\sqrt{x}}{x^2+3}$ however, IS defined at $x = 0$. So, to write $\left(\frac{f}{g}\right)(x) = \dfrac{\sqrt{x}}{x^2+3}$ is not correct. You must write "$\left(\frac{f}{g}\right)(x) = \dfrac{\sqrt{x}}{x^2+3}$, if $x \neq 0$."

A similar domain issue occurs when dealing with simplifying $(f \circ g)(x)$. $(\sqrt{x})^2 = x$ *only* for $x \geq 0$. If x is negative, $(\sqrt{x})^2$ does not exist. To simplify $(f \circ g)(x)$, then you must be sure that the domain is clear. "$(f \circ g)(x) = \frac{1}{x+3}$ if $x \geq 0$."

Algebra We are now ready for the algebra review.

Exponents It is assumed that you are familiar with basic exponent rules and are proficient in using them, at least for positive integer exponents. This section is to remind you how to interpret other exponents and to reinforce the idea that the rules for other exponents are essentially the same as those for positive integer exponents.

1. $a^0 = 1$ for all numbers a, EXCEPT $a = 0$. 0^0 is not defined.
2. $a^{-n} = \dfrac{1}{a^n}$, EXCEPT when $a = 0$ because a denominator can never be zero. Two immediate consequences of this rule are:
   (a) $a^{-1} = \dfrac{1}{a}$
   (b) $\dfrac{1}{a^{-n}} = a^n$

[8] i.e. you don't get the same thing when you interchange f and g.

3. $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$. Of course a cannot be negative if n is even. When $m = 1$, this rule simplifies to $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Example 2.4. Here are some examples for using these rules.

1. $\left(\frac{2}{5}\right)^{-1} = \frac{1}{2/5} = \frac{5}{2}$
2. $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$
3. $\sqrt[4]{\sqrt[3]{x}} = \left(x^{\frac{1}{3}}\right)^{\frac{1}{4}} = x^{\frac{1}{12}} = \sqrt[12]{x}$
4. $\dfrac{x^2y^{-5}z}{x^{-4}yz^3} = x^6y^{-6}z^{-2} = \dfrac{x^6}{y^6z^2}$

Be Careful with Parentheses

1. $ab^n = a \cdot b^n$
2. $(ab)^n = a^n \cdot b^n$
3. $(a + b)^n \neq a^n + b^n$

You might think that these are obvious, but don't be insulted that they are here. Many an error has been made when an expression like -5^2 is equated to 25. While it is true that $-5 \cdot -5 = 25$, it is not true that -5^2 means $-5 \cdot -5$. To correctly write $-5 \cdot -5$, one would need parentheses: $(-5)^2$. The correct evaluation of -5^2 is $-(5 \cdot 5) = -25$.

There are even more frequent abuses of the third rule. When $n = 2$, there isn't much problem. You would never think to write $(a+b)^2 = a^2+b^2$ because you know to "FOIL" the $(a+b)^2$. However, when n is a value other than 2 there is a sorry eagerness to "distribute" the power through the parentheses. Sometimes the n is disguised as a root so the crime is not so obvious.

Here are some typical errors involving parentheses and exponents.

ERROR: $\sqrt{a^2 + b^2} = a + b$ No! This is saying $(a + b)^{\frac{1}{2}} = a^{\frac{1}{2}} + b^{\frac{1}{2}}$

ERROR: $\left(\frac{1}{a} + \frac{1}{b}\right)^{-1} = a + b$ No! This is saying $(a + b)^{-1} = a^{-1} + b^{-1}$

ERROR: $\sqrt[3]{x^3 + 8} = x + 2$ No! What is this saying?

A Reminder about Even Roots

If $a > 0$ then $\sqrt{a}$ means "the positive square root of a.. The other square rootof a is $-\sqrt{a}$ which is the negative square root of a. It is incorrect to say: $\sqrt{16} = \pm 4$; rather $\sqrt{16} = 4$ and the other square root of 16 is -4.

The number 0 has only one square root, namely 0. If $a < 0$ then a does not have a square root, because "minus by minus" and "plus by plus" are both positive.//

The Absolute Value Function

The absolute value of a number a is written $|a|$. Here.s the rule: If $a \geq 0$ then $|a| = a$. If $a < 0$ then $|a| = -a$. Remember -(negative) is positive.. Example $|-3| = -(-3) = 3$.

Absolute values are never negative.

We will be using the function $f(x) = |x|$ several times in this course. It is given by

Definition 2.1. *The* <u>*absolute value*</u> *of x is defined to be:*

$$f(x) = |x| = \sqrt{x^2} = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

One definition uses square roots, the other defines $f(x)$ piecewise.

Finding solutions to $f(x) = 0$

Often we will need to find the values of x which make $f(x)$ equal to 0. We call this “finding the zeros.. Here are some examples [9]:

Example 2.5. Find the zeros of the following functions:

1. $f(x) = x^2 + 3x + 2$
   Solution: $f(x) = (x+1)(x+2)$, so $x + 1 = 0$ or $x + 2 = 0$. $x = -1$ or $x = -2$.

2. $g(x) = \dfrac{2x-5}{x^2-7}$
   Solution: $2x - 5 = 0$. $x = \frac{5}{2}$

3. $h(x) = \dfrac{x^2-9}{x+3}$
   Solution: $x^2 - 9 = (x+3)(x-3) = 0$, so $x + 3 = 0$ or $x - 3 = 0$. $x = -3$ or $x = 3$. However, the domain of this function does not include $x = -3$, so $x = 3$ is the only root.

4. $F(x) = (x+2)^{\frac{1}{2}} + \frac{1}{2}(x+2)^{-\frac{1}{2}}$
   Solution: $F(x) = (x+2)^{-\frac{1}{2}}[(x+2)^1 + \frac{1}{2}] = (x+2)^{-\frac{1}{2}}(x + \frac{5}{2}) = 0$. So, $x + \frac{5}{2} = 0$. $x = -\frac{5}{2}$
   Notice that we do not have $x = -2$ as a solution. It is not in the domain of the function.

Example 2.6. Solve the following equations for x.

1. $x^3 = x$
   Solution:
   $$\begin{aligned} x^3 &= x \\ x^3 - x &= 0 \\ x(x^2-1) &= 0 \\ x(x+1)(x-1) &= 0 \end{aligned}$$
   So, $x = 0$, $x = -1$, or $x = 1$ Notice that we do not begin by “canceling” an x from both sides. If we did that, we would have $x^2 = 1$ and not find the root $x = 0$.

2. $x^2 - 4x - 5 = 7$
   $$\begin{aligned} x^2 - 4x - 5 &= 7 \\ x^2 - 4x - 12 &= 0 \\ (x+2)(x-6) &= 0 \end{aligned}$$

[9] In finding the zeros, you often find yourself using the following basic facts about numbers:
If $a \cdot b = 0$, then $a = 0$ or $b = 0$
If $\dfrac{a}{b} = 0$, then $a = 0$. Reminder: $b \neq 0$

So, $x = -2$ or $x = 6$. Notice that we did not factor the left side immediately. If we did that, we would have $(x+1)(x-5) = 7$. This is not useful because we cannot conclude: "$x + 1 = 7$ or $x - 5 = 7$." The fact "If $a \cdot b = 0$ then $a = 0$ or $b = 0$" only works for zero.

Section 2 - Exercises (answers follow)

1. Specify the domain of the given function.

   (a) $f(x) = x^3 - 3x^2 + 2x + 5$

   (b) $y = \dfrac{2x-4}{2x+5}$

   (c) $f(x) = \dfrac{4x+2}{x^2}$

   (d) $g(t) = \sqrt{t^2+4}$

   (e) $f(x) = -\sqrt{\dfrac{5}{x^2+6}}$

   (f) $f(x) = \sqrt{\dfrac{-5}{x^2+6}}$

   (g) $f(x) = -\sqrt{\dfrac{5}{x+6}}$

2. For the following functions find the domain and all roots.

   (a) $f(x) = \sqrt{2x-7}$

   (b) $f(x) = \sqrt{5-x}$

   (c) $f(x) = \frac{x^2+x-2}{x^2+7x+10}$

   (d) $f(x) = \frac{x^2+2}{2x+1}$

   (e) $f(x) = \frac{x^2+3x}{x}$

   (f) $f(x) = \sqrt[3]{\frac{x-2}{x+6}}$

   (g) $f(x) = \sqrt{\frac{x}{x+1}}$

   (h) $f(x) = \sqrt{\frac{x-2}{x+6}}$

   (i) $f(x) = \frac{x^2+2x-15}{x-3}$

   (j) $f(x) = \sqrt{16-x^2} - \frac{12}{\sqrt{16-x^2}}$

3. For piecewise defined function f, find: domain, $f(0)$, $f(1)$, $f(5)$

   $$f(x) = \begin{cases} 2x+2, & x < 1, \\ 4x, & 1 < x < 3, \\ \frac{3+x}{3-x}, & x > 3. \end{cases}$$

   Challenge: Does this function have any roots? If so, what are they?

4. Given $f(x) = \frac{3}{x+1}$ and $g(x) = \frac{x+2}{x-1}$.
   Find the following functions and their domains: $(f+g)(x)$, $(fg)(x)$, $\left(\frac{f}{g}\right)(x)$.

5. Given $f(x) = 2 - 3x^2$ and $g(x) = x - 1$.
   Find: $(f \circ g)(x)$, $(g \circ f)(x)$, $(g \circ g)(x)$ and $(f \circ f)(2)$.

6. Given $f(x) = 1 - x$ and $g(x) = \begin{cases} 2x & x < 0 \\ x^2 & x \geq 0 \end{cases}$

   Find: $(f \circ g)(4)$, $(f \circ g)(-4)$, $(g \circ f)(4)$ and $(g \circ f)(-4)$.

7. For each function F, find two functions f and g such that $F = (f \circ g)$.
   Do not use the trivial $f(x) = x$ or $g(x) = x$.

   (a) $F(x) = \frac{3}{x+5}$

   (b) $F(x) = \sqrt{x^2 + x - 2}$

8. $32^{\frac{4}{5}}$ can be written as $\sqrt[5]{32^4}$ or as $(\sqrt[5]{32})^4$.
   Which expression is easier to evaluate? Evaluate $32^{\frac{4}{5}}$ without a calculator.

9. Without a calculator, evaluate the following:
   (a) 17^0 (b) $8^{-\frac{1}{3}}$ (c) $4^{\frac{3}{2}}$ (d) $100^{\frac{1}{2}} - 64^{\frac{1}{2}}$ (e) $(100-64)^{\frac{1}{2}}$ (f) -3^2
   (g) $\sqrt{25}$ (h) $\sqrt{-9}$

10. Change the following to exponential form (eliminate the radical sign). Simplify.
    (a) $\sqrt[3]{x^5}$ (b) $\left(\sqrt[5]{2x}\right)^3$ (c) $\left(\sqrt{\frac{x}{y^3}}\right)^5$ (d) $\frac{x}{\sqrt[5]{x^3}}$ (e) $\sqrt[6]{\sqrt[3]{x^4}}$

11. Change the following to radical form:
    (a) $x^{\frac{1}{3}}$ (b) $-x^{\frac{1}{2}}$ (c) $(-x)^{\frac{1}{2}}$ (d) $x^{\frac{9}{5}}$ (e) $-3x^{\frac{2}{3}}$ (f) $2(xy)^{-\frac{3}{4}}$

12. For which values of x is each of the following defined?
    (a) $\sqrt{x}$ (b) $\sqrt{-x}$ (c) $\sqrt{x^2}$ (d) $\frac{1}{\sqrt{x}}$ (e) $\sqrt{x-6}$ (f) $\sqrt{6-x}$ (g) $\sqrt[3]{x}$

13. Which expressions, if any, are equivalent to $\sqrt{(-x)^5}$?
    (a) $x^{-\frac{5}{2}}$ (b) $(-x)^{\frac{5}{2}}$ (c) $-x^{\frac{2}{5}}$ (d) $(-x)^{\frac{2}{5}}$ (e) $-\sqrt{x^5}$ (f) $\sqrt{-x^5}$ (g) $(\sqrt{-x})^5$

14. Rewrite into exponential form with only positive exponents. Simplify.
    (a) $\left(x^{\frac{1}{2}}\right)^{-\frac{1}{3}}$ (b) $\left(\frac{3x}{y}\right)^{-2}$ (c) $x^{\frac{1}{2}}x^{-\frac{2}{3}}$ (d) $\sqrt{x^{-7}}$ (e) $\left(\frac{a^{-2}}{b^{-2}} + \frac{b^{-2}}{a^{-1}}\right)^{-1}$

    (f) $\left(\frac{x^{m^2}}{x^{2m-1}}\right)^{\frac{1}{m-1}}$ where m is a constant and $m > 1$

15. Rewrite into radical form. Simplify as much as possible.
    (a) $\left(\frac{x^6 y}{z^3}\right)^{\frac{1}{2}}$ (b) $\left(\frac{x^2 + y^2}{x^4}\right)^{\frac{1}{2}}$

Section 2 - Answers

1. (a) $\mathbb{R}$

   (b) $(-\infty, -\frac{5}{2}) \cup (-\frac{5}{2}, \infty)$

   (c) $(-\infty, 0) \cup (0, \infty)$

   (d) $\mathbb{R}$

(e) $\mathbb{R}$

(f) $\emptyset$ (No real numbers are valid in this expression)

(g) $(-6, \infty)$

2. (a) Domain: $[\frac{7}{2}, \infty)$ Roots: $\frac{7}{2}$

   (b) Domain:$(-\infty, 5]$ Roots: 5

   (c) Domain:$(-\infty, -5) \cup (-5, -2) \cup (-2, \infty)$ Roots: 1

   (d) Domain:$(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$ Roots: none

   (e) Domain:$(-\infty, 0) \cup (0, \infty)$ Roots: -3

   (f) Domain:$(-\infty, -6) \cup (-6, \infty)$ Roots: 2

   (g) Domain:$(-\infty, -1) \cup [0, \infty)$ Roots: 0

   (h) Domain:$(-\infty, -6) \cup [2, \infty)$ Roots: 2

   (i) Domain:$(-\infty, 3) \cup (3, \infty)$ Roots: -5

   (j) Domain:$(-4, 4)$ Roots: $2, -2$

3. Domain: $(-\infty, 1) \cup (1, 3) \cup (3, \infty)$ $f(0) = 2$, $f(1)$ does not exist, $f(5) = -4$ Roots: -1

4. $(f+g)(x) = \frac{x^2+6x-1}{x^2-1}$, $D_{f+g} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
   $(fg)(x) = \frac{3x+6}{x^2-1}$, $D_{fg} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
   $\left(\frac{f}{g}\right)(x) = \frac{3x-3}{x^2+3x+2}$, $D_{f/g} = (-\infty, -2) \cup (-2, -1) \cup (-1, 1) \cup (1, \infty)$

5. $(f \circ g)(x) = -3x^2 + 6x - 1$ $(g \circ f)(x) = -3x^2 + 1$ $(g \circ g)(x) = x - 2$ $(f \circ f)(2) = -298$

6. $(f \circ g)(4) = -15$ $(f \circ g)(-4) = 9$ $(g \circ f)(4) = -6$ $(g \circ f)(-4) = 25$

7. Answers are not unique. Possible answers are:
   (a) $f(x) = \frac{3}{x}$; $g(x) = x + 5$ (b) $f(x) = \sqrt{x}$; $g(x) = x^2 + x - 2$

8. 16

9. (a) 1 (b) $\frac{1}{2}$ (c) 8 (d) 2 (e) 6 (f) -9 (g) 5 only (h) Does not exist.

10. (a) $x^{\frac{5}{3}}$ (b) $(2x)^{\frac{3}{5}}$ (c) $x^{\frac{5}{2}} y^{-\frac{15}{2}}$ (d) $x^{\frac{2}{5}}$ (e) $x^{\frac{2}{9}}$

11. (a) $\sqrt[3]{x}$ (b) $-\sqrt{x}$ (c) $\sqrt{-x}$ (d) $\sqrt[5]{x^9}$ or $(\sqrt[5]{x})^9$ (e) $-3\sqrt[3]{x^2}$ or $-3(\sqrt[3]{x})^2$
    (f) $\dfrac{2}{\sqrt[4]{(xy)^3}}$ or $\dfrac{2}{(\sqrt[4]{xy})^3}$

12. (a) $x \geq 0$ (b) $x \leq 0$ (c) $\mathbb{R}$ (d) $x > 0$ (e) $x \geq 6$ (f) $x \leq 6$ (g) $\mathbb{R}$

13. b, f, g

14. (a) $\dfrac{1}{x^{\frac{1}{6}}}$ (b) $\dfrac{y^2}{9x^2}$ (c) $\dfrac{1}{x^{\frac{1}{6}}}$ (d) $\dfrac{1}{x^{\frac{7}{2}}}$ (e) $\dfrac{a^2b^2}{a^3 + b^4}$ (f) x^{m-1}

15. (a) $\dfrac{|x|^3\sqrt{y}}{z\sqrt{z}}$ Note: x can be negative; z cannot be negative. (b) $\dfrac{\sqrt{x^2+y^2}}{x^2}$

3 Polynomials and Rational Functions

In this section we review the definitions of two important types of functions: polynomial functions and rational functions. Then we look at some specific examples of these fuctions, focusing primarily on linear functions.

Example 3.1. Some examples of polynomials, and their degrees:

1. $x^5 + 3x^4 - x + 12$ degree: 5
2. $\frac{1}{3}x^9 + \sqrt{5}x^2 - \pi$ degree: 9
3. $2x$ degree: 1
4. 7 degree: 0
5. $x^2 - x^3 + x$ degree: 3
6. $(x^2 + 4)(x^4 + x^3 - 1)$ degree: 6
7. $\dfrac{3x^4 + 6x}{-2}$ degree: 4

And here is the general definition:

Definition 3.1. *A degree n polynomial is an expression of the form:*

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

where the a_i are real number constants with $a_n \neq 0$.

Some examples of expressions that are *not* polynomials[10] are:

1. $3\sqrt{x} + 7$
2. $\dfrac{x^3 + 5x^2}{x - 7}$
3. $3^x + 3^{x-1} + 9$
4. $\sin(x)$

A *polynomial function* is simply a function whose "rule" for calculating the y value is a polynomial. The distinction between a polynomial and a polynomial function is not of consequence here and we will use the terms interchangeably. The domain of any polynomial is $\mathbb{R}$.

You have probably learned that a polynomial of degree n has at most n roots. When we study graphing in Section 19 the reason for this should become clear.

A *rational function* is a function that is the ratio of two polynomials: $f(x) = \dfrac{P(x)}{Q(x)}$, where P and Q are polynomials.

[10] Any non-zero constant (i.e. number) is regarded as a polynomial of degree 0.

The domain of f is all values of x for which $Q(x) \neq 0$. Since the denominator is a polynomial, and the number of roots of a polynomial is at most the degree of the polynomial, the domain of a rational function is $\mathbb{R}$ with finitely many points removed.

Every polynomial is a rational function since $Q(x)$ can be the constant polynomial $Q(x) = 1$.

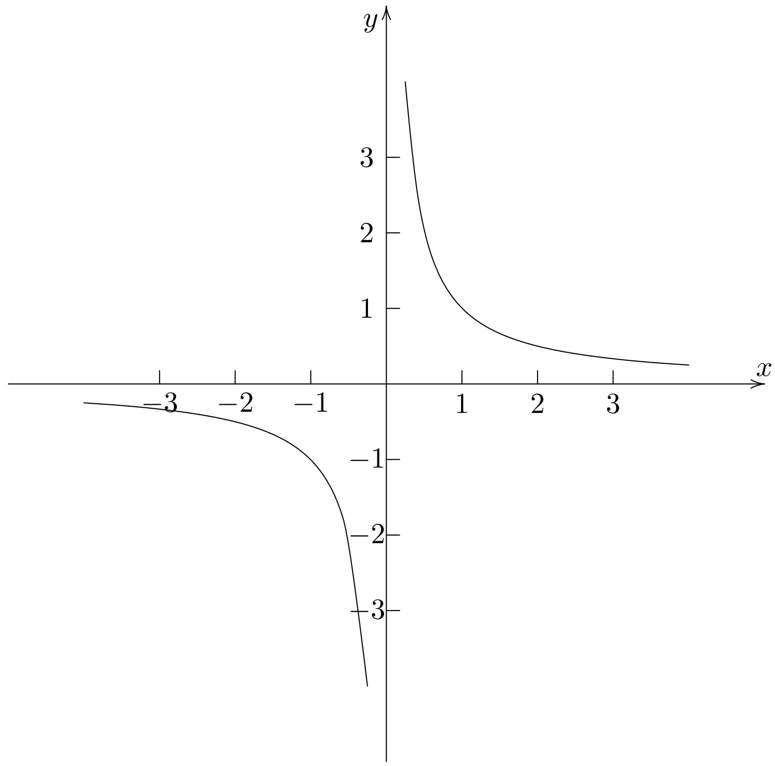
Example 3.2. Some examples of rational functions are:

$$\frac{x^3 + x - 4}{x^2 - x} \qquad 6(x+3)^{-2}$$

The Rational Function $f(x) = \dfrac{1}{x}$

In this course we will frequently use the rational function $f(x) = \frac{1}{x}$ as an example. You should be familiar with its properties and its graph. The domain of the function is $(-\infty, 0) \cup (0, \infty)$. The range of the function is also $(-\infty, 0) \cup (0, \infty)$. This will become more evident as you think about the possible y values of the function. What are some of the ordered pairs that make up this function? Here are a few: $(1, 1)$, $(2, \frac{1}{2})$, $(10, \frac{1}{10})$, $(-1, -1)$, $(-2, -\frac{1}{2})$, $(-10, -\frac{1}{10})$, $(\frac{1}{2}, 2)$, $(\frac{1}{10}, 10)$, $(-\frac{1}{2}, -2)$, $(-\frac{1}{10}, -1)$.

Plot these points on an evenly calibrated set of axes. Drawing a graph is not simply a matter of plotting a few points and then playing "dot-to-dot." Consider your function. Does it make sense that when x is positive, y must also be positive? and when x is negative, y must be negative? Can you see why y can never be zero? This means that there are no x-intercepts. When x is positive, can you justify the fact that the bigger x gets, the smaller y gets? and the smaller x gets, the bigger y gets? What is the situation when x is negative? Use these observations to justify the way that you sketch your graph. It should look like this:



$$y = \frac{1}{x}$$

Quadratic Functions

Definition 3.2. *A quadratic function is a polynomial function of degree* 2*. So, the general form of a quadratic is* $a_2x^2 + a_1x + a_0$*, or more commonly written as* $ax^2 + bx + c$*, where* $a \neq 0$*.*

Example 3.3. Here are some examples of quadratic functions:

1. $f(x) = x^2 + 3x + 2$
2. $f(x) = 3x^2 + 2x - 8$
3. $f(x) = x^2 - 2x$
4. $f(x) = x^2 - 3x + 1$
5. $f(x) = 2x^2 + 5$

The graph of a quadratic function is a parabola.

Quadratic functions are nice because it is always possible to find all of the roots of a quadratic. In the first three quadratic functions above we can find the roots by factoring and setting each factor equal to zero:

1. $0 = x^2 + 3x + 2 = (x + 2)(x + 1)$, so $x = -2$ or $x = -1$

2. $0 = 3x^2 + 2x - 8 = (3x-4)(x+2)$, so $x = \frac{4}{3}$ or $x = -2$

3. $0 = x^2 - 2x = x(x-2)$, so $x = 0$ or $x = 2$

The last two quadratic functions in Example 3.3 do not factor. To find the roots for these functions, we use the quadratic formula:

$$\text{If } ax^2 + bx + c = 0 \quad (\text{where } a \neq 0) \quad \text{then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{3.1}$$

This formula isn't magic. It's algebra. If you divide across the equation by the (non-zero) number a it becomes

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Factoring the left hand side we get

$$\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = 0.$$

So, you see that the process of "finding factors" is just a way of bypassing the quadratic formula when you can spot the factors.

As a polynomial of degree 2, a quadratic function can have at most two roots. If the radicand[11] of the quadratic formula is negative, then the function has no roots. If the radicand is positive, there are two roots. If the radicand is zero, then both factors are the same, so there is only one distinct root.

Caution: Occasionally a student will attempt to use the quadratic formula on functions like $f(x) = x^3 + x - 7$. This does not work. Even though there are only three terms, the degree of this polynomial is 3, not 2. It is not a quadratic function. The quadratic formula applies only to quadratic functions.

Linear Functions and Equations

Definition 3.3. *A linear function is a polynomial function of degree 1 or degree 0. So, the general form of a linear function is $a_1x + a_0$ (degree 1 when $a_1 \neq 0$) or just a_0 (degree 0). More commonly, a linear function is written as $mx + b$, where $m \neq 0$ in the degree 1 case, and $m = 0$ in the degree 0 case. A degree 0 linear function is, of course, just a constant function, such as $y = b$ or $y = 26$. The graph of a linear function is a non-vertical (straight) line*[12].

Here we recall some ideas about lines (from analytic geometry) that you will have seen before coming into this course.

The most general equation of a line is:

$$px + qy + r = 0 \text{ where } p, q, r \text{ are constants.} \tag{3.2}$$

Draw axes in the plane. A line is *vertical* if it is parallel to the y-axis. A vertical line has equation $x = k$; here k is the point on the x-axis where the line crosses the x-axis. As its equation

[11] "Radicand" refers to the expression under a radical. In this case it is $b^2 - 4ac$.
[12] We always mean a "line" to be straight. We use the word "curve" otherwise.

indicates, it is exactly the set of ordered pairs (x, y) where the x-coordinate is always the number k. So, some of the points are $(k, 3)$, $(k, -1)$, $(k, \sqrt{5})$, etc. Clearly, this is not the graph of a function.

When equation 3.2 has $p = 1$, $q = 0$ and $r = -k$, we have exactly the equation $x = k$.

All other lines are *non-vertical.* They have slopes. The *slope* of the non-vertical line joining two different points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. The denominator is not zero because on a non-vertical line, no two points can have the same x coordinate.[13]

Any non-vertical line will cross the y-axis at some number, b. The number b is the *y-intercept* of the line.

When Equation 3.2 has $p = m$ (the slope), $q = -1$ and $r = b$ (the y=intercept), then we have $mx - y + b = 0$. This easily becomes the more familiar equation of a non-vertical line:

$$y = mx + b \tag{3.3}$$

This is called the *slope-intercept* form of the linear equation.

There is another way of writing the equation of a non-vertical line which we'll find useful. The equation of the line through the specific point (x_1, y_1) with slope m is

$$y - y_1 = m(x - x_1). \tag{3.4}$$

This is called the *point-slope* form of the linear equation. This line is the graph of the linear function $f(x) = mx + b$.

When Equation 3.2 has $p = m$, $q = -1$ and $r = y_1 - mx_1$ then, with minor algebraic manipulations, we have the linear equation in point-slope form.

All of equations 3.2, 3.3 and 3.4 are interchangeable when writing the equation of a function whose graph is a non-vertical line.

In short, the equation $px + qy + r = 0$ describes all linear equations, both vertical and non-vertical (vertical when $q = 0$ and $p \neq 0$ and non-vertical when $q \neq 0$). The familiar $y = mx + b$ and the useful $y - y_1 = m(x - x_1)$ do not apply to vertical lines because for a vertical line, m has no meaning.

Here are some other things to recall from high school math:

1. If the slope of a line is positive, the line is increasing (going up) when viewed left-to-right in the xy plane.

2. If the slope of a line is negative, the line is decreasing (going down) when viewed from left-to-right in the xy plane.

3. A horizontal line has slope 0, a vertical line has no slope, and these lines are perpendicular to each other.

4. If a non-horizontal, non-vertical line has slope m $(m \neq 0)$ the lines perpendicular to it all have slope $-\dfrac{1}{m}$. Question: Why must we insert $(m \neq 0)$?

[13] By the same reasoning, vertical lines do not have a slope. Any two points on a vertical line *will* have the same x coordinate. So the denominator of the slope fraction would be zero. The fraction is then meaningless.

5. Parallel lines have the same slope.

Vocabulary

During our study of economic applications we will be using some terms that you might not have seen before:

Suppose you decide to make some money by selling cold drinks on a hot day. You buy a cooler (\$8), ice (\$3) and a variety of canned beverages (100 cans at 25 cents each, for a total of \$25). You sell your drinks, charging 75 cents for each can. Suppose you successfully sell all 100 cans. For simplicity, we will suppose that you live in Delaware where there is no sales tax.

Cost – This is the amount of money that the seller, vendor, manufacturer, etc. has to spend to make and market the product. This is the expense that you had for purchasing the cooler, ice and drinks (\$36). Often cost will include a *fixed cost* that doesn't change regardless of how many items are being produced for sale, and a *variable cost* that does depend on that quantity. For your beverage enterprise the fixed cost is the expense for the cooler and ice (\$11). Your variable cost is the cost of the cans of drink that you buy (\$25).

Price – This is the amount of money that is charged by the seller for each item. Your price is 75 cents.

Demand [14] – This is the quantity of items sold. Demand is sometimes called *Quantity sold*. Your demand is 100 (cans).

Revenue – This is the total money that the seller receives from the customers.
Revenue = (Price $\times$ Demand). Since you sold all of your drinks, your revenue is $.75 \times 100 = \$75$.

Profit – This is the amount of money that the seller has after all of the costs are paid.
Profit = Revenue − Cost. Your business had a profit of $75 - 36 = \$39$. A business is said to have a loss if the profit is negative (i.e., Cost > Revenue). A business is said to break even when the profit is zero (i.e., Cost = Revenue).

We will use these terms in the following examples so that you become familiar with them.

Example 3.4. If production of chairs has a fixed cost of \$25,000 and a per chair cost of \$200 then the *cost* of producing x chairs is the linear function

$$C(x) = 200x + 25,000.$$

$C(x)$ is a number of dollars, x is a number of chairs. We usually omit the units ("dollars" or "number of chairs") in doing the math but it's a good idea to keep them in mind and they should be written as part of the answer to a word problem.

If the manufacturer charges p dollars per item then the *revenue* from selling x items will be the linear function

$$R(x) = px$$

(Units: p dollars, $R(x)$ dollars, x is a number.) The manufacturer's *profit* from selling x chairs will be the linear function

$$P(x) = R(x) - C(x) = px - 200x - 25,000.$$

[14] This term can have several different meanings. In an economics course, make sure that you understand the definition of *demand* that the course is using.

The *break even point* occurs when

$$
\begin{aligned}
px - 200x - 25,000 &= 0 \\
(p - 200)x &= 25,000 \\
\text{or} \quad x &= \frac{25,000}{p - 200}
\end{aligned}
\tag{3.5}
$$

That is, if you are going to sell at \$$p$ per item you need to sell this number of chairs to break even.

The last paragraph answered the question: How many chairs must be produced in order to break even if you charge a pre-determined price \$$p$ per chair? But the business problem might be different. Perhaps you are definitely going to produce 500 chairs. Then you would ask: How much should be charged per chair in order to break even? Now we must solve the profit function for p in terms of x rather than (as above) for x in terms of p.

$$
\begin{aligned}
px - 200x - 25,000 &= 0 \\
px &= 25,000 + 200x \\
p &= \frac{25,000}{x} + 200
\end{aligned}
$$

and if $x = 500$, we get: $\quad p = \dfrac{25,000}{500} + 200 = 50 + 200 = 250$

Answer: Charge \$250 per chair to break even.

A Graphical Representation

Following is a graphical representation of what we have been doing concerning linear cost and revenue functions. What can we identify in the graph?

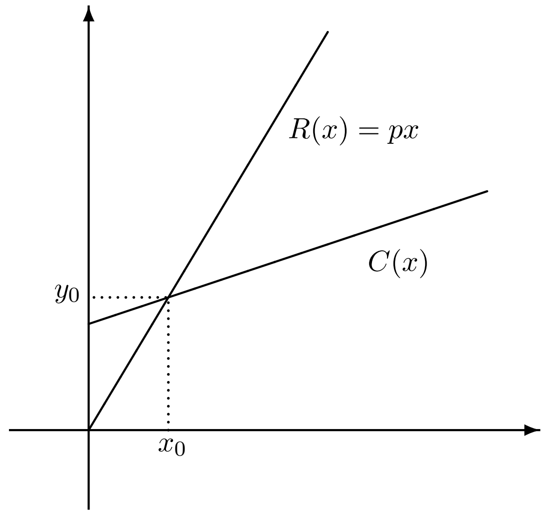
The lines $y = C(x)$ and $y = R(x)$ cross at a point (x_0, y_0). This is where the cost and revenue are equal. This is the break even point. The number x_0 is the number of items to be produced for breaking even. The number $y_0 = C(x_0) = R(x_0)$ is the cost, and also the revenue, for that number of items.

For a fixed value of p, the function $R(x) = px$ is known. The number p is the slope of $R(x)$. When p is large, the graph is steep, so the intersection with $C(x)$ will occur closer to the y axis (the x coordinate is smaller). When p is smaller, the graph is less steep, so the intersection with $C(x)$ is further from the y axis (the x coordinate is larger). Recall that x represents the quantity sold. Does it make sense that when the price is higher you need to sell fewer items to break even, and vice-versa?

The revenue function has y-intercept zero. When $x = 0$ (no items are sold), the revenue will be zero.

The y-intercept for $C(x)$ represents the fixed cost. This is the expense incurred even when no product is made (when x=0).

When the cost function $C(x)$ is linear, the slope of the graph of $C(x)$ is called the *marginal* *cost*[15]. You can think of the marginal cost as the cost per item when fixed costs are ignored; it is the amount by which your cost increases each tme you produce an extra item.



Linear Cost and Revenue

Example 3.5. Sally has found a way to help finance her family's vacation at the beach. Sally pays her children 25 cents for each nice shell they find. Then she gets a vendor's license and she sells sea shells by the seashore. The license costs her \$350.

1. Write a linear cost function $C(x)$ to describe Sally's cost as a function of the number of shells (x) she buys from her children.
2. Sally sells her shells for \$1.35 each. Write a linear revenue function, $R(x)$ to describe this. How many shells must she sell for a profit of \$1,000?
3. Suppose Sally's children are lazy and only bring her 100 shells. What price must she charge per shell in order to break even?

Solution:

1. Sally has a fixed cost of of \$350 for the license. She pays this even if she sells no shells. She has a per shell cost of \$.25 that she pays to obtain the shells. Thus, $C(x) = .25x + 350$, where x is the number of shells that she buys from her children and $C(x)$ is her total cost.
2. Sally sells her shells for \$1.35 each, so her revenue function is $R(x) = 1.35x$.
   Profit, $P(x), = R(x) - C(x) = 1.35x - (.25x + 350) = 1.10x - 350$.
   If $P(x) = 1,000$, we get: $1,000 = 1.10x - 350 \implies 1,350 = 1.1x \implies x = 1,227.\overline{27}$.
   So, she must sell 1,228 shells.
3. Here, we are given $x = 100$ and we want to find p such that $P(p) = 0$.
   $0 = P(p) = R(p) - C(p) = 100p - (.25 \cdot 100 + 350) = 100p - 375$.
   So, $p = \$3.75$

[15] Once we have studied the derivative in Section 8 we will have a way to find the marginal cost of non-linear functions that is consistent with the slope calculation here.

Section 3 - Exercises (answers follow)

1. For each of the following, decide whether the expression is a polynomial, a rational expression, or neither. If it is a polynomial, give the degree of the polynomial.

   (a) $\dfrac{6x^2+1}{6x-3}$

   (b) $3x^6+2x^4-x+2$

   (c) $\dfrac{x^2-4}{x+2}$

   (d) $\frac{1}{2}x^3-4x$

   (e) $\dfrac{\sqrt{x}}{\sqrt{x+1}}$

   (f) $(x+1)(x+2)(x^3+4)$

   (g) $x-x^3-8$

   (h) 2^x+x^2

   (i) $\sqrt{2}x-\sqrt{3}$

   (j) 10

   (k) $x^{\frac{1}{2}}+x^{\frac{1}{3}}+x^{\frac{1}{4}}+1$

2. Apply the quadratic formula to find the roots for each of the functions in Example 3.3.

3. Find the slope of the line that passes through each pair of points.

   (a) $(2,5)$ and $(1,3)$

   (b) $(4,5)$ and $(-1,-2)$

   (c) $(\frac{2}{3},-\frac{1}{4})$ and $(\frac{1}{7},\frac{1}{5})$

4. Find an equation for each line.

   (a) Through $(2,2)$ and $(-1,4)$

   (b) Through $(2,-4)$; $m=3$

   (c) Through $(1,4)$ and parallel to the x axis

   (d) Through $(-1,4)$, parallel to $2x-y=6$

   (e) Through $(-1,4)$ and perpendicular to $y=2x+6$

   (f) Through (a,b) with slope k

   (g) x-intercept $-2/3$ and perpendicular to $x+y+1=0$

5. Given that the point $(2,9)$ lies on the line $kx+3y+4=0$, find k.

6. A newsletter has fixed production costs of \$400 per edition and marginal printing and distribution costs of 40¢ per copy. It sells for 50¢ per copy.
   (a) Write the cost, revenue, and profit functions.
   (b) What profit (or loss) results from the sale of 500 copies of the newsletter?
   (c) How many copies should be sold in order to break even?
7. Assume that each of the following can be expressed as a linear cost function. Find the cost function in each case.
   (a) Fixed cost \$150; 10 items cost \$300 to produce
   (b) Marginal cost: \$100; 10 items cost \$2237 to produce
8. What is the marginal cost in Problem 7a?
9. Your marginal cost to produce one item is \$2.50. Your total cost to produce 100 items is \$300, and you sell them for \$6 each.
   (a) Find the linear cost function for item production.
   (b) How many items must you produce and sell in order to break even?
   (c) How many items must you produce and sell to make a profit of \$500?
10. Each unit of a certain commodity sells for $p = 5x + 20$ cents when x units are produced. If all x units are sold at this price, express the revenue derived from the sales as a function of x.
11. A manufacturer has a monthly fixed cost of \$10,000 and a variable cost of \$.50/unit. Find a function C that gives the total cost incurred in the manufacture of x units/month.
12. Producing x desserts costs $C(x) = 7x + 21$; revenue is $R(x) = 14x$, where $C(x)$ and $R(x)$ are in dollars.
    (a) What is the break-even quantity?
    (b) What is the profit from 100 desserts?
    (c) How many desserts will produce a profit of \$500?
13. The sales of a company were \$20,000 in its third year of operation and \$55,000 in its fifth year. Let y denote sales in the xth year of operation. Assume that the points (x, y) all lie on a line.
    (a) Find the slope of the sales line, and give an equation for the line in the form $y = mx + b$.
    (b) Use your answer from part (a) to find out how many years must pass before the sales surpass \$200,000.
14. A manufacturer's total cost consists of a fixed cost of \$4,000 and a production cost of \$40 per unit. Express the total cost as a function of the number of units produced and draw the graph.

15. A car gets 30 miles to the gallon and has a 15 gallon tank. It starts the trip with x gallons in the tank. Write down a linear function $f(x)$ giving the number of miles it can go without needing more gas.

16. A thinking question: in the previous exercise what does $f(16.5)$ mean?

Section 3 - Answers

1. (a) rational (b) poly, degree 6 (c) rational (d) poly, degree 3
   (e) neither (f) poly, degree 5 (g) poly, degree 3 (h) neither
   (i) poly, degree 1 (j) poly, degree 0 (k) neither

2. 1. $x = \dfrac{-3 \pm \sqrt{9-8}}{2} \Longrightarrow x = -1$ or $x = -2$
   2. $x = \dfrac{-2 \pm \sqrt{4+96}}{6} \Longrightarrow x = \frac{4}{3}$ or $x = -2$
   3. $x = \dfrac{2 \pm \sqrt{4-0}}{2} \Longrightarrow x = 0$ or $x = 2$
   4. $x = \dfrac{3 \pm \sqrt{9-4}}{2} \Longrightarrow x = \dfrac{3+\sqrt{5}}{2}$ or $x = \dfrac{3-\sqrt{5}}{2}$
   5. No roots.

3. (a) 2 (b) $\frac{7}{5}$ (c) $-\frac{189}{220}$

4. (a) $y = -\frac{2}{3}x + \frac{10}{3}$ (b) $y = 3x - 10$ (c) $y = 4$ (d) $y = 2x + 6$
   (e) $y = -\frac{1}{2}x + \frac{7}{2}$ (f) $y = kx + (b - ak)$ (g) $y = x + \frac{2}{3}$

5. $k = -\frac{31}{2}$

6. (a) $C(x) = .4x + 400$ $R(x) = .5x$ $P(x) = .1x - 400$
   (b) loss of \$350
   (c) 4000

7. (a) $C(x) = 15x + 150$ (b) $C(x) = 100x + 1237$

8. 15

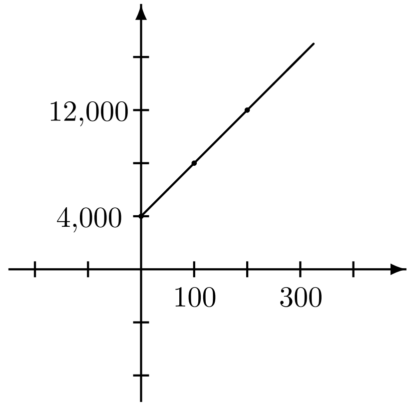
9. (a) $C(x) = 2.50x + 50$ (b) 15 items (rounded up) (c) 158 items (rounded up)

10. $R(x) = 5x^2 + 20x$

11. $C(x) = .50x + 10,000$

12. (a) 3 desserts (b) \$679 (c) 75 desserts (rounded up)

13. (a) $m = 17,500$
    (b) $y = 17,500x - 32,500$
    (c) $\frac{2325}{175}$ (approx. 13 years, $3\frac{1}{2}$ months)



$y = 40x + 4,000$

14. $y = 40x + 4,000$

15. $f(x) = 30x$

16. Hint: What does x represent in your function? Which values of x make sense in the problem?

4 Exponential and Logarithmic Functions

In this section we review exponential and logarithmic functions. You have studied these in high school. This section is more detailed, and the exercises more extensive, than in previous sections because the functions are so important for us.

Definition 4.1. *An exponential function is a function in the form $f(x) = a^x$ where a is a constant real number and $a > 0$ and $a \neq 1$. The number a is called the base of the function.*

Essentially, an exponential function is one where the variable is the exponent.

So, $f(x) = 2^x$ and $g(x) = (\frac{3}{7})^x$ and $h(x) = (\sqrt{5})^x$ are all examples of exponential functions.

Why do you suppose that we do not include $a = 1$ or $a \leq 0$ in the definition?

Characteristics and Graphs

The domain of $f(x) = a^x$ is $\mathbb{R}$, the set of all real numbers.

Let's look in detail at the particular exponential function $f(x) = 2^x$. We'll start by making a table of some specific values of the function:

x	$f(x) = 2^x$	x	$f(x) = 2^x$
-20	$\frac{1}{1{,}048{,}576}$	$\frac{1}{10}$	1.072
-10	$\frac{1}{1{,}024}$	$\frac{1}{4}$	1.189
-4	$\frac{1}{16}$	$\frac{1}{3}$	1.260
-3	$\frac{1}{8}$	$\frac{1}{2}$	1.414
-2	$\frac{1}{4}$	1	2
-1	$\frac{1}{2}$	2	4
$-\frac{1}{2}$	0.707	3	8
$-\frac{1}{3}$	0.794	4	16
$-\frac{1}{4}$	0.841	10	$1,024$
$-\frac{1}{10}$	0.933	20	$1,048,576$
0	1		

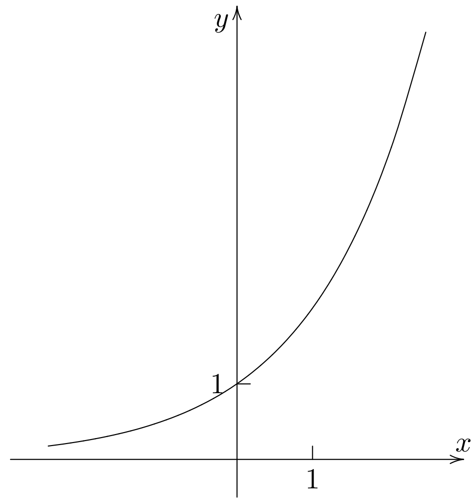
Look carefully at this list of function values. The data suggests some conclusions. The data is not a proof, but in fact the following statements are true:

1. The range of the function is $y > 0$. So, there is no x-intercept and the function is unbounded in the positive direction.

2. The y-intercept is 1.

3. The function is strictly increasing. As the x values get larger, the y values get larger.

We can use (some of) the data above to draw a sketch of the graph of $f(x) = 2^x$. It has been stated earlier that graphing a function is more than just plotting a few points and then "playing dot-to-dot." However, it does turn out this time that the graph for this function is indeed a smooth curve that behaves nicely (it is all connected; doesn't jump around or have gaps in it).



$y = 2^x$

The above graph of $f(x) = 2^x$ is representative of the overall shape of graphs of exponential functions where the base a is greater than 1. Thus, $y = 3^x$, $y = \pi^x$ and $y = (\frac{3}{2})^x$ are all positve, increasing, unbounded functions with y-intercept at $(0, 1)$ and no x-intercept. All have a horizontal asymptote on the left; as x gets larger and larger in the negative direction, y gets closer and closer to zero. The variation in base only alters the steepness of the curve, not the basic shape. It is good to be familiar with this graph.

Irrational Exponents

We have said that the domain of $f(x) = 2^x$ is $\mathbb{R}$. Yet all of the data points that we plotted were rational x values. We had a variety of values: positive, negative, integer, non-integer. From our review of exponents in Section 2 we know how to deal with these. But, how do we deal with x values that are irrational, such as $x = \sqrt{3}$ or $x = \pi$? What does 2^π mean? It does not mean 2 multiplied times itself π times. It does not mean the "πth" root of 2. We can use the graph of $f(x) = 2^x$ to get a value for 2^π. 2^π is simply the y-value on the graph when the x value is π.

You might think that this is a "cheating" way to answer the question, "What do we mean by 2^π?" But, at least it is a reasonable answer. Since the graph of $f(x) = 2^x$ is smooth and increasing, and since $3.1 < \pi < 3.2$, we would want the value of 2^π to be somewhere between the values of $2^{3.1}$ and $2^{3.2}$. We know how to interpret $2^{3.1}$ and $2^{3.2}$ because these exponents are rational. Since $3.1 = \frac{31}{10}$ and $3.2 = \frac{32}{10}$ we understand $2^{3.1} = 2^{\frac{31}{10}} = \sqrt[10]{2^{31}} \approx 8.574$ and $2^{3.2} = 2^{\frac{32}{10}} = \sqrt[10]{2^{32}} \approx 9.198$. Our smooth, connected graph tells us that there IS a value for 2^π and we have figured that it must be between 8.574 and 9.198. We can get closer to the actual value of 2^π by simply choosing rational numbers closer to π than are 3.1 and 3.2. With calculus we have a way of squeezing the interval so closely around π that we say we can know the actual value of 2^π.

So, what is the point of all of this? Certainly from a practical standpoint we will simply use a calculator to get $2^\pi \approx 8.825$. But what we have here is a way to interpret an irrational exponent. The smooth connectedness of the graph of $f(x) = a^x$ gives us a way to understand the values of numbers in the form a^x where the exponent is irrational. We can consider all irrational exponents in the same way that we did here with π.

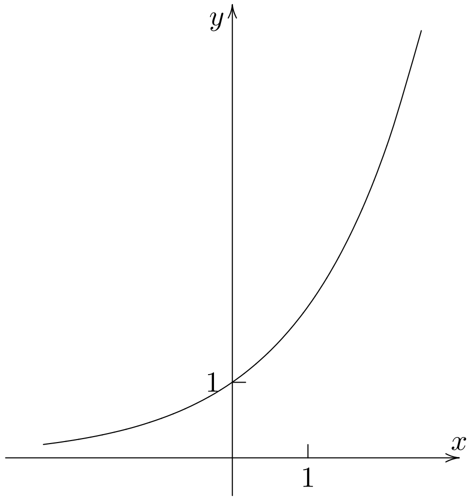
Exponential Functions with base $0 < a < 1$

You will recall that the definition of exponential function allows for the base a to be any positive number except 1. So far we have dealt only with a values greater than 1. Let's consider the exponential function $f(x) = (\frac{1}{2})^x$. If we make a table of values for this function and look at its graph we see that it is a little different from the ones studied previously, although some of the numbers look quite familiar.

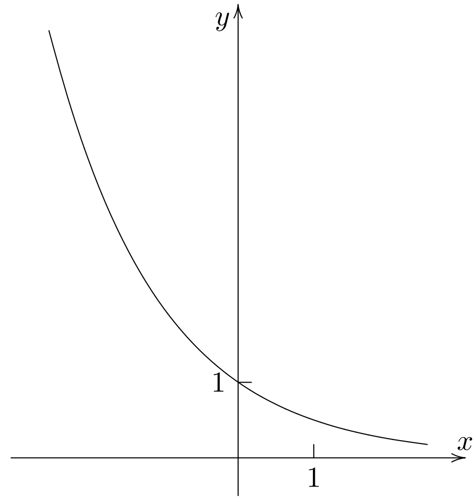
x	$f(x) = (\frac{1}{2})^x$	x	$f(x) = (\frac{1}{2})^x$
-20	$1,048,576$	$\frac{1}{10}$	0.933
-10	$1,024$	$\frac{1}{4}$	0.841
-4	16	$\frac{1}{3}$	0.794
-3	8	$\frac{1}{2}$	0.707
-2	4	1	$\frac{1}{2}$
-1	2	2	$\frac{1}{4}$
$-\frac{1}{2}$	1.414	3	$\frac{1}{8}$
$-\frac{1}{3}$	1.260	4	$\frac{1}{16}$
$-\frac{1}{4}$	1.189	10	$\frac{1}{1{,}024}$
$-\frac{1}{10}$	1.072	20	$\frac{1}{1{,}048{,}576}$
0	1		

The values for the exponential function with base $a = \frac{1}{2}$ and the values for the exponential function with base $a = 2$ are simply reciprocals of each other. We can explain this algebraically: $(\frac{1}{2})^x = \frac{1^x}{2^x} = \frac{1}{2^x}$. We can go a step further and write $(\frac{1}{2})^x = 2^{-x}$.

Compare the graphs of both exponential functions below.



$y = 2^x$



$y = (\frac{1}{2})^x = 2^{-x}$

It is a fact that for any number b where $0 < b < 1$, there is a number $a > 1$ such that $b = \frac{1}{a}$. (Convince yourself that this is true.) Since $\frac{1}{a} = a^{-1}$, we can write $b = a^{-1}$, or more generally, $b^x = a^{-x}$. This says that any exponential function can be written as an exponential function with a base greater than 1. We will mostly be dealing with functions that have a base greater than 1, so we will usually be thinking of the graphs of $y = a^x$ and $y = a^{-x}$ as the two general shapes above, where $a = 2$. However, we will certainly not always be dealing with $a > 1$ so it is important that you *understand* the graphs.

Solving Simple Exponential Equations

We can see from the graphs of the exponential functions that these functions must be one-to-one, that is, for every y value, there is exactly one x value. From this we immediately get the following result:

$$\text{If } a^x = a^y, \text{ then } x = y. \qquad \text{AND} \qquad \text{If } x = y, \text{ then } a^x = a^y.$$

We can use this idea to solve equations that involve simple exponential functions.

Example 4.1. Solve for x.

$$\begin{aligned} 2^{3x+1} &= \sqrt{2} \\ 2^{3x+1} &= 2^{\frac{1}{2}} \\ 3x+1 &= \frac{1}{2} \\ 3x &= -\frac{1}{2} \\ x &= -\frac{1}{6} \end{aligned}$$

Factoring can also help us find roots of functions that involve exponential terms:

Example 4.2. Solve for x.

$$x^2(5^{x+2}) - 9(5^x) = 0$$
$$5^x(x^2 \cdot 5^2 - 9) = 0$$
$$5^x(25x^2 - 9) = 0$$
$$5^x(5x+3)(5x-3) = 0$$

So, we get: $5^x = 0$ or $(5x+3) = 0$ or $(5x-3) = 0$.

Since there are no values for x where $5^x = 0$, our final solutions are $x = -\frac{3}{5}$ or $x = \frac{3}{5}$.

Logarithmic Functions

Suppose you had the equation $x = 3 + y$ and you wanted to solve it for y. You can rewrite this equation into the equivalent equation $y = x - 3$. These two statements say the same thing. Similarly, if you had $3y = x$, you could rewrite this into $y = \frac{1}{3}x$. These two statements say the same thing.

But what if you had $x = 5^y$? How can you solve this for y? The answer is that you use a *logarithm*: $y = \log_5 x$. The two statements $x = 5^y$ and $y = \log_5 x$ say the same thing.

$$\log_a x = y \text{ means } a^y = x \tag{4.1}$$

The expression $\log_a x$ is read "log, base a, of x." The two equations, $\log_a x = y$ and $a^y = x$, mean exactly the same thing. They are two ways of expressing the same relationship between x and y.

In other words, $\log_a x$ is equal to the number y such that $a^y = x$. So, for example, the value of $\log_{10} 1,000$ must be $y = 3$ because 3 is the exponent that satisfies $10^y = 1,000$. The statement $10^3 = 1,000$ can be equivalently written as $\log_{10} 1,000 = 3$.

Example 4.3.

1. $\log_3 9 = 2$ because $3^2 = 9$
2. $\log_5 1 = 0$ because $5^0 = 1$
3. $\log_2 \frac{1}{4} = -2$ because $2^{-2} = \frac{1}{4}$
4. $\log_\pi \sqrt{\pi} = \frac{1}{2}$ because $\pi^{\frac{1}{2}} = \sqrt{\pi}$

A <u>*common logarithm*</u> is a logarithm with a base of 10. It is fairly standard notation, and we will use if from this point on, that we do not bother to explicitly write the subscript "10" when we mean a common logarithm. So, when we write "$\log x$" we mean "$\log_{10} x$".

While we can have logarithms with base between 0 and 1, we only need to concern ourselves with logarithms that have base $a > 1$. As mentioned before, any number between 0 and 1 can be written as $\frac{1}{a}$ where a is some number greater than 1. So, suppose we have $y = \log_{\frac{1}{a}} x$ This says the same thing as $(\frac{1}{a})^y = x$. This is the same thing as $a^{-y} = x$, which in turn is the same as $\log_a x = -y$. If we combine the first and last expressions, substituting for y, we get $\log_{\frac{1}{a}} x = -\log_a x$. So, anything that we need to do with logarithms that have a base between 0 and 1 we can do by using the reciprocal base (which is greater than 1) if we negate the logarithm.

Example 4.4.

1. Problem: Rewrite the logarithmic equations into their equivalent exponential forms:

   $\log_3 81 = 4 \qquad \log\ .01 = -2 \qquad \log_8 1 = 0 \qquad \log_7 13 = x$

   Answers: $3^4 = 81 \qquad 10^{-2} = .01 \qquad 8^0 = 1 \qquad 7^x = 13$

2. Problem: Rewrite the exponential equations into their equivalent logarithmic forms:

   $2^{-1} = \frac{1}{2} \qquad 5^{\frac{1}{3}} = \sqrt[3]{5} \qquad 10^2 = 100 \qquad 3^{20} = x$

   Answers: $\log_2 \frac{1}{2} = -1 \qquad \log_5 \sqrt[3]{5} = \frac{1}{3} \qquad \log\ 100 = 2 \qquad \log_3 x = 20$

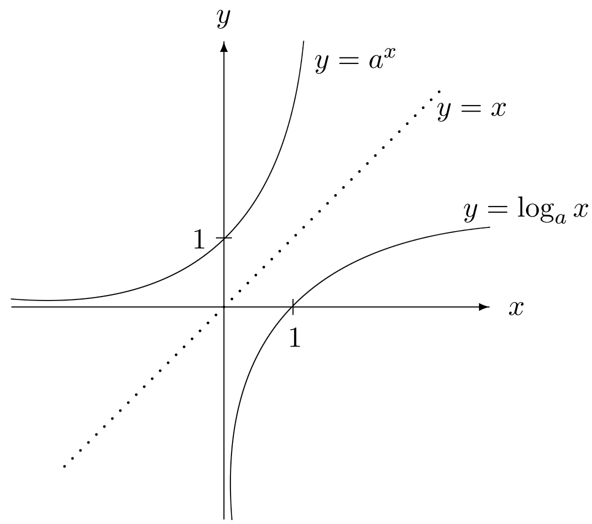
3. Problem: Rewrite the logarithms with base less than one to their equivalent logarithms with base greater than one: $\log_{\frac{1}{3}} 60 = x \qquad \log_{\frac{2}{3}} 60 = x$

   Answers: $-\log_3 60 = x \qquad -\log_{\frac{3}{2}} 60 = x$

We now look at the *logarithm function.* $y = \log_a x$ is a function. For each value of x that you put in, there is only one value of y that can result. So, we will write $f(x) = \log_a x$. The domain of this function is $(0, \infty)$. This is easy to see if you rewrite the expression into the exponential form $a^y = x$. The x value must always be positive. What about the y value? It can be any real number. So, the range for the exponential function is $\mathbb{R}$.

The graph of the logarithm function is a reflection of the graph of the corresponding (same base) exponential function about the line $y = x$. Below we show the graphs for $y = a^x$ and $y = \log_a x$ when $a > 1$.

It will be very useful for you to remember what the graphs of each of these functions look like. There are several properties of the logarithmic and exponential functions that you need to know well. They are written following the graphs. Compare the list to the graph. If you can remember the graphs, you don't need to memorize this information because the graphs contain it all.



Exponential and Logarithmic Functions when $a > 1$

	$y = a^x$	$y = \log_a x$
Domain:	$\mathbb{R}$	$(0, \infty)$
Range:	$(0, \infty)$	$\mathbb{R}$
x-intercepts;	none	$(1, 0)$
y-intercepts:	$(0, 1)$	none
Asymptotes:	negative x-axis	negative y-axis

Algebraic Properties of Logarithms

The algebra for logarithms is consistent with the rules that one uses when dealing with exponents. However, the notation for logarithms can make this somewhat difficult to see. Often it is useful to rewrite the logarithmic expression into its exponential equivalent in order to understand the properties of logarithms. Several of the most important properties are listed next, followed by examples for each one.

Here we assume that a, m, n, p are values consistent with the domain of the logarithm function. So, a, m, n, p are real numbers, a, m, n are positive and $a \neq 1$.

1. $\log_a 1 = 0$
   Examples: $\log_5 1 = 0$ $\qquad \log 1 = 0$

2. $\log_a a = 1$
   Examples: $\log_2 2 = 1$ $\qquad \log 10 = 1$

3. $\log_a a^p = p$
   Examples: $\log_4 16 = \log_4 4^2 = 2$ $\qquad \log \sqrt{10} = \log 10^{\frac{1}{2}} = \frac{1}{2}$

4. $a^{\log_a m} = m$
   Examples: $6^{\log_6 17} = 17$ $\qquad 10^{\log 4} = 4$

5. $\log_a mn = \log_a m + \log_a n$
   Examples: $3 = \log_2 8 = \log_2(4 \cdot 2) = \log_2 4 + \log_2 2 = 2 + 1 = 3$
   $\log_4(3x) = \log_4 3 + \log_4 x$

6. $\log_a \frac{m}{n} = \log_a m - \log_a n$
   Examples: $3 = \log_2 8 = \log_2 \frac{32}{4} = \log_2 32 - \log_2 4 = 5 - 2 = 3$
   $\log_5 \frac{2}{x} = \log_5 2 - \log_5 x$

7. $\log_a m^p = p\ \log_a m$
   Examples: $6 = \log_2 64 = \log_2 4^3 = 3\log_2 4 = 3 \cdot 2 = 6$
   $\log_7 \sqrt[3]{x^2} = \log_7 (x)^{\frac{2}{3}} = \frac{2}{3}\log_7 x$

We have to be careful to only use the Properties when we have the proper domains for our functions. Sometimes this requires us to be alert. For example, we could easily misuse Property 7. If simply given the function $f(x) = \log_a x^2$ we would be tempted to say that it is equal to function $g(x) = 2\log_a x$. It is not. The domain of f is $\{x : x \neq 0\}$. The domain of g is $\{x : x > 0\}$. So, these two functions are not the same. Property 7 claims equality of the expressions because it restricts the domain, only allowing m to be positive.

That having been said, in all further examples and in the homework problems we will assume that the given variables are consistent with the domains of the Properties above.

Example 4.5. Here we take a single logarithmic expression and expand it into an equivalent expression that uses several logarithms.

1. $$\log_2(4x^2y) = \log_2 4 + \log_2 x^2 + \log_2 y = 2 + 2\log_2 x + \log_2 y$$

2. $$\log_7\left(\frac{6}{\sqrt{x^2+1}}\right) = \log_7 6 - \log_7 \sqrt{x^2+1} = \log_7 6 - \frac{1}{2}\log_7(x^2+1)$$

3. $$\log\left(\frac{x^2}{y^5z^3}\right)^4 = 4\log\left(\frac{x^2}{y^5z^3}\right) = 4(\log x^2 - \log(y^5z^3)) = 4(\log x^2 - \log\ y^5 - \log z^3)$$
   $$= 4(2\log x - 5\log y - 3\log z) = 8\log x - 20\log y - 12\log z$$

Example 4.6. Here we take a combination of logrithmic expressions and condense them into a single expression with a coefficient of 1.

1. $$\log_3(x+2y) - \log_3(x-y) = \log_3 \frac{x+2y}{x-y}$$

2. $$\log x^2 + \frac{1}{2}\log y - \log z = \log x^2 + \log\sqrt{y} - \log z = \log \frac{x^2\sqrt{y}}{z}$$

3. $$\frac{1}{3}(\log_5 x - 2\log_5 y) + 5\log_5 z = \frac{1}{3}\log_5\left(\frac{x}{y^2}\right) + \log_5 z^5$$
   $$= \log_5 \sqrt[3]{\frac{x}{y^2}} + \log_5 z^5 = \log_5\left(z^5\sqrt[3]{\frac{x}{y^2}}\right)$$

4.

$$\log_6 9 + \log_6 4 = \log_6(9 \cdot 4) = \log_6 36 = 2$$

5.

$$\log_9 25 - \log_9 75 = \log_9 \frac{25}{75} = \log_9 \frac{1}{3} = -\frac{1}{2}$$

6.

$$\frac{2}{3}\log_a 27 + 2\log_a 2 - \log_a 3 = \log_a 27^{\frac{2}{3}} + \log_a 2^2 - \log_a 3$$
$$= \log_a 9 + \log_a 4 - \log_a 3 = \log_a \frac{9 \cdot 4}{3} = \log_a 12$$

Changing Bases

Suppose you would like to know the approximate value of x for $2^x = 100$. You know that the number is somewhere between 6 and 7 (why?), but you need to be more precise than that. You are looking for an exponent value. So, you are looking for a logarithm. In particular, you want to know the value of $\log_2 100$. This is good so far. But when you then go to your calculator you realize that it doesn't handle logarithms with a base of 2. However, your calculator does have a "log" button for common logs (base 10).[16] You need to be able to change a base 2 logarithm into a base 10 logarithm.

There is a straightforward way to change from one base to another. It uses the algebra that we already know for logarithms, and the following fact:

If $x = y$, then $\log_a x = \log_a y$ AND If $\log_a x = \log_a y$, then $x = y$.

Follow the steps as we change from a base a logarithm $\log_a x$ into an expression involving base b logarithms.

We will call our base a logarithm y. So,

$$y = \log_a x$$
$$a^y = x$$
$$\log_b a^y = \log_b x$$
$$y \log_b a = \log_b x$$
$$y = \frac{\log_b x}{\log_b a}$$

Now, substituting back for y, we get the **Change of Base Formula for Logarithms**:

(4.2) $$\log_a x = \frac{\log_b x}{\log_b a}$$

Look carefully at the placement of the a's, b's, x's.

[16] Your calculator will also have a button for "ln" which is a "natural logarithm." We will deal with this in Section 5.

For our particular example we can change $\log_2 100$ into $\dfrac{\log\ 100}{\log\ 2}$. A calculator will give approximate values: $\dfrac{2}{0.30103} \approx 6.64386$

In the example above we found it useful to change a base 2 logarithm into a common log. However, the change of base formula works for changing to any base. $\log_2 100$ is in fact equal to $\dfrac{\log_7 100}{\log_7 2}$, $\dfrac{\log_{13} 100}{\log_{13} 2}$, $\dfrac{\log_{88} 100}{\log_{88} 2}$, and even $\dfrac{\log_2 100}{\log_2 2}$. All of these fractions are approximately 6.64386.

It would be reasonable to ask if there is a change of base formula for exponential expressions. In other words, if we are given the expression a^x is there some y so that $a^x = b^y$ for a desired base b? The answer is "yes." We find it by simply solving the equation $a^x = b^y$ for y.

$$\begin{aligned} a^x &= b^y \\ \log_b a^x &= \log_b b^y \\ x \log_b a &= y \end{aligned}$$

Thus we have the **Change of Base Formula for Exponents**:

$$a^x = b^{x \log_b a} \tag{4.3}$$

Look carefully at the placement of the a's, b's, x's.

Example 4.7.

We change the function $f(x) = 2^x$ to an equivalent function that uses 5 as its base.

$$f(x) = 2^x = 5^{x \log_5 2}$$

Solving Logarithmic and Exponential Equations

We will start with some examples of using logarithms to solve exponential functions. In particular, we are making use of Property 3 on page 36 to bring the x out of the exponent. Sometimes we use this operation without really realizing it. We can mentally do the rewriting of the logarithm into its equivalent exponential form. We are still using the logarithm concept, just not writing it down or even acknowledging it. This point is illustrated in Example 4.8. Compare the problems in this example. Where in the first problem are we implicitly using the logarithm?

Example 4.8. Solve for x: $2(1 + 4^x) = 6$ and $2(1 + 4^x) = 12$

$$\begin{array}{ccc} 2(1+4^x) = 6 & 2(1+4^x) = 12 & 2(1+4^x) = 12 \\ 1 + 4^x = 3 & 1 + 4^x = 6 & 1 + 4^x = 6 \\ 4^x = 2 & 4^x = 5 & 4^x = 5 \\ x = \frac{1}{2} & \log_4 4^x = \log_4 5 & \log 4^x = \log 5 \\ & x = \log_4 5 & x \log 4 = \log 5 \\ & & x = \frac{\log 5}{\log 4} \end{array}$$

Notice that in the second problem of Example 4.8 we leave our answer in terms of a base 4 logarithm. While that is correct, it is not very useful for giving us an idea of the value of the answer. We could use the change of base formula and write the answer equivalently as $x = \dfrac{\log 5}{\log 4}$. Another possibility is to solve this problem using a common logarithm from the beginning, as is done in the alternative solution of Example 4.8.

When you "take the log" of both sides of an equation you must use the same-based log on both sides. If you have an exponential equation with multiple bases then you need to decide which logarithm base you wish to use. You could choose any of the bases that are in the problem or you could choose something completely different, such as a common log or a natural log. Again, we offer an example with multiple, but equivalent, solutions.

Example 4.9. Solve for x: $5^x = 6^{x-1}$

$$\begin{array}{ccc}
5^x = 6^{x-1} & 5^x = 6^{x-1} & 5^x = 6^{x-1} \\
\log_5 5^x = \log_5 6^{x-1} & \log_6 5^x = \log_6 6^{x-1} & \log 5^x = \log 6^{x-1} \\
x = (x-1)\log_5 6 & x\log_6 5 = x - 1 & x\log 5 = (x-1)\log 6 \\
x = x\log_5 6 - \log_5 6 & x\log_6 5 - x = -1 & x\log 5 = x\log 6 - \log 6 \\
x - x\log_5 6 = -\log_5 6 & x(\log_6 5 - 1) = -1 & x\log 5 - x\log 6 = -\log 6 \\
x(1 - \log_5 6) = -\log_5 6 & x = \dfrac{-1}{\log_6 5 - 1} & x(\log 5 - \log 6) = -\log 6 \\
x = \dfrac{-\log_5 6}{1 - \log_5 6} & & x = \dfrac{-\log 6}{\log 5 - \log 6}
\end{array}$$

We now start with some logarithmic equations and will use the operation of raising the expression to a power in order to solve for x. We are really using Property 4 from 36.

Example 4.10. Solve for x: $\log_5(x+3) = 2$

$$\log_5(x+3) = 2$$
$$5^{\log_5(x+3)} = 5^2$$
$$x + 3 = 25$$
$$x = 22$$

Notice that the base we choose for the "raising both sides" is the same as the logarithm base.

Example 4.11. Solve for x: $2 + 3\log(x-5) = 0$

$$2 + 3\log(x-5) = 0$$
$$3\log(x-5) = -2$$
$$\log(x-5) = \frac{-2}{3}$$
$$10^{\log(x-5)} = 10^{\frac{-2}{3}}$$
$$x - 5 = 10^{\frac{-2}{3}}$$
$$x = 10^{\frac{-2}{3}} + 5$$

Notice that we isolate the logarithm before applying the “raising both sides” operation. If you have more than one logarithmic expression you should combine them first.

Example 4.12. Solve for x: $\log_3 x + \log_3(x+2) = 1$

$$\log_3 x + \log_3(x+2) = 1$$

$$\log_3(x \cdot (x+2)) = 1$$

$$\log_3(x^2 + 2x) = 1$$

$$3^{\log_3(x^2+2x)} = 3^1$$

$$x^2 + 2x = 3$$

$$x^2 + 2x - 3 = 0$$

$$(x+3)(x-1) = 0$$

$x = -3$ and $x = 1$ appear to be solutions. However, $x = -3$ is not a solution because it is not in the domain of the original problem.

When solving logarithmic equations always check your solution in the original problem to make sure that you have no domain violations.

Remember that the domain of a logarithm function can only be values that make its argument positive. In Example 4.12 neither $\log_3 x$ nor $\log_3(x+2)$ can accept $x = -3$ as input. However, you only need to have one given logarithmic expression undefined by your “solution” to make that “solution” invalid.

Go back to Examples 4.10 and 4.11 and make sure that the solutions presented are valid.

Section 4 - Exercises (answers follow)

1. Solve each equation for x.

(a) $2(1+4^x)=6$ (b) $2^{x+3}=4^{x-1}$ (c) $\dfrac{5^{x+3}}{5^{2x}}=25$

(d) $x^3 6^x - 6^x = 0$ (e) $5^{2x}-5^{-x+1}=0$ (f) $9^{x-1}=3^{1+x}$

2. Change each logarithmic statement to its equivalent exponential form.

(a) $5=\log_2 x$ (b) $y=\log 27$ (c) $12=\log_a 5$

3. Change each exponential statement to its logarithmic equivalent form.

(a) $3^x=2$ (b) $10^5=y$ (c) $x^4=9$

4. Evaluate the following numbers without using a calculator.

(a) $\log_9 1$ (b) $\log(.01)$ (c) $\log_3 81$

(d) $\log_4 2$ (e) $\log_3 \frac{1}{27}$ (f) $\log_{\frac{1}{2}} 8$

(g) $\log_4 2+\log_4 16$ (h) $\log_2 \sqrt[3]{\sqrt[4]{2}}$ (i) $2\log_7 7^3+\log_7 7^{-5}+7^{\log_7 3}$

5. Without using a calculator, find the value of x.

(a) $\log_9 x=\frac{1}{2}$ (b) $\log_x 11=1$ (c) $\log_x 27=\frac{3}{2}$ (d) $\log_6 x=-2$

6. Which is larger: $\log_6 37$ or $\log_7 48$? Why? Do not use a calculator.

7. Suppose $3^x=100$.

(a) On a number line, between which two consecutive integers would you expect to find x?

(b) Solve for x. Use a calculator to find x. Was your answer to part (a) correct?

8. Find the domain of each of the following functions.

(a) $f(x)=10^{\frac{x-2}{x^2-3x+2}}$ (b) $f(x)=4^{\sqrt{x-1}}+3^{-\sqrt{2-x}}$ (c) $f(x)=\dfrac{7^{x^2}+3x\cdot 6^x}{5^{x^2-1}}$

(d) $f(x)=\dfrac{7^{x^2}+3x\cdot 6^x}{5^{x^2-1}-25}$ (e) $f(x)=\log 3^x$ (f) $f(x)=\log_5(5-x)$

(g) $f(x)=\log_3(x-7)-\log_3(x+2)$ (h) $f(x)=\log_4(x^2-x-2)$

9. (a) Explain in words the difference between $(\log_a x)(\log_a y)$ and $\log_a(xy)$. Which one is used in Property 5?

   (b) Explain in words the difference between $\dfrac{\log_a x}{\log_a y}$ and $\log_a\left(\dfrac{x}{y}\right)$. Which one is used in Property 6?

   (c) Explain in words the difference between $(\log_a x^p)$ and $(\log_a x)^p$. Which one is used in Property 7?

10. Use the Properties of Logarithms to condense each of these expressions into a single logarithmic expression with a positive exponent and a coefficient of 1.

    (a) $\log_3 x + \log_3 2$ (b) $\log_2 9 - \log_2 y$ (c) $2\log x - 5\log y$

    (d) $\frac{1}{2}\log_4(x+5)$ (e) $-4\log_6(2x)$ (f) $3\log x + 4\log y - 4\log z$

    (g) $\frac{1}{3}[\log_2 x + \log_2(x+1)]$

11. Use the Properties of Logarithms to expand each of these single logarithms into expressions with multiple logarithms having single character arguments.

    (a) $\log_3\left(\frac{y}{2}\right)$ (b) $\log(10x)$ (c) $\log_6\left(\frac{1}{z^3}\right)$ (d) $\log_4(4x^2y)$

    (e) $\log_4(4xy)^2$ (f) $\log\left(\frac{x^2-1}{x^3}\right)$ (g) $\log_7\sqrt[5]{\frac{x^2}{y^3}}$ (h) $\log_2\frac{\sqrt{x}}{z^4}$

12. Use the Properties of Logarithms to evaluate the following:

    (a) $\log_6 12 + \log_6 3 - \ln 1$ (b) $\frac{2}{3}\log_4 8 + \frac{1}{2}\log_4 9 - \log_4 6$

13. (a) Use Property 6 to show that $\log_a \frac{1}{n} = -\log_a n$

    (b) Use Property 7 to show that $\log_a \frac{1}{n} = -\log_a n$

14. Rewrite $\log_7 9$ as an equivalent logarithm using: (a) base 5 (b) base 10

15. Use the Change of Base Formula for Logarithms to show that:

    (a) $\log_a b = \dfrac{1}{\log_b a}$ (b) $\log_2 5 = 2\log_4 5$

16. Use the Change of Base Formula for Logarithms to show that the three solutions in Example 4.9 are equivalent.

17. Use the Change of Base Formula for Exponential Expressions to change each expression below to its equivalent expression in base 10: (a) 3^x (b) 6^x

18. Given: $\log 3 \approx 0.477$, find approximate values for the following numbers without using a calculator:

    (a) $\log 30$ (b) $\log 3000$ (c) $\log 9$ (d) $\log_3 10$

19. Given the number $\log_9 21$:
    (a) Rewrite it as a ratio of common logs
    (b) Show that it is equal to $\log_3 \sqrt{21}$

20. Solve the following exponential equations for x

(a) $2^{x-3} = 32$	(b) $2(5^x) = 32$	(c) $3^{6x-5} = 27$
(d) $6^x + 10 = 47$	(e) $4^x = 3^{2x-1}$ Use a common log	(f) $3^{2x} - 5(3^x) - 6 = 0$
(g) $7(4^{6x-2}) + 13 = 41$	(h) $2^x = 2(5^x)$ Use a common log	

21. Solve the following logarithmic equations for x

(a) $2\log_5(3x) = 4$	(b) $3 + 2\log x = 15$	(c) $\log(\log x) = 2$
(d) $\log_3(x+1) - \log_3 x = 2$	(e) $\log_{12}(x-6) - \log_{12} 4 = 1$	(f) $\log_3(2x-1) = 2\log_3 x$
(g) $\log_6(x+2) + \log_6(x+7) = 2$		

Section 4 - Answers

1.
(a) $\frac{1}{2}$ (b) 5 (c) 1 (d) 1 (e) $\frac{1}{3}$ (f) 3

2.
(a) $2^5 = x$ (b) $10^y = 27$ (c) $a^{12} = 5$

3.
(a) $\log_3 2 = x$ (b) $\log y = 5$ (c) $\log_x 9 = 4$

4.
(a) 0 (b) -2 (c) 4 (d) $\dfrac{1}{2}$ (e) -3

(f) -3 (g) $\dfrac{5}{2}$ (h) $\dfrac{1}{12}$ (i) 4

5.
(a) 3 (b) 11 (c) 9 (d) $\dfrac{1}{36}$

6. $\log_6 37$ is larger because $\log_6 37 > 2$ and $\log_7 48 < 2$.

7. (a) Between 4 and 5 (b) $x = \frac{1}{2}\log 3 \approx 4.19180655$

8.

(a) $(-\infty, 1) \cup (1, 2) \cup (2, \infty)$	(b) $[1, 2]$	(c) $\mathbb{R}$
(d) $(-\infty, -\sqrt{3}) \cup (-\sqrt{3}, \sqrt{3}) \cup (\sqrt{3}, \infty)$	(e) $\mathbb{R}$	(f) $(-\infty, 5)$
(g) $(7, \infty)$	(h) $(-\infty, -1) \cup (2, \infty)$	

9. (a) The first is a product of logarithms. The second is the logarithm of a product. The second is used in Property 5.
(b) The first is a quotient of logarithms. The second is the logarithm of a quotient. The second is used in Property 6.
(c) Ther first is the logarithm of a power. The second is a power of a logarithm. The first is used in Property 7.

10.

(a) $\log_3(2x)$ (b) $\log_2\left(\frac{9}{y}\right)$ (c) $\log\left(\frac{x^2}{y^5}\right)$ (d) $\log_4\sqrt{x+5}$

(e) $\log_6\left(\frac{1}{16x^4}\right)$ (f) $\log\left(\frac{x^3y^4}{z^4}\right)$ (g) $\log_2\sqrt[3]{x^2+x}$

11.

(a) $\log_3 y - \log_3 2$ (b) $1+\log x$ (c) $-3\log_6 z$

(d) $1+2\log_4 x+\log_4 y$ (e) $2+2\log_4 x+2\log_4 y$ (f) $\log(x^2-1)-3\log x$

(g) $\frac{2}{5}\log_7 x-\frac{3}{5}\log_7 y$ (h) $\frac{1}{2}\log_2 x-4\log_2 z$

12. (a) 2 (b) $\frac{1}{2}$

13. (a) $\log_a\left(\frac{1}{n}\right)=\log_a 1-\log_a n=0-\log_a n=-\log_a n$
(b) $\log_a\left(\frac{1}{n}\right)=\log_a n^{-1}=-1\log_a n=-\log_a n$

14. (a) $\dfrac{\log_5 9}{\log_5 7}$ (b) $\dfrac{\log_{10} 9}{\log_{10} 7}$

15. (a) $\log_a b=\dfrac{\log_b b}{\log_b a}=\dfrac{1}{\log_b a}$ (b) $\log_2 5=\dfrac{\log_4 5}{\log_4 2}=\dfrac{\log_4 5}{\frac{1}{2}}=2\log_4 5$

16. Show that each of the first two expressions are equal to the third one. The first is done for you. The second is done similarly and is not shown:
$$\frac{-\log_5 6}{1-\log_5 6}=\frac{-\frac{\log 6}{\log 5}}{\frac{\log 5}{\log 5}-\frac{\log 6}{\log 5}}=\frac{-\frac{\log 6}{\log 5}}{\frac{\log 5-\log 6}{\log 5}}=\frac{-\log 6}{\log 5-\log 6}$$

17. (a) $10^{x\log 3}$ (b) $10^{x\log 6}$

18.

(a) 1.477 (b) 3.477 (c) 0.954 (d) $\dfrac{1}{0.477}$

19. (a) $\dfrac{\log 21}{\log 9}$ (b) Proof not shown. Hint: Rewrite the problem using base 3.

20. (a) 8 (b) $\log_5 16$ (c) $\frac{4}{3}$ (d) $\log_6 37$ (e) $\dfrac{-\log 3}{\log 4 - 2\log 3}$
(f) $\log_3 6$ (g) $\frac{1}{2}$ (h) $\dfrac{\log 2}{\log 2 - \log 5}$

21. (a) $\dfrac{25}{3}$ (b) $1,000,000$ (c) 10^{10^2} (d) $\frac{1}{8}$ (e) 54 (f) 1 (g) 2

Part II
DIFFERENTIAL CALCULUS

Differential calculus is about rates of change. We all experience change. Differential calculus is the mathematics which precisely quantifies change. In Sections 6-29 we explain this, first for functions having one independent variable, and then for functions with more than one independent variable.

5 Interest Rates and the Number e

Compound Interest

Suppose you have P dollars to invest (here we use P for "principal," not "profit") that you deposit in a bank. The bank offers a 3% interest rate, compounded annually. By "interest rate" we mean the annual rate without considering compounding. By "compounding" we mean the frequency of interest distribution. So, "compounded annually" means that interest is earned once per year.

How much money would you have in this bank one year after you make your deposit? You would have your original deposit, P, plus the 3% interest earned, $.03P$, for a total of $P + .03P = P(1 + .03)$ dollars.

Suppose your bank compounded the 3% interest monthly instead of annually? How much would you have at the end of one year? The answer to this question is a little more involved. We will break it into time periods. We first ask, "How much money would you have at the end of one month?"

Since your annual interest rate is .03, your monthly rate would be $\frac{.03}{12}$. So, using the same idea as above, at the end of one month you would have:

$$P + P\left(\tfrac{.03}{12}\right) = P\left(1 + \tfrac{.03}{12}\right) \text{ dollars.}$$

How much money would you have at the end of two months? You would have the money that you had at the end of the first month, $P\left(1 + \frac{.03}{12}\right)$, plus the interest earned on that money, $\left(P\left(1 + \frac{.03}{12}\right)\right)\left(\frac{.03}{12}\right)$. The total would be:

$$P\left(1 + \tfrac{.03}{12}\right) + \left(P\left(1 + \tfrac{.03}{12}\right)\right)\left(\tfrac{.03}{12}\right) = P\left(1 + \tfrac{.03}{12}\right)\left[1 + \tfrac{.03}{12}\right] = P\left(1 + \tfrac{.03}{12}\right)^2 \text{ dollars.}$$

How much money would you have at the end of 3 months? You would have the money that you had at the end of two months plus the interest earned on that money:

$$P\left(1 + \tfrac{.03}{12}\right)^2 + P\left(1 + \tfrac{.03}{12}\right)^2\left(\tfrac{.03}{12}\right) = P\left(1 + \tfrac{.03}{12}\right)^2\left[1 + \tfrac{.03}{12}\right] = P\left(1 + \tfrac{.03}{12}\right)^3 \text{ dollars.}$$

Do you see the pattern? At the end of six months you would have $P\left(1 + \frac{.03}{12}\right)^6$dollars; at the end of eleven months you would have: $P\left(1 + \frac{.03}{12}\right)^{11}$dollars. We can now answer the original question. At the end of one year we would have $P\left(1 + \frac{.03}{12}\right)^{12}$ dollars.

We could actually continue this process beyond a one year investment. If we had an interest rate of 3% compounded monthly and we kept the principal P in the bank for one year and nine months ($= 21$ months) we would have a total of $P\left(1 + \frac{.03}{12}\right)^{21}$ dollars. Notice that the value for the monthly interest rate, $\frac{.03}{12}$, doesn't change. The exponent on the expression is the number of times that interest is applied during the length of the investment.

How much money would you have after one year if you invested P dollars at an interest rate of 3% compounded weekly? Assuming exactly 52 weeks per year, we would have: $P\left(1 + \frac{.03}{52}\right)^{52}$ dollars.

How much money would you have after one year if you invested P dollars at an interest rate of 3% compounded daily? Assuming exactly 365 days per year, we would have: $P\left(1 + \frac{.03}{365}\right)^{365}$ dollars.

Example 5.1. Suppose you invest $2,000 in a bank that pays 5% interest compounded quarterly. How much money would you have after seven years? Use a calculator to see if your result is reasonable.

Answer: You have interest applied four times per year, so you have a multiplier of $\frac{.05}{4}$ each time you get interest. You keep your money in for 4 x 7 = 28 interest periods. So, your value at the end of three years will be: $2,000\left(1+\frac{.05}{4}\right)^{28} \approx \$2,831.98$

Example 5.2. You want to buy a $23,000 car four years from now. If you find a bank that offers an interest rate of $2\frac{1}{2}\%$, compounded monthly, how much money would you need to have today in order to meet your goal? Use a calculator to see if your result is reasonable.

Answer: We are looking for P so that $P\left(1+\frac{.025}{12}\right)^{48} = 23,000.$

So, $P = \dfrac{23,000}{\left(1+\frac{.025}{12}\right)^{48}} \approx \$20,813.43$

Example 5.3. Suppose you have $4,000 now and you don't want to spend any money until you have accumulated $5,000. If your bank offers 6% interest, compounded semi-annually, how long will you need to keep your money in the bank in order to meet your goal? Use a calculator to make sure that your answer is reasonable.

Answer: We are looking for a time t years, so that $4,000\left(1+\frac{.06}{2}\right)^{2t} = 5,000$. We solve for t.

$$\begin{aligned}
4,000\left(1+\frac{.06}{2}\right)^{2t} &= 5,000 \\
(1+.03)^{2t} &= \frac{5000}{4000} = 1.25 \\
\log\left(1.03^{2t}\right) &= \log(1.25) \\
2t\log(1.03) &= \log(1.25) \\
t &= \frac{\log(1.25)}{2\log(1.03)} \approx 3.77\text{years}
\end{aligned}$$

Present Value and Future Value

We use the term *Present Value* for the amount of money at the beginning of an investment time period. The term *Future Value* is used to indicate the amount of money at the end of an investment time period. In Example 5.1 the Present Value is $2,000 and the Future Value is $2,831.98. In Example 5.2 the Present Value is $20,813.43 and the Future Value is $23,000. In Example 5.3 the Present Value is $4,000 and the Future Value is $5,000.

We could summarize this section so far by the formulas for periodic compounding of interest:

$$F = P\left(1+\frac{r}{n}\right)^{nt} \tag{5.1}$$
$$P = \frac{F}{\left(1+\frac{r}{n}\right)^{nt}} = F\left(1+\frac{r}{n}\right)^{-nt}$$

$$F = P\left(1+\frac{r}{n}\right)^{nt} \tag{5.2}$$
$$P = \frac{F}{\left(1+\frac{r}{n}\right)^{nt}} = F\left(1+\frac{r}{n}\right)^{-nt}$$

$$\begin{aligned} where: r &= \text{annual interest rate expressed as a fraction (i.e., 5\% = .05)}\\ t &= \text{number of years of the investment}\\ n &= \text{number of periods per year that interest is paid}\\ P &= \text{ present value or ``Principal''}\\ F &= \text{future value or ``Final''} \end{aligned}$$

The Effects of Compounding

You probably know that if all else is the same, then the more frequently interest is compounded, the higher will be the future value of an investment. The question is, "How *much* higher return would one get as the frequency of compounding increases?" Let's look at the scenario we had at the beginning of the section, with principal P and annual interest rate 3%. We'll see how some of the compounding figures compare.

Example 5.4. Compounding Calculations (accurate to eight decimal places):

$$\begin{array}{lllll} \text{annually} & P(1+.03) & = P(1.03) & = P(1+.03) & = P+P(.03)\\ \text{quarterly} & P\left(1+\frac{.03}{4}\right)^4 & = P(1.030339191) & = P(1+.030339191) & = P+P(.030339191)\\ \text{monthly} & P\left(1+\frac{.03}{12}\right)^{12} & = P(1.030415957) & = P(1+.030415957) & = P+P(.030415957)\\ \text{weekly} & P\left(1+\frac{.03}{52}\right)^{52} & = P(1.03044562) & = P(1+.03044562) & = P+P(.03044562)\\ \text{daily} & P\left(1+\frac{.03}{365}\right)^{365} & = P(1.030453264) & = P(1+.030453264) & = P+P(.030453264) \end{array}$$

The last number in each row (.030xxxxxx) is the *Effective Interest Rate.* This number represents the annual interest rate *if* the interest were compounded annually. This is what we can use to look at the effect of compounding. This table verifies our idea that increased compounding will increase return. The effective interest rates increase, but how high will they go? Do you think that they will ever reach .04? It looks from the table that they are increasing less and less each time. Just for fun, let's look at the effective interest rate if the interest was compounded every second. That would be approximately 365 x 24 x 60 x 60 = $31,536,000$ compoundings per year. The result turns out to be an effective interest rate of .030454412. This isn't much higher than what we had for the daily compounding rate. Now do you think that the effective interest rate could hit .04? It doesn't. We could increase the frequency of compounding to every billionth of a nanosecond and the effective interest rate will not exceed .030454534 (rounded to eight places). We will argue why this is true.

The number "e"

If you look at the numbers in Example 5.4 you can see that the effective interest rate is calculated by $\left(1+\frac{r}{n}\right)^n - 1$. With the interest rate r fixed, we ask the question, "What happens when the number of compoundings per year, n, gets larger and larger?" We will address this question more formally in Section 20, but we can get some grasp on it now.

Think about $\left(1+\frac{r}{n}\right)^n$. The positive number r is fixed. As n gets larger and larger, the quantity $\frac{r}{n}$ gets smaller and smaller, so $\left(1+\frac{r}{n}\right)$ gets closer and closer to the number 1. However, if any constant c is greater than 1, then c^n gets very very large as n gets large. We have a "tug of war" going on here. As we increase n, the value inside the parentheses is getting smaller, pulling to make the entire expression smaller. At the same time, the exponent outside the parentheses is pulling to make the entire expression larger. How do these pressures play off against each other?

We first answer this question for the special case $r = 1$ (i.e., interest rate of 100%). What happens to $\left(1+\frac{1}{n}\right)^n$ as n gets very large? Here is a table of a few values (to eight decimal places):

n	$(1+\frac{1}{n})^n$
1	2
5	2.48832
20	2.653297705
100	2.704813829
200	2.711517123
1,000	2.716923932
10,000	2.718145927
50,000	2.7182544646
1,000,000	2.718280469
1,000,000,000	2.718281827

This suggests that $(1+\frac{1}{n})^n$ increases with n but might not go to infinity. In fact if you know the Binomial Theorem and are willing to try, you can prove that $(1+\frac{1}{n})^n$ increases with n but it is always between 2 and 3.

It shouldn't surprise you then that once n gets large the decimal entries of $(1+\frac{1}{n})^n$ settle down, only changing in far-out decimal places, and the larger n is, the more decimal places settle down to fixed values. The name given to the value that is the stalemate point in this "tug of war" is "e." The number e is not an approximation; it is the actual real number where this "tug of war" balances [17]. We use a letter to represent this number because it is irrational and so can't be written as a finite or repeating decimal.

Here is e correct to 22 decimal places:

$$2.7182818284590452353602875$$

[17] It would not be surprising if you are bothered by the idea that "e is not an approximation." We have not yet studied enough material to be rigorous about this. We will. In the meantime, if you want to feel more comfortable about it, read the optional paragraphs about π at the end of this section.

This is the correct answer to within one ten thousand billion billionth.

You don't need to remember this long number! Just call it e and perhaps remember that to three decimal places e is 2.718.

Now we can return to our original question, where r is fixed, but not necessarily equal to 1. Consider the following algebra:

$$(1+\frac{r}{n})^n = ((1+\frac{r}{n})^{\frac{n}{r}})^r = ((1+\frac{1}{m})^m)^r, \text{ where } m = \frac{n}{r}. \tag{5.3}$$

Since r is fixed, m is getting very large as n gets large. So we have the same situation as before: As n gets very large, $\left(1+\frac{1}{m}\right)^m$ approaches e. Substituting back in, we then get e^r for $\left(1+\frac{r}{n}\right)^n$.

Again, later we will write this more formally. The intention here is to introduce the number "e" at this point in the course because it has some nice qualities and is used for things other than interest rates.

Compounding Continuously

We saw that after one year (compounding interest n times per year at the rate r) \$$P$ becomes \$$P(1+\frac{r}{n})^n$. In t years \$$P$ becomes \$$P(1+\frac{r}{n})^{nt}$. In the 1970's many banks claimed in their ads that they would compute interest on your money "every second of every day". What they meant was that they would let n become unboundedly large. In other words, \$$P$ would become \$$Pe^r$ after one year and \$$Pe^{rt}$ after t years. This is called *continuous compounding.*

If we go back once again to the scenario of principal P and interest rate 3%, (recall Example 5.4) but this time compound continuously we would have a future value after one year of $Pe^{.03} \approx P(1.030454534)$. So, the effective interest rate is .030454534. You can find an e^x function key on any scientific calculator.

Example 5.5. Suppose \$600 is invested for seven years at an annual interest rate of 4%, compounded continuously. Find the future value for this investment. What is the effective interest rate? Use a calculator to make sure that your answers are reasonable.

Answer: $F = Pe^{rt}$ becomes $F = 600e^{(.04)7} = 600e^{.28} \approx \$793.88$

The effective interest rate is $e^{.04} - 1 \approx .04081077419$.

Example 5.6. Sal invests \$3000 for 2 years at an interest rate of 10%, compounded continuously. How much money would Sue have to invest if her bank offered 10.5 % compounded annually (i.e., as simple interest) and she wanted to have the same amount in two years as Sal will then have.

Answer: We need to find the present value of Sue.s investment so its future value will be equal to Sal's. $r = .10$ for Sal's investment and $r = .105$ for Sue's; $t = 2$ for both.

Use the first formula 5.4 to find the future value of Sal's investment: $F = Pe^{rt} = 3000e^{(.10)(2)} = \$3664.21$. Then use the second formula 5.2 to find present value of Sue's investment: $P = F(1+\frac{r,n}{)}^{nt} = 3664.21(1+\frac{.105,1}{)}^{(1)(2)} = \$3000.93$, or about \$3001. So, the difference is negligible at this interest rate over this period of time.

Formulas for Continuous Compounding that are analogous to those for Periodic Compounding (Formulas 5.2) are:

$$F = Pe^{rt}$$
$$P = \frac{F}{e^{rt}} = Fe^{-rt} \tag{5.4}$$

***The* Exponential Function**

In Section 4 we reviewed exponential and logarithmic functions. We said that an exponential function is one of the form $f(x) = a^x$ where $a > 0$ and $a \neq 1$. We looked at how the functions where $0 < a < 1$ differed from those where $a > 1$. Most of the examples we dealt with had a as a rational number, but all of the concepts and algebra work for irrational values of a as well. One exponential function that is prevelant in mathematics is the one where a is the irrational number e. In fact, when "*the*" exponential function is mentioned, it refers to $f(x) = e^x$. Occasionally you will see this function written as $f(x) = exp(x)$.

Since $e \approx 2.718$, the function $f(x) = e^x$ has the same characteristics as other exponential functions where the base is greater than one. The domain of f is $\mathbb{R}$; the range is $(0, \infty)$ The y-intercept of the function is $(0, 1)$. The x-axis is a horizontal asymptote on the left. The graph resembles that of $g(x) = 2^x$ (see page 31), except that it is steeper when $x > 0$ and flatter when $x < 0$.

Don't let the "e" bother you. Just like with any other base, it is true that:

$$e^0{=}1 \qquad e^r \cdot e^s = e^{r+s} \qquad (e^r)^s = e^{rs} \qquad e^{\frac{1}{2}} = \sqrt{e} \qquad e^{-x} = \frac{1}{e^x}$$

Also just like with any other base, the exponential function $f(x) = e^x$ has a corresponding logarithmic function.

The Natural Logarithm

Recall that a logarithm with base 10 is called a *common log* and has the shortcut notation that it can be written without the 10. "$\log x$" is understood to mean $\log_{10} x$. Similarly, a logarithm with base e is given the name *natural log*[18] and is written $\ln x$. "$\ln x$" is understood to mean $\log_e x$.

The graphs of $g(x) = e^x$ and $f(x) = \ln x$ are symmetric about the line $y = x$. The general form of the graphs of these functions can been seen on page 36. Notice that the domain of $f(x) = \ln x$ is $(0, \infty)$ and the range of $f(x) = \ln x$ is $\mathbb{R}$. In contrast, the domain of $g(x) = e^x$ is $\mathbb{R}$ and the range of $g(x) = e^x$ is $(0, \infty)$. The natural logarithm function has x-intercept $(1, 0)$, and the negative y-axis forms a vertical asymptote for the graph. The exponential function has y-intercept $(0, 1)$ and the negative x-axis forms a horizontal asymptote for its graph.

It will be very useful for you to keep in mind the shapes of these graphs.

Don't let the "ln" notation scare you. The natural logarithm works just like any other logarithm. We have seen that its graph is just like any other logarithm with base greater than 1.

Below are some logarithm statements from Section 4 and their corresponding statements for natural logarithms. Convince yourself that the natural logarithm statement is just the original

[18] French: "logarithm naturel," abbreviated to "ln"

statement applied to the specific base e.

Statements:

$$\begin{array}{ll} y = a^x \text{ means } \log_a y = x & y = e^x \text{ means } \ln y = x \\ a^{\log_a x} = x & e^{\ln x} = x \\ \log_a a^x = x & \ln e^x = x \end{array}$$

Algebraic manipulation rules:

$$\begin{array}{ll} \log_a 1 = 0 & \ln 1 = 0 \\ \log_a(xy) = \log_a x + \log_a y & \ln(xy) = \ln x + \ln y \\ \log_a \frac{x}{y} = \log_a x - \log_a y & \ln \frac{x}{y} = \ln x - \ln y \\ \log_a \frac{1}{x} = -\log_a x & \ln \frac{1}{x} = -\ln x \\ \log_a x^p = p \log_a x & \ln x^p = p \ln x \end{array}$$

Change of base formulas:

$$\begin{array}{ll} \log_a x = \dfrac{\log_b x}{\log_b a} & \log_a x = \dfrac{\ln x}{\ln a} \\ a^x = b^{x \log_b a} & a^x = e^{x \ln a} \end{array}$$

In calculus the most commonly used exponential function is e^x and the most commonly used logarithm function is the base e logarithm, $\ln x$. Since we have the abiltiy to change bases, our work with any base can be done by changing to base e and then working there. The benefits of doing so will become obvious as we continue though the course. Scientific calculators directly handle natural logarithms, so you won't need to change everything to common logarithms for numeric evaluation.

In the exercises for this section there are problems similar to those for Section 4. It is important that you become adept at manipulating logarithmic and exponential expressions. They will be used throughout the course.

Discussion on π (optional)

Suppose you have a circle with radius measure 1 unit. If you inscribe an equilateral (all sides have the same length) triangle inside that circle, you can use your geometry knowledge to show that the area of the triangle is $\frac{3\sqrt{3}}{4} \approx 1.299038106$. If you inscribe a square inside the circle, you can calculate the area. It is exactly 2. We could then inscribe a regular (all sides have the same length) polygon with five sides, then six sides, then 60 sides, 120 sides, etc. and calculate their areas[19]. Imagine an inscribed polygon with 180 thousand sides, or even 180 million sides! As we continue to increase the number of sides, these polygons get closer and closer to the shape of the circle. But since all of them are still enclosed by the circle, they all have an area that is less than the area of the circle.

n=Number of Sides	Area of Polygon
3	1.299038106
4	2
5	2.377641291
6	2.598076211
60	3.135853898
180	3.140954703
720	3.141552779
180,000	3.141592653
180,000,000	3.141592654

What IS the area of the circle? $A = \pi r^2$. Since $r = 1$, we have $A = \pi$. The area of the circle is *exactly* π, not *approximately* π. The area of each of our inscribed polygons is less than π but we say that as we let the number of sides, n get bigger and bigger and bigger and bigger...their areas can get as close to π as we want. In a sense, π IS the ever-approachable boundary. The number π is real, but irrational, so it cannot be written as a finite decimal. We can approximate it to any accuracy of decimal that we want, depending on the accuracy required for our particular application. When we want the exact number we have to write "π."

We are doing the same thing with e, except that we don't have a nice geometric picture to go with it (yet). Instead of having a formula for areas that has an approachable boundary of π when n is large, we have a formula for interest rates that has an approachable boundary of e when n (the number of compoundings per year) is large. Neither of these boundaries is an approximation.

Probably when you first worked with π it seemed strange to you, but by now you are comfortable with it. You accept π as an irrational number that is approximately 3.14. Don't let e scare you. It is just a number too, an irrational number that is approximately 2.72. When we want the exact value, we write "e."

[19] The area of a regular polygon inscribed in a circle of radius 1 with n sides is: $n\left(\sin\left(\frac{180°}{n}\right)\right)\left(\cos\left(\frac{180°}{n}\right)\right)$. No, you aren't expected to know this.

Section 5 - Exercises (answers follow)

For the investment problems below, except problem 7, you should use a calculator to get a decimal answer.

1. Suppose \$1,000 is invested at an annual interest rate of 7 percent. Compute the future value of the investment after 10 years if the interest is compounded:
   (a) Annually (b) Quarterly (c) Monthly (d) Continuously

2. (a) Find the *effective interest rate* for an investment of \$50,000 with annual interest rate of 4%, compounded:
      (a) Annually (b) Quarterly (c) Monthly (d) Continuously

   (b) Find the *effective interest rate* for an investment of \$90,000 with annual interest rate of 4%, compounded:
      (a) Annually (b) Quarterly (c) Monthly (d) Continuously

3. Find the *interest* earned on \$10,000 invested for 5 yr. at 6% interest compounded:
   (a) Annually (b) Semiannually (twice per year) (c) Quarterly (d) Monthly

4. You invest \$3,000 in an account that offers an annual interest rate of 5%, compounded continuously. How long must you leave the money in the account if you wish to earn \$400 in interest?

5. How long will it take money to double if it is placed in an account that gives 3.5 % compounded (a) Annually? (b) Continuously?

6. You are to choose between two investments: one pays 8% compounded seminannualy, and the other pays $7\frac{1}{2}$% compounded monthly. If you plan to invest \$18,000 for 18 months, which investment should you choose? How much extra interest will you earn by making the better choice?

7. You invest \$300 at an annual interest rate of 5% for 20 years. If the interest is compounded continuously, will your investment reach \$1,000 by the end of the 20 years? Answer this without a calculator.

The problems below are to give you practice using exponential and logarithmic functions with a base of e. If you have trouble, review the comparable exercises in Section 4.

1. Change each logarithmic statement to its equivalent exponential form:
   (a) $y = \ln 3$ (b) $\ln 1 = 0$ (c) $\ln x = 42$

2. Change each exponential statement to its equivalent logarithmic form:
   (a) $e^5 = y$ (b) $\frac{1}{e} = e^{-1}$ (c) $y = e^x$

3. Evaluate without using a calculator:
   (a) $\ln \sqrt{e}$ (b) $e^{\ln 7}$ (c) $\ln e$ (d) $2 \ln e^3 + \ln e^{-4}$

4. Given $\ln 12 \approx 2.5$ and $\ln 4 \approx 1.4$, find the approximate value of: $\ln 16$, $\ln 48$, $\ln 3$

5. Find the domain of the following functions:
   $f(x) = \ln(4x-5)$ $\quad g(x) = \ln(x^2-x-2)$ $\quad h(x) = e^{3x+1}$

6. Simplify: (a) $(e^x + e^{-x})^2 + (e^x - e^{-x})^2$ $\quad$ (b) $\dfrac{e^{-x}(e^x - e^{-x}) + e^{-x}(e^x + e^{-x})}{e^{-2x}}$

7. Do not use a calculator. Between which two consecutive integers would you find:
   (a) $\frac{1}{e}$ (b) $\ln 6$ (c) $10e$

8. Condense to write as a single natural logarithm:
   (a)$3\ln x + \ln y - \ln\sqrt{z}$ $\quad$ (b) $\frac{1}{2}\ln 4 - (\ln 5 + 2\ln 3)$

9. Change each expression into its equivalent using base e.
   (a) 3^x (b) 6^x (c) a^x (d) 5^{x^2}

10. Change each expression to its equivalent natural logarithm expression:
    (a) $\log_a x$ (b) $\log_6 8$

11. Show that $\ln 10 = \dfrac{1}{\log e}$.

12. Solve the following equations for x.
    (a) $e^{2x} \cdot e^{3x-1} = 1$
    (b) $x^3e^x + e^x = 0$
    (c) $e^x - 5e^{2x} = 0$
    (d) $2\ln(3x) = 4$
    (e) $\ln(x-6) - \ln 4 = 2$
    (f) $\ln(\ln x) = 3$

Section 5 - Answers

Investment Problems:

1. (a) \$1967.15 (b) \$2001.60 (c) \$2009.66 (d) \$2013.75

2. (a) (a) .04 (b) .040604 (c) .040742 (d) .040811
   (b) These are the same as for part (a). The amount of money invested affects the *amount* of interest earned, but does not affect the interest *rate*.

3. (a) \$3382.26 (b) \$3439.16 (c) \$3468.55 (d) \$3488.50

4. $t = \dfrac{\ln(\frac{17}{15})}{.05} \approx 2.503$ years

5. (a) 20.149 years (b) 19.804 years

6. Choose the 8% investment, which would yield \$111.30 additional interest.

7. No. $\quad 300e^{.05\cdot 20} = 300e < \$900$

Exponential and Logarithmic Problems

1. (a) $e^y = 3$ (b) $e^0 = 1$ (c) $e^{42} = x$
2. (a) $\ln y = 5$ (b) $\ln(\frac{1}{e}) = -1$ (c) $\ln y = x$
3. (a) $\frac{1}{2}$ (b) 7 . (c) 1 (d) 2
4. (a) 2.8 (b) 3.9 (c) 1.1
5. (a) $D_f = \left(\frac{5}{4}, \infty\right)$ (b) $D_g = (-\infty, -1) \cup (2, \infty)$ (c) $D_h = \mathbb{R}$
6. (a) $2e^{2x} + 2e^{-2x}$ (b) $2e^{2x}$
7. (a) Between 0 and 1 (b) Between 1 and 2 (c) Between 27 and 28
8. (a) $\ln\left(\dfrac{x^3y}{z^{\frac{1}{2}}}\right)$ (b) $\ln\left(\dfrac{2}{45}\right)$
9. (a) $e^{x\ln 3}$ (b) $e^{x\ln 6}$ (c) $e^{x\ln a}$ (d) $e^{x^2\ln 5}$
10. (a) $\dfrac{\ln x}{\ln a}$ (b) $\dfrac{\ln 8}{\ln 6}$
11. $\ln 10 = \dfrac{\log 10}{\log e} = \dfrac{1}{\log e}$
12. (a) $\frac{1}{5}$ (b) -1 (c) $-\ln 5$ (d) $\frac{1}{3}e^2$ (e) $4e^2 + 6$ (f) e^{e^3}

6 Limits

Finite Limits

The subject of calculus revolves around the idea of a *limit* Mathematicians have precisely defined the term "limit," but we will not be so formal in this course. However, it is important that you have an accurate understanding of the concept. In order to do this, you must first understand the idea of *arbitrarily close.* An illustration, albeit rather contrived, should help with this.

Suppose you work for a strange company and it is your job to draw squares. Fortunately, you have a machine that draws squares of any size; all you have to do is tell the machine the length of the side desired. On Monday, your boss comes to you and says that he needs a square with an area of 25 square inches. Now you know that $A = s^2$ (area = side × side), so you go to set your machine for a side length of 5 inches. Unfortunately, you find that there is a malfunction in your machine, and the only side length that it isn't handling correctly is a length of 5 inches. You report this to your boss.

"Can't you just use a different length?"

"No, sir. To have a square with an area of exactly 25 in^2, I must use a side length of exactly 5 inches." (Your expertise in the matter is why you get paid the big bucks.)

"Well, maybe I could get by with a square that is little bit bigger...but I want an area less than 26 in^2. Can you do that?"

" Yes, sir. If I make the length of the side 5.05 inches, then I could give you a square with area 25.5025 in^2."

"Well, that would be OK, I guess, but maybe smaller would be better. Can you get me a square with area less than, say, 25.2 in^2?"

"Yes. I could make the length of a side 5.01 inches. That would give you an area of 25.1001 in^2."

"Well, I'm satisfied with that, but I'm not sure how my manager will react. Could you make a square with area less than 25.0001 in^2?"

"Certainly. Any side with length less than 5.00001 inches will work."

"Great! Great! You are a marvel! Oh, but wait...the CEO is coming today to check out this project. Just how close *can* you get to a square of area 25 in^2? What shall I tell her? What is the very best you can do?"

"Sir, you can tell the chief that although I cannot make a square with area exactly 25 in^2, I can make a square that has an area as close to 25 in^2 as she wishes. The difference in area between my square and a square of area 25 in^2 can be just as small as she wants. All I need to do is make sure that the length of the side is sufficiently small."

There are two concepts to be gleaned from the previous illustration. One is the idea of being *arbitrarily close.* The other is the idea of a *limit.*

In the process of making the side of the square closer and closer to 5, you make the area closer and closer to 25. However, having the area simply get closer and closer to 25 isn't enough for *arbitrarily* close.[20] Arbitrary closeness requires that no matter how small (or arbitrary) a positive difference you choose, you can obtain that difference and less. A *limit* is the entity to which you can become arbitrarily close. In this case, the unattainable area of 25 in^2 is the limit.

Let's apply these concepts to functions.

We say *the limit of $f(x)$ as x approaches the number a is L* if $f(x)$ gets arbitrarily close to L as x gets closer and closer to a. When this is so, we write $\lim_{x\to a} f(x) = L$.

In this definition, L is a real number, and a may or may not be in the domain of f.

Example 6.1. $\lim_{x\to 3}(2x+4) = 10$. Here, $f(x) = 2x+4$, $a = 3$ and $L = 10$.

We are claiming that 10 is the limit of $(2x+4)$ as x gets close to the number 3 because 10 is the number to which $(2x+4)$ becomes arbitrarily close as x gets close to 3. No matter how small you want the difference between $(2x+4)$ and 10 to be, you can achieve that difference by making x sufficiently close to 3. For instance, if you want the difference between $(2x+4)$ and 10 to be less than .01, you only have to make sure that your x is within .005 of the number 3. Check it out.

There were a lot of words in Example 6.1 about differences and closeness. Let's look at the example again. If x values get closer and closer to 3 what values does $(2x+4)$ take on? The following table shows some values. Notice that x values could be less than 3 or greater than 3 as they get closer to 3.

$x < 3$	$2x+4$	$x > 3$	$2x+4$
2	8	4	12
2.5	9	3.5	11
2.9	9.8	3.1	10.2
2.99	9.98	3.01	10.02
2.99999	9.99998	3.00001	10.00002

Can you see from the table that $2x+4$ can get arbitrarily close to 10...you need only get x sufficiently close to 3?

In Example 6.1 it is true that $f(3) = 2(3)+4 = 10$. It cannot be overstated that this is irrelevant to the limit. When we write: $\lim_{x\to a} f(x) = L$, we do not consider the actual value of the function at $x = a$. We are making a statement that says that the y values of the function are getting arbitrarily close to the number L as the x values approach a. We are saying nothing about $f(a)$.

[20] The numbers in the pattern 25.21, 25.201, 25.2001, 25.20001, etc. are getting closer and closer to the number 25. But they are not getting arbitrarily close. They are all maintaining a difference of of at least 0.2. In point of fact, this sequence of numbers is getting arbitrarily close to 25.2.

Example 6.2. $g(x) = \begin{cases} 2x+4, & x<3 \\ 2x+4, & x>3 \end{cases}$ $\qquad h(x) = \begin{cases} 2x+4, & x<3 \\ 5, & x=3 \\ 2x+4, & x>3 \end{cases}$

$g(3)$ does not exist. $h(3) = 5$. However $\lim_{x\to 3} g(x) = 10$ and $\lim_{x\to 3} h(x) = 10$ because the limit as $x \to 3$ is not concerned about the existence or value of the function *at* $x = 3$.

You can see, then, that finding a limit as $x \to a$ is not a matter of finding $f(a)$. This is emphasized again in the following, more complicated example.

Example 6.3. $\lim_{x\to 1} \left(\dfrac{x^2 - 2x + 1}{x - 1} \right) = 0.$

Certainly this limit was not found by evaluating the function at $x = 1$. The function is not defined at $x = 1$.

Think about this one: First note that $x^2 - 2x + 1 = (x-1)(x-1)$. When $x \neq 1$ we can divide above and below to get:

$$\frac{x^2 - 2x + 1}{x - 1} = x - 1.$$

By the definition of limit we are not interested in what happens when x is 1 but rather in what the value of $\dfrac{x^2 - 2x + 1}{x - 1}$ is as x approaches 1. And as x approaches 1 the number $x - 1$ approaches 0. Hence $\lim_{x\to 1} \dfrac{x^2 - 2x + 1}{x - 1} = \lim_{x\to 1} (x - 1) = 0$.

Example 6.4.

$$f(x) = \begin{cases} 1 & \text{if } x \le 5 \\ -2 & \text{if } x > 5 \end{cases}.$$

Here $\lim_{x\to 5} f(x)$ does not exist. If $x \to 5$ using x values decreasing to 5, $f(x) \to -2$. But if $x \to 5$ using x values increasing to 5, $f(x) \to 1$. So there is no one number L such that $f(x)$ approaches L as x approaches 5. The fact that $f(5)$ makes sense in this example (we defined $f(5) = 1$) is irrelevant. There is no limit as x approaches 5.

Example 6.5.

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -2 & \text{if } x \text{ is irrational} \end{cases}.$$

Here $\lim_{x\to 0} f(x)$ does not exist. As x approaches 0, there are always some x values that are rational and some that are irrational. Therefore, there are always $f(x)$ values of 1 and of -2. There is no number L to which the y values become arbitrarily close.

Example 6.6.

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is an integer} \\ -2 & \text{if } x \text{ is not an integer} \end{cases}.$$

In this case, $\lim_{x\to 0} f(x)$ does exist and is equal to -2. As the x values get very close to 1, there are no integers, so all of the $f(x)$ values are -2. Thus $f(x)$ approaches -2 as x approaches zero. The fact that $f(0) = 1$ is irrelevant. Actually, for this function Here $\lim_{x\to c} f(x) = -2$ for all values of c in $\mathbb{R}$.

In Example 6.4 we discussed x approaching 5 from two directions. There is a notation for this. For Example 6.4 we would write: $\lim_{x\to 5^-} f(x) = 1$ and $\lim_{x\to 5^+} f(x) = -2$. The first is called the *left hand limit* (LHL) and the second is called the *right hand limit* (RHL). For $\lim_{x\to 5} f(x)$ to exist, it must be true that LHL=RHL. Make note that the small $-$ and $+$ superscripts do not indicate that x is positive or negative. They indicate that x is less than 5 or x is greater than 5 respectively.

A very reasonable question at this point is, "So, how do we find limits?" We do NOT find limits by repeatedly substituting in x values closer and closer to see what pattern of y values comes out. The chart for Example 6.1 was an illustration to help with understanding. It isn't how one solves limit problems.

In slower moving calculus courses there would be time for a detailed discussion of limits of sums, differences, products and quotients. Here, we'll go straight to the facts, which result from strict application of mathematical definitions and proof processes. These are some Limit Laws for Finite Limits that you can use to evaluate limits.

Theorem. *If all limits mentioned on each of the following lines exist then*

1. $\lim_{x\to a} c = c$ *for any constant* c.
2. $\lim_{x\to a} x = a$
3. $\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
4. $\lim_{x\to a} (f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
5. $\lim_{x\to a} (f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
6. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ *provided* $\lim_{x\to a} g(x) \neq 0$.

By combining parts 1 and 5 of the theorem above we can see that for any constant c we have $\lim_{x\to a} cf(x) = c \cdot (\lim_{x\to a} f(x))$ provided $\lim_{x\to a} f(x)$ exists.

By combining parts 1, 2, 3, 4 and 5 we get the very useful result that $\lim_{x\to a} p(x) = p(a)$ for any polynomial $p(x)$.

Example 6.7. Evaluate the limit: $\lim_{x\to 2}(x^5 - 3x^4 - x^2 + 7)$.

$$\lim_{x\to 2}(x^5 - 3x^4 - x^2 + 7) = 2^5 - 3(2)^4 - (2)^2 + 7 = 32 - 48 - 4 + 7 = -13.$$

There are some other Limit Laws for Finite Limits that require special attention to conditions, particularly to conditions of domain and the existence of some intermediate limits. But, you may use these laws as long as some care is taken to make sure that their usage makes sense.

a. $\lim_{x\to a} b^{f(x)} = b^{\lim_{x\to a} f(x)}$

b. $\lim_{x\to a}[f(x)^n] = [\lim_{x\to a} f(x)]^n$

You'll find we use these facts about limits often.

Example 6.8. Evaluate the limit: $\lim_{x\to 4}\left(\dfrac{\frac{1}{4}-\frac{1}{x}}{x-4}\right)$

We cannot use Limit Law 6 because the limit in the denominator would be zero. So, we algebraically rewrite our function by combining the fractions in the numerator and simplifying:

$$\lim_{x\to 4}\left(\frac{\frac{1}{4}-\frac{1}{x}}{x-4}\right) = \lim_{x\to 4}\left(\frac{\frac{x-4}{4x}}{\frac{x-4}{1}}\right) = \lim_{x\to 4}\left(\frac{1}{4x}\right) = \frac{1}{16}$$

Example 6.9. Evaluate the limit: $\lim_{x\to 1}\dfrac{x-1}{\sqrt{x+3}-2}$

We cannot use Limit Law 6 because the denominator would be zero. So we algebraically rewrite the function, using the conjugate to get rid of the radical.

$$\lim_{x\to 1}\frac{x-1}{\sqrt{x+3}-2} = \lim_{x\to 1}\left(\frac{x-1}{\sqrt{x+3}-2}\cdot\frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}\right) = \lim_{x\to 1}\frac{(x-1)(\sqrt{x+3}+2)}{(x+3)-4} =$$
$$\lim_{x\to 1}\frac{(x-1)(\sqrt{x+3}+2)}{(x-1)} = \lim_{x\to 1}(\sqrt{x+3}+2) = \sqrt{1+3}+2 = 4$$

Example 6.10. Find $\lim_{x\to 3} f(x)$, $\lim_{x\to 2} f(x)$ and $\lim_{x\to 0} f(x)$ for: $f(x) = \begin{cases} x+3 & x<2 \\ 2x-1 & 2<x<3 \\ x+2 & x>3 \end{cases}$.

$\lim_{x\to 3^+} f(x) = 3+2 = 5$ and $\lim_{x\to 3^-} f(x) = 2(3)-1 = 5$. Since RHL = LHL, we have $\lim_{x\to 3} f(x) = 5$.
$\lim_{x\to 2^+} f(x) = (2)2-1 = 3$, and $\lim_{x\to 2^-} f(x) = 2+3 = 5$. Since RHL $\neq$ LHL, $\lim_{x\to 2} f(x)$ doesn't exist.
$\lim_{x\to 0} f(x) = 0+3 = 3$. We do not have to use one-sided limits when $x\to 0$ because all values of x very close to 0 are in the interval $x<2$.

Example 6.11. Find: $\lim_{x \to 2} \frac{3x-6}{x^2-4x+4}$. Limit Law 6 doesn't apply so we try a rewrite:

$$\lim_{x \to 2} \frac{3x-6}{x^2-4x+4} = \lim_{x \to 2} \frac{3(x-2)}{(x-2)(x-2)} = \lim_{x \to 2} \frac{1}{x-2}$$

Here we are stuck. This function cannot be simplified further. We still cannot apply Limit Law 6. There is no real number L to which the function values come arbitrarily close. This limit does not exist [21].

Infinite Limits

Let's look back at Example 6.11. We concluded that there was no real number L to which the function values become arbitrarily close. So, by our understanding of "limit," this limit does not exist. But, let's see what *is* happening with the function values as $x \to 2$ for this function.

$x < 2$	$\frac{1}{x-2}$	$x > 2$	$\frac{1}{x-2}$
1	-1	3	1
1.5	-2	2.5	2
1.9	-10	2.1	10
1.99	-100	2.01	100
1.99999	-100,000	2.00001	100,000
1.9999999	-10,000,000	2.0000001	10,000,000

Look at the values for $x > 2$. As the x values get closer and closer to 2, the denominator $(x-2)$ gets closer and closer to zero. So the function itself, the reciprocal of $(x-2)$, gets larger and larger. How large will $\frac{1}{x-2}$ get? Will the function value ever get to be a trillion (12 zeros after the 1)? Yes. From the pattern, you can see that $f(x)$ will be a trillion when $x = 2.000000000001$. Is there a maximum value that $\frac{1}{x-2}$ will attain? No. Do you see that for *any* large number you can pick, the function $\frac{1}{x-2}$ can exceed that number if you choose an x value sufficiently close to 2?

We say that $\lim_{x \to a} f(x) = \infty$ *if* $f(x)$ *becomes unboundedly large as* x *approaches* a*.*

Our function $f(x) = \dfrac{1}{x-2}$ becomes unboundedly large as x approaches 2 from the right, so we can make the corresponding one-sided limit statement: $\lim_{x \to 2^+} \frac{1}{x-2} = \infty$.

Now look at the table values for $x < 2$. A similar thing is happening, except that these function values are negative. As $x \to 2$ from the left, the function values are unbounded in the negative direction. We write: $\lim_{x \to 2^-} \frac{1}{x-2} = -\infty$.

Since $\lim_{x \to 2^+} \frac{1}{x-2}$ is not the same as $\lim_{x \to 2^-} \frac{1}{x-2}$, we would say that $\lim_{x \to 2} \frac{1}{x-2}$ does not exist.

Most calculus text books define limits to be real numbers, as we did in the first part of this section. Infinity and minus-infinity are not real numbers, so limits like those in Example 6.11 would

[21] There is no FINITE limit for this function. Read the subsection immediately following, entitled "Infinite Limits..

be said to not exist. However, most books then admit that it is convenient to use the limit notation to describe the behavior of functions whose values become unbounded (+ or −) as x approaches some value a. We shall do this too. To merely state that $\lim_{x\to 2} \frac{1}{x-2}$ does not exist is insufficient. One should use the one-sided limit notation to relate the unbounded behavior of the function as $x \to 2$ from each side. More examples will follow, so this should become clear.

The Limit Laws for Finite Limits in the first half of this section do not apply to infinite limits.

So, how do we know we have an infinite limit? Consider $\lim_{x\to a} \frac{f(x)}{g(x)}$ where f and g are functions. If $\lim_{x\to a} f(x) = c \neq 0$ and $\lim_{x\to a} g(x) = 0$ then we have a situation where the denominator of the fraction is getting very small as x approaches a, but the numerator is not. The value of the function then is becoming unbounded as x approaches a. It is necessary to check the signs of both the numerator and the denominator to see if the unboundness of the quotient is positive or negative.

Example 6.12. Evaluate $\lim_{x\to -3} \frac{x+2}{x+3}$.

Answer: $\lim_{x\to -3}(x+2) = -1$ and $\lim_{x\to -3}(x+3) = 0$. Since the denominator is getting close to zero but the numerator is not, the value of the function is becoming unbounded as x approaches -3.

As x approaches 3 from the left $x < -3$, so the values of $(x+3)$ are negative.

As x approaches 3 from the right $x > -3$, so the values of $(x+3)$ are positive.

As x approaches 3 from either direction, $(x+2)$ approaches -1, which is negative.

So, we conclude: $\lim_{x\to -3^-} \frac{x+2}{x+3} = \infty$ and $\lim_{x\to -3^+} \frac{x+2}{x+3} = -\infty$, so $\lim_{x\to -3} \frac{x+2}{x+3}$ does not exist.

Example 6.13. Evaluate: $\lim_{x\to 0} \frac{1}{x^2}$.

Answer: $\lim_{x\to 0} 1 = 1$, and $\lim_{x\to 0} x^2 = 0$. Since the denominator is getting close to zero but the numerator is not, the value of the function is becoming unbounded as x approaches 0.

As x approaches 0 from either direction, the values of x^2 are positive.

As x apporaches 0 from either direction, the numerator is 1, which is positive.

So, we conclude: $\lim_{x\to 0^-} \frac{1}{x^2} = \infty$ and $\lim_{x\to 0^+} \frac{1}{x^2} = \infty$, so $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

Example 6.14. For $f(x) = \frac{3x+6}{x^2-3x-10}$, find $\lim_{x\to -2} f(x)$ and $\lim_{x\to 5} f(x)$.

Answer: $f(x) = \frac{3x+6}{x^2-3x-10} = \frac{3(x+2)}{(x+2)(x-5)} = \frac{3}{x-5}$ when $x \neq -2$ and $x \neq 5$.

$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{3}{x-5} = -\frac{3}{7}$

$\lim_{x\to 5} 3 = 3$ and $\lim_{x\to 5}(x-5) = 0$, so the value of the function is becoming unbounded as x approaches 5.

As x approaches 5 from the left, $(x-5)$ is negative

As x approaches 5 from the right, $(x-5)$ is positive.

The numerator, 3, is always positive.

So, we conclude $\lim_{x\to 5^-} f(x) = -\infty$ and $\lim_{x\to 5^+} f(x) = \infty$, so $\lim_{x\to 5} f(x)$ does not exist.

Section 6 - Exercises (answers follow)

Find the indicated limit.

1. $\lim_{x\to -2} 6$

2. $\lim_{x\to 1}(3x^2+5x+2)$

3. $\lim_{s\to 0}(2s^3-1)(2s^2+4)$

4. $\lim_{x\to 0}\dfrac{2x-3}{2x-1}$

5. $\lim_{x\to 3}\dfrac{x^2-16}{x-4}$

6. $\lim_{x\to 2}\dfrac{x^2+x-6}{x-2}$

7. $\lim_{x\to 2}\dfrac{\sqrt{x}-2}{x-4}$

8. $\lim_{x\to 1}\dfrac{[-1/(x+3)]+1/4}{x}$

9. $\lim_{x\to 2} x$

10. $\lim_{x\to 3} e^{2x-1}$

11. $\lim_{x\to 3}\dfrac{x-3}{x-3}$

12. $\lim_{h\to 0}\dfrac{(x+h)^2-x^2}{h}$

13. $\lim_{h\to 0}\dfrac{\sqrt{h+9}-3}{h}$

14. $\lim_{y\to 0}\dfrac{6y-9}{y^3-12y+3}$

15. $\lim_{x\to 2}\dfrac{2-x}{\sqrt{7+6x^2}}$

16. $f(x)=\begin{cases}\dfrac{x^2-1}{x+1} & \text{if } x<-1\\ \\ x^2-3 & \text{if } x\geq -1\end{cases}$ Find $\lim_{x\to -1} f(x)$

17. $f(x)=\begin{cases}3+x & \text{if } x<2\\ 3x+1 & \text{if } x\geq 2\end{cases}$ Find $\lim_{x\to 2} f(x)$

18. $f(x)=\begin{cases}3+x & \text{if } x<2\\ 3x-1 & \text{if } x>2\end{cases}$ Find $\lim_{x\to 2} f(x)$

19. $\lim_{x\to 4}\dfrac{x^2-16}{x-4}$

20. $\lim_{z\to 6}\dfrac{z-6}{z^2-36}$

21. $\lim_{z\to 6}\dfrac{z+6}{z^2-36}$

22. $\lim_{x\to 1^-}\dfrac{1+x^2}{1-x^2}$

23. $\lim_{x\to 0^-}\left(\dfrac{1}{x}+\dfrac{1}{x^2-x}\right)$

24. $\lim_{t\to 9}\dfrac{t-9}{\sqrt{t}-3}$

25. $\lim_{u\to 0}\dfrac{\sqrt{u^2+4}-2}{4}$

26. $\lim_{x\to 4}\dfrac{x^2-16}{\sqrt{x+5}-3}$

27. $\lim_{x\to 5}\dfrac{x^2+x-30}{2x-10}$

28. $\lim_{x\to 3}\dfrac{\frac{1}{x}-\frac{1}{3}}{x^2-9}$

29. $\lim_{x\to -1}\dfrac{x^2+3x+2}{x^2+x-6}$

30. $\lim_{x\to 6^-}\dfrac{-4x+3}{x-6}$

31. Consider the graph of $f(x) = \ln x$ (page 36).

(a) What is $\lim\limits_{x\to 0^+} \ln x$? (b) What about $\lim\limits_{x\to 0^-} \ln x$?

32. Find: $\lim\limits_{x\to 0} \dfrac{|x|}{x}$. Hint: Rewrite the function as a piecewise defined function (see page 13).

33. The statement: $\dfrac{x^2-9}{x-3} = x+3$ is false, but the statement: $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3} = \lim\limits_{x\to 3}(x+3)$ is true. Explain.

34. Express the situation described at the beginning of this section (the story about the square drawing machine) as a one-sided limit.

Section 6 - Answers

1. 6 2. 10 3. -4 4. 3 5. 7 6. 5 7. $\frac{1}{2+\sqrt{2}}$ 8. 0 9. 2

10. e^5 11. 1 12. $2x$ 13. $\frac{1}{6}$ 14. -3 15. 0 16. -2

17. $\lim\limits_{x\to 2^-} f(x) = 5$ and $\lim\limits_{x\to 2^+} f(x) = 7$, so $\lim\limits_{x\to 2} f(x)$ does not exist 18. 5 19. 8

20. $\frac{1}{12}$ 21. $\lim\limits_{z\to 6^-} f(x) = -\infty$ and $\lim\limits_{z\to 6^+} f(x) = \infty$, so $\lim\limits_{z\to 6} f(x)$ does not exist 22. ∞

23. -1 24. 6 25. 0 26. 48 27. $\frac{11}{2}$ 28. $-\frac{1}{54}$ 29. 0 30. ∞

31. (a) $-\infty$ (b) This limit makes no sense. There are no values of x less than zero in the domain, so x can't approach from the left.

32. $\lim\limits_{x\to 0^-} \dfrac{|x|}{x} = -1$ and $\lim\limits_{x\to 0^+} \dfrac{|x|}{x} = 1$, so $\lim\limits_{x\to 0} \dfrac{|x|}{x}$ does not exist.

33. The first statement is not true if $x = 3$. The second statement is a limit where $x \to 3$, so we know that x is not 3.

34. $\lim\limits_{s\to 5^+} s^2 = 25$

7 The Slope of the Tangent to a Graph

We'll start with an example. The graph of $y = x^2 + 1$ is a parabola whose lowest point is $(0, 1)$. The point $(3, 10)$ is on the graph. For any number h, the point $(3 + h, (3 + h)^2 + 1)$ is also on the graph. When $h \neq 0$ this is a different point from $(3, 10)$ because its x-coordinate is different. Assuming $h \neq 0$ we ask:

Question 1. *What is the slope of the line*[22] *joining* $(3, 10)$ *to* $(3 + h, (3 + h)^2 + 1)$*?*

Answer.

$$\begin{aligned} \frac{[(3+h)^2+1]-10}{(3+h)-3} &= \frac{9+6h+h^2+1-10}{h} \\ &= \frac{h^2+6h}{h} \\ &= h+6 \end{aligned}$$

Question 2. *Towards what does this slope tend as h approaches 0, and how should we interpret the answer?*

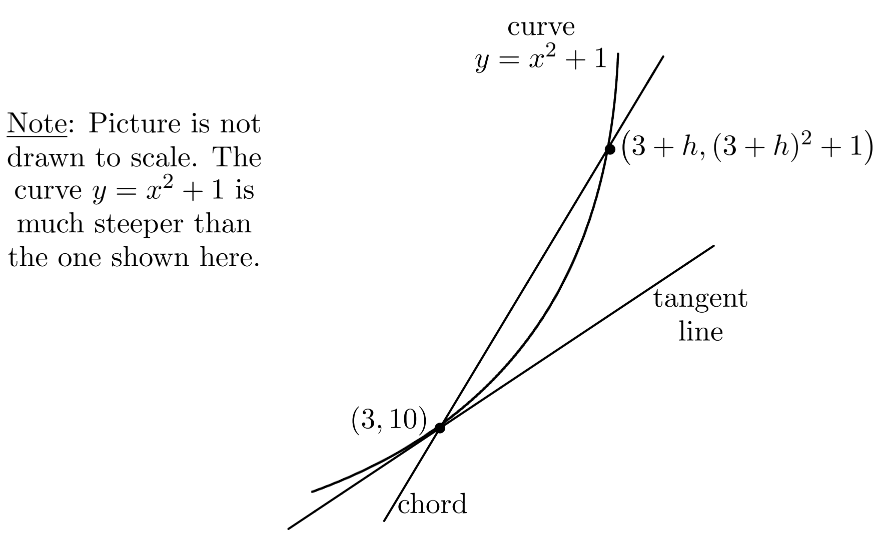
Answer. $h + 6 \to 6$ as $h \to 0$. Interpretation of this lies at the core of calculus:

First interpretation (wrong!): When $h = 0$ the point $(3+h, (3+h)^2+1)$ *is* the point $(3, 10)$ so the slope of the line joining $(3, 10)$ to $(3, 10)$ is 6. This is nonsense. There are lots of lines through $(3, 10)$, not just one. Indeed, for any number m the line $y - 10 = m(x - 3)$ has slope m and passes through $(3, 10)$; and the vertical line $x = 3$ also passes through $(3, 10)$.

Second interpretation (right!): The line through $(3, 10)$ with slope 6 is the line

$$y - 10 = 6(x - 3)$$

and this must be a very special line in relation to the graph of $y = x^2 + 1$. We call it the *tangent* to the graph at $(3, 10)$. See "Tangent Line Illustration" below.



Tangent Line Illustration

[22]The line segment joining two points on a curve is sometimes called a *chord*.

Now let's do the same thing more generally. Consider the function $f(x)$ and, for a moment, let's assume the domain of f is $(-\infty, \infty)$. Pick an x value, say $x = a$. Then $(a, f(a))$ is on the graph of f. For any $h \neq 0$ the slope of the line joining $(a, f(a))$ to the (different) point $(a+h, f(a+h))$ is

$$\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

Question 3. *What happens to this number as $h \to 0$?*

Answer. EITHER: $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ is equal to some *finite* number, in which case we call that limit $f'(a)$. The number $f'(a)$ is defined to be the *slope* of the unique line tangent to the graph at the point $(a, f(a))$.

OR: $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ is undefined, so there is no number towards which $\dfrac{f(a+h) - f(a)}{h}$ tends as $h \to 0$, in which case we do not have a slope for a line tangent to the graph at $(a, f(a))$. This means that the tangent line at this point is vertical, or there is no tangent line.

Example 7.1. Find the equation of the tangent line to the curve $f(x) = x^2 + 7x + 1$ at the point $(2, 19)$.

Answer: To find the equation of a tangent line we need a point and a slope. The point given is $(2, 19)$ and the slope is $f'(2)$. We need to find $f'(2)$:

$$\begin{aligned} f(x) &= x^2 + 7x + 1 \\ f(2+h) - f(2) &= [(2+h)^2 + 7(2+h) + 1] - [2^2 + 7 \cdot 2 + 1] \\ &= 4 + 4h + h^2 + 14 + 7h + 1 - 4 - 14 - 1 \\ &= h^2 + 11h \\ \frac{f(2+h) - f(2)}{h} &= h + 11 \end{aligned}$$

So $f'(2) = \lim\limits_{h \to 0} \dfrac{f(2+h) - f(2)}{h} = \lim\limits_{h \to 0} (h + 11) = 11$.

The equation of the tangent line, then is $y - 19 = 11(x - 2)$.

We now look at three examples where $f'(0)$ does not exist. In the first two examples, there is no tangent line to the graph at $(0, f(0)$. In the third example, there is a line tangent to the graph at $(0, f(0))$, but it is vertical. A sketch of the graphs of these three functions (below) can help you to see the difference.

Example 7.2. Find $f'(0)$ for $f(x) = \begin{cases} -2 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

Answer: Here, $a = 0$.

For $h > 0$, $\dfrac{f(0+h) - f(0)}{h} = \dfrac{1 - 1}{h} = 0$. So, $\lim\limits_{h \to 0^+} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^+} 0 = 0$

For $h < 0$, $\dfrac{f(0+h) - f(0)}{h} = \dfrac{-2 - 1}{h} = \dfrac{-3}{h}$. So, $\lim\limits_{h \to 0^-} \dfrac{f(0+h) - f(0)}{h} = \lim\limits_{h \to 0^-} \dfrac{-3}{h} = \infty$

For $f'(0)$ to exist, the one-sided limits would have to be finite and equal. This is not the case. Also, it doesn't make sense for a slope to be ∞, so certainly $f'(0)$ does not exist.

Example 7.3. Find $f'(0)$ for $f(x) = \sqrt[3]{x}$.
Answer: Here, $a = 0$.

$$\frac{f(0+h) - f(0)}{h} = \frac{\sqrt[3]{0+h} - \sqrt[3]{0}}{h} = \frac{\sqrt[3]{h}}{h} = \frac{1}{\sqrt[3]{h^2}}$$

$\displaystyle\lim_{h\to 0^-} \frac{1}{\sqrt[3]{h^2}} = \infty$ and $\displaystyle\lim_{h\to 0^+} \frac{1}{\sqrt[3]{h^2}} = \infty$. So, we can say that $\displaystyle\lim_{h\to 0} \frac{1}{\sqrt[3]{h^2}} = \infty$. However, it makes no sense for a line to have a slope of ∞, so we say that $f'(0)$ does not exist.

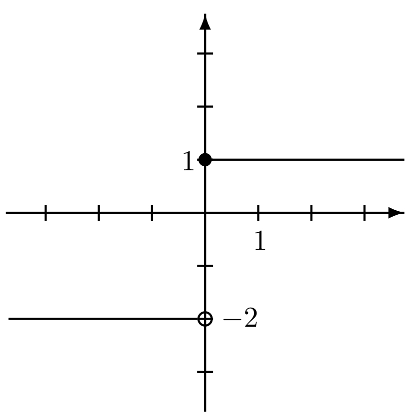
Example 7.4. Find $f'(0)$ for $f(x) = |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

Answer: Here, $a = 0$.

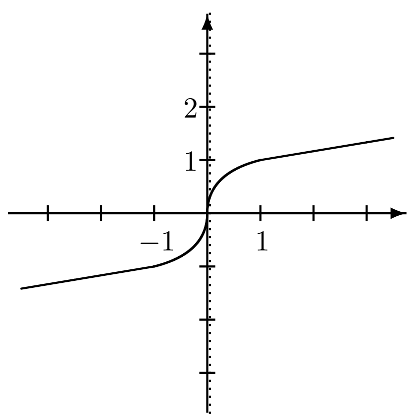
For $h > 0$, $\displaystyle\frac{f(0+h) - f(0)}{h} = \frac{(0+h) - 0}{h} = \frac{h}{h} = 1$ So, $\displaystyle\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} 1 = 1$

For $h < 0$, $\displaystyle\frac{f(0+h) - f(0)}{h} = \frac{-(0+h) - 0}{h} = \frac{-h}{h} = -1$. So, $\displaystyle\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} -1 = -1$.

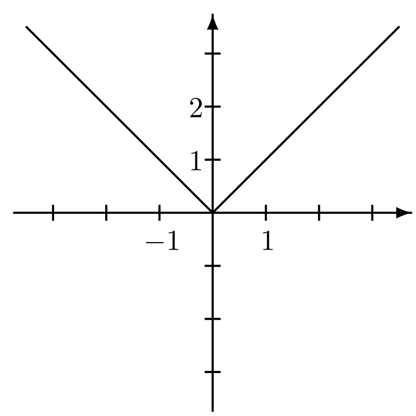
We see that the one-sided limits are not the same so $\displaystyle\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = f'(0)$ does not exist.



Example 7.2
No tangent line at $(0, 1)$



Example 7.3
Vertical tangent line at $(0, 0)$



Example 7.4
No *unique* tangent line at (0,0)

We say that $f'(a)$ exists only if $\displaystyle\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ exists and is finite.

Section 7 - Exercises (answers follow)

For exercises 1-3 use the methods of this section. Do not use any short-cut methods that you may have learned previously.

1. Find the *slope* of the tangent line to the graph of each function at the given x value.
   (a) $f(x) = 6$; $x = 2$
   (b) $f(x) = 7 - 5x$; $x = 12$
   (c) $f(x) = \frac{3}{x}$; $x = \frac{1}{2}$
   (d) $f(x) = 2x^2 + x^3$; $x = 5$
   (e) $f(x) = \dfrac{x}{x-1}$; $x = 2$
   (f) $f(x) = \sqrt{x}$; $x = 4$

2. Find the *equation* of the tangent line at the given x values.
   (a) $f(x) = \frac{2}{3x-7}$; $x = 2$
   (b) $f(x) = x^2 - 3x$; $x = -1$
   (c) $f(x) = x + \frac{1}{x}$; $x = 3$

3. $f(x) = \dfrac{1}{x}$
   (a) Find the slope of the line tangent to f at the point $(1, 1)$.
   (b) Look at the graph of f on page 20. At what other point on the graph would you expect the slope of the tangent line to be -1? Check your answer using the appropriate limit.
   (c) Find the slope of the line tangent to f at the point $(4, \frac{1}{4})$.
   (d) Find the slope of the line tangent to f at the point $(\frac{1}{2}, 2)$.
   (e) You have calculated $f'(1)$, $f'(-1)$, $f'(4)$ and $f'(\frac{1}{2})$, the slopes of f at four different points. Did you find the algebra to be repetitive? We can do this in general: Show that for x value a, the slope of the tangent line, $f'(a)$, is $-\frac{1}{a^2}$.
   (f) Check your answers for parts (a) through (d) in the formula for $f'(a)$ given in part (e).
   (g) Observe that $f'(a) = -\dfrac{1}{a^2}$ is always negative. Look again at the graph of $f(x) = \dfrac{1}{x}$. Are there any places on the graph where you would expect the tangent line to have a positive slope?

Section 7 - Answers

1. (a) 0 (b) -5 (c) -12 (d) 95 (e) -1 (f) $\frac{1}{4}$
2. (a) $y + 2 = -6(x - 2)$ (b) $y - 4 = -5(x + 1)$ (c) $y - \frac{10}{3} = \frac{8}{9}(x - 3)$
3. (a) -1 (b) $(-1, -1)$ (c) $-\frac{1}{16}$ (d) -4 (g) No. All tangent lines will slant downward, consistent with a line of negative slope. Any line with positive slope would rise upwards, crossing the graph of f, not touching it tangentially.

8 Derivatives

The Derivative Function

If you have not done Exercise 3 in Section 7, now would be a good time.

In that exercise you learned that you can find an expression for the slope of the tangent line to a function without specifically identifying the point on the function. That is, you could find an expression for $f'(a)$ in terms of a, and then use this expression to find the slope of the tangent line for any specific value of a. We plug un a specific value of a and we get out the slope of the tangent line at $(a, f(a))$. This sounds very much like the behavior of a function. In our discussion so far, a was treated as a constant. It was arbitrary, but constant. Now we will write this limit in function notation, using x as the independent variable. This function has a special name, "derivative."

Definition 8.1. *The* <u>*derivative of f*</u>*, denoted f', is the function defined as:*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The domain of f' is the set of all numbers x in the domain of f for which this limit exists.

Vocabulary:

1. The process of finding the derivative of f is called *differentiating* f. "To differentiate" is to find the derivative function.
2. In Section 12 we will meet the "derivative of the derivative" which is usually called the "second derivative." So, the derivative introduced here in this section is sometimes called the *first derivative.*
3. If a is in the domain of f and $f'(a)$ exists, we say that f is *differentiable at a.*
4. If I is an open interval lying in the domain of f and if $f'(x)$ exists for all x in I, we say that f is *differentiable on I.*

Example 8.1. For $f(x) = \dfrac{1}{x}$, find $f'(x)$ and the equation of the line tangent to f at the point $\left(-2, -\frac{1}{2}\right)$.

Answer: From Exercise 3 in Section 7, we get $f'(x) = -\dfrac{1}{x^2}$.

The slope of the tangent line is $f'(-2) = -\dfrac{1}{(-2)^2} = -\dfrac{1}{4}$. So, the equation of the tangent line is $y + \frac{1}{2} = -\frac{1}{4}(x+2)$.

Example 8.2. For $f(x) = \sqrt{x}$, find $f'(x)$, the domain of $f'(x)$, and the slope of the line tangent to f at the point $(4, 2)$.

Answer:

$$\begin{aligned} f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \lim_{h\to 0} \left(\frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \right) \\ &= \lim_{h\to 0} \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}} \\ &= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

The domain of f is $[0, \infty)$ but $f'(x)$ does not exist for $x = 0$, so the domain of f' is only $(0, \infty)$.

The slope of the tangent line at $(4, 2)$ is $f'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$.

The Derivative is an Instantaneous Rate of Change

Section 7 was about geometry – the slope of the line tangent to a graph at a specific point. Here we interpret the same mathematics quite differently. Look again at the number

$$\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

for some fixed x value a and some number $h \neq 0$. The numerator measures the amount of change (positive or negative or zero), in the value (the y-coordinate) of the function as you move from $x = a$ to $x = (a+h)$. The denominator is the number $(a+h)-a$ and so measures the change in the x-coordinate as you move from a to $a+h$ (a positive change if $h > 0$, negative if $h < 0$). The above quotient comes from the specific function points $(a, f(a))$ and $(a+h, f(a+h))$. It does not take into account how f behaves at function points between a and $a+h$. We say that $\dfrac{f(a+h)-f(a)}{h}$ is the *average rate of change* of f between $x = a$ and $x = (a+h)$.

Now consider

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

It is the limit of this average rate of change as $(a+h)$ gets closer and closer to a. We say that this number $f'(a)$ is the *instantaneous rate of change of f at a*. This is an important idea because we are often as interested in the rate at which a function is changing (say, cost or revenue or profit) as we are in the function itself.

The difference between average rate of change and instantaneous rate of change can be thought of this way: Suppose a train is traveling on a track, in one direction. At 3:00 p.m. the train is 10 miles from the station. At 6:00 p.m. the train is 100 miles from the station. The number $\dfrac{100-10}{6-3} = 30$ tells us that the *average* speed (rate of change of distance compared to time) is 30 mph. This does not tell us anything about the speed of the train at any specific point during those three hours.

In contrast, the *instantaneous rate of change at a point* would give us the speed on the speedometer at a specific instant in time during the three hour trip. That is the value that we would get from the derivative.

YOU NEED TO REMEMBER THAT THE DERIVATIVE MEASURES THE INSTANTANEOUS RATE OF CHANGE. THIS IS A KEY CONCEPT OF CALCULUS.

Recall (from Section 7) that the derivative measures the slope of the tangent line to the graph of f at the point $(a, f(a))$. A large positive derivative suggests a steeply climbing graph, i.e. a fast positive rate of change. A slightly negative derivative suggests a gently falling graph, i.e. a slow negative rate of change[23]. This will be made precise in Section 17.

As x varies, the values of $f'(x)$ tell the whole "rate of change" story of the function f.

Example 8.3. Consider the graph of $f(x) = \dfrac{1}{x}$ on the interval $\frac{1}{3} < x < 3$.

The *average* rate of change of f over this interval is: $\dfrac{f(\frac{1}{3}) - f(3)}{\frac{1}{3} - 3} = \dfrac{3 - \frac{1}{3}}{-\frac{8}{3}} = \dfrac{\frac{8}{3}}{-\frac{8}{3}} = -1$

If you look at the graph of f on page 20 you can see that -1 is a reasonable value for the slope of the line that would go through the points $(\frac{1}{3}, 3)$ and $(3, \frac{1}{3})$. This tells you nothing about the behavior of the graph between these two points.

The *instantaneous* rate of change, the derivative, tells you how the graph is changing at any point in the interval $\frac{1}{3} < x < 3$. We know from Exercise 3 in Section 7 that $f'(x) = -\dfrac{1}{x^2}$. We see that $f'(\frac{1}{2}) = -4$, $f'(\frac{3}{4}) = -\frac{16}{9}$, $f'(1) = -1$, $f'(\frac{3}{2}) = -\frac{4}{9}$, and $f'(2) = -\frac{1}{4}$. The graph is consistent with these derivative values and the idea of a sharply falling graph becoming a more gently falling graph as we increase in x value.

Example 8.4. Rats are infesting City Hall. The Zap-a-Rat company analyzes the situation and claims that they can rid the building of rats within 30 hours. The company shows the mayor the following function: $R(t) = -t^2 + 20t + 290$ where R is the number of rats remaining t hours after the extermination begins. The mayor is impressed by the equation and hires the company. Assuming that Zap-a-Rat's analysis and equation are correct,...

How many rats are currently in City Hall?

Answer: $R(0) = 290$ rats

What is the average rate of change in the quatity of rats from the end of the 5^{th} hour to the end of the 20^{th} hour of the treatment?

Answer: $\dfrac{\Delta R}{\Delta t} = \dfrac{R(20) - R(5)}{20 - 5} = \dfrac{290 - 365}{15} = \dfrac{-75}{15} = -5$ rats/hour.

At what rate is the rat population declining at the end of the 25^{th} hour?

Answer: $R'(t) = -2t + 20$ (verification of this is left as an exercise) $R'(25) = -2(25) + 20 = -30$ rats/hour.

During the extermination process, was the number of rats ever increasing?

[23]Reminder: we always read from left to right and that's how words like "climbing" and "falling" should be understood.

Answer: Yes. $R'(t) > 0$ during the first 10 hours. If the change in the number of rats at any time is positive, it means that the number of rats is increasing at that time.

Will the rats in fact be gone in 30 hours?

Answer: Yes. $R(30) = -(30)^2 + 20(3) + 290 = -10$. In fact, the number of rats is zero when $R(t) = 0$, which is when $t = 10 + \frac{1}{2}\sqrt{1560} \approx 29.75$ hours. Yay, Zap-a-Rat!

Example 8.5. The cost, in dollars, to produce a product is given as a function of the quantity, q of the product produced: $C(q) = 50,000 + 5q + .01q^2$.

What is the average change in cost if the quantity of product is increased from 100 items to 200 items?

Answer: $\dfrac{\Delta C}{\Delta q} = \dfrac{C(200) - C(100)}{200 - 100} = \dfrac{51,400 - 50,600}{200 - 100} = \dfrac{800}{100} = 8$ dollars/unit.

At what rate is the cost increasing when 120 units are being produced?

Answer: $C'(q) = 5 + .02q$ (again, an exercise). $C'(120) = 5 + .02(120) = \$7.40$ per unit.

In Section 3 we introduced the term *marginal*. It referred to the slope of a linear function. We expand this concept to include the instantaneous rate of change of a function. In Example 8.5 the *marginal cost* is the function $C'(q) = 5 + .02q$. This is consistent also with the interpretation of derivative as slope, as done in Section 7.

Rectilinear (Straight-Line) Motion

The derivative is used in physics for an object that moves in a straight line. Conventionally we think of the path of the object as a horizontal line for side-to-side motion (such as a running person) or a vertical line for up and down motion (such as a rocket shooting skyward and/or falling back to Earth).

The function $s(t)$ gives the position of the object, relative to a fixed point, at time t. We could model our train illustration above: the track is a horizontal line calibrated so that one unit is one mile; the fixed reference point is the station; time is measured in hours past noon. We would then have $s(3) = 10$ and $s(6) = 100$. We could further suppose that at 1:00 p.m. the train is *approaching* the station, and is 20 miles from it. This would give us $s(1) = -20$. What would $s(-2) = -50$ mean? [Answer: It means that at 10:00 a.m. the train is approaching the station and is 50 miles away].

We use the term *velocity* to mean the rate of change of position compared to time. The *average velocity* over time period $t_1 \le t \le t_2$ is $\dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$ and the *instantaneous velocity* at time t is given by the derivative function $s'(t) = v(t)$. Like any average or instantaneous rate of change, velocity can be negative. On a horizontal line, average velocity would be negative if the ending position were to the left of the starting position. On a vertical line, average velocity would be negative if the ending position were lower than the starting position. The instantaneous velocity would be negative if the movement is to the left (or down) and would be positive if the movement is to the right (or up).

While velocity can be negative, *speed* is always positive. Speed is the absolute value of velocity.

Speed $= |v(t)|$. In ordinary life we're more inclined to talk of speed[24] than of velocity ("I drove at 55 mph") but velocity is easier to deal with in math and physics because it isn't an absolute value. Also, it contains directional information which is useful.

Example 8.6. At a carnival shooting range, a target duck moves horizontally for 8 seconds. Its position at time t, measured in centimeters from the center of the target path, is given by the equation $s(t) = t^3 - 3t - 100$.

(a) What is the position of the duck when it begins its motion? What is its position when it ends its motion?
Answers: $s(0) = -100$. The duck is 100 cms. to the left of center.
$s(8) = 388$ The duck is 388 cms. to the right of center.

(b) What is the average velocity of the duck over its entire time of motion?
Answer: $\dfrac{s(8) - s(0)}{8 - 0} = \dfrac{388 - -100}{8} = 61$ cm/sec.

(c) What is the velocity of the duck at $t = 0$? $t = 2$? at $t = 8$?
Answers: $v(t) = s'(t) = 3t^2 - 3$. (This time you can just take my word for it; there are already sufficient exercises for you). $v(0) = -3$. The duck is moving to the left at a speed of 3 cm/sec $v(2) = 3(2)^2 - 3 = 9$ cm/sec. The duck is moving to the right at a speed of 9 cm/sec. $v(8) = 3(8)^2 - 3 = 45$ cm/sec. The duck is moving to the right at the speed of 45 cm/sec.

(d) At what times is the duck moving to the left? right?
Answer: $v(t) = 3t^2 - 3 = 3(t^2 - 1)$ is negative when $t < 1$ and positive when $t > 1$. So, the duck moves to the left for the first second and then moves to the right the rest of the time.

Example 8.7. A ball is shot straight up from the ground with a velocity of 48 ft./sec. Its position above the ground at time t seconds after being launched is given by the equation $s(t) = -16t^2 + 48t$.

(a) When will the ball hit the ground again?
Answer: The ball will be on the ground when $s(t) = 0$. $s(t) = -16t^2 + 48t = -16t(t - 3)$. So, $s(t) = 0$ at $t = 0$ (the initial launch) and $t = 3$ (when it hits the ground again).

(b) What is the average velocity of the ball for the duration of its trip?
Answer: $\dfrac{s(3) - s(0)}{3 - 0} = 0$ ft./sec.

(c) How long was the ball moving upward?
Answer: The ball is moving upward when $v(t) = s'(t) > 0$. $s'(t) = -32t + 48$ (trust me...no exercise). $-32t + 48 = 0$ when $t = \frac{3}{2}$. So the ball rises for 1.5 seconds.

(d) How far does the ball travel all together?
Answer: The distance traveled up is $s(\frac{3}{2}) - s(0) = [-16(\frac{3}{2})^2 + 48(\frac{3}{2})] - 0 = 36$ ft. The distance traveled down is the same as the distance traveled up, so the total distance traveled is 72 ft.

[24] It was correct to use the term "speed" in the original illustration of the train because the train was always traveling in the same positive direction so the velocity was always positive.

Section 8 - Exercises (answers follow)

For all exercises below, compute the derivative of the given function using the method discussed in this section. Do not use short cut formulas you may have learned elsewhere.

1. For each function f, find $f'(x)$

   (a) $f(x) = 4x$

   (b) $f(x) = 6x^2 - 4x$

   (c) $R(t) = -t^2 + 20t + 290$

   (d) $C(q) = 50,000 + 5q + .01q^2$

2. For each function f, find $f'(x)$ and then find $f'(0)$ and $f'(1)$

   (a) $f(x) = x^3 - 2$

   (b) $f(x) = \frac{8}{x}$

   (c) $f(x) = \sqrt{x}$

3. Find the equation of the tangent line to each curve when x has the given value.

   (a) $f(x) = x^2 - 6x^3$; $x = 3$

   (b) $f(x) = 2/x$; $x = 2$

   (c) $f(x) = 11\sqrt{x}$; $x = 5$

4. Suppose the demand (quantity sold) for a certain item is given by $q(p) = -3p^2 + 2p + 1$, where p represents the price of the item in dollars.

   (a) What is the average rate of change in demand when the price is increased from \$7 to \$10?

   (b) Find the rate of change of demand with respect to price.

   (c) Find the rate of change of demand when the price is \$10.

5. An object moves along the x-axis. Its position, in inches relative to the origin, at time t seconds is given by $s(t) = 6t^2 - 4t$. Notice that you have found the derivative for this function in problem 1b above.

   (a) What is the velocity function?

   (b) When is the object moving in the positive direction? negative direction?

   (c) What is the speed of the object at $t = 0$? at $t = 4$?

   (d) What is the total distance traveled (back and forth) by the object between $t = 0$ and $t = 4$?

6. Speed is the absolute value of velocity. Is average speed the absolute value of average velocity? Explain. Hint: Look at Example 8.7.

7. Given $f(x) = x^3 - 5$. Show that the line tangent to the graph of f at the point $(2, 3)$ is parallel to the line tangent to the graph of f at the point $(-2, -13)$.

Section 8 - Answers

1. (a) $f'(x) = 4$
   (b) $f'(x) = 12x - 4$
   (c) $R'(t) = -2t + 20$
   (d) $C'(q) = 5 + .02q$
2. (a) $f'(x) = 3x^2$; $f'(0) = 0$, $f'(1) = 3$
   (b) $f'(x) = \frac{-8}{x^2}$; $f'(0)$ is not defined; $f'(1) = -8$
   (c) $f'(x) = \frac{1}{2\sqrt{x}}$; $f'(0)$ is not defined; $f'(1) = \frac{1}{2}$
3. (a) $y + 153 = -156(x - 3)$
   (b) $y - 1 = -\frac{1}{2}(x - 2)$
   (c) $y - 11\sqrt{5} = \frac{11}{2\sqrt{5}}(x - 5)$
4. (a) -49 items/dollar
   (b) $q'(p) = -6p + 2$
   (c) -58 items/dollar
5. (a) $v(t) = 12t - 4$
   (b) positive direction when $t > \frac{1}{3}$; negative direction when $t < \frac{1}{3}$
   (c) 4 inches/sec   44 inches/sec.
   (d) $81\frac{1}{3}$ inches ($\frac{2}{3}$ inches to the left and then $80\frac{2}{3}$ inches to the right)
6. No. The average speed would only be the absolute value of the average velocity if the velocity was always positive or always negative over the time interval. In the case of Example 8.7, the average velocity is zero. That would only be the average speed if the ball didn't move.
7. Hint: Show that the tangent lines both have the same slope.

9 Continuity

Definition 9.1. *Let $f(x)$ be a function and let a be in the domain of f. We say f is continuous at a if $\lim_{x \to a} f(x) = f(a)$.*

In words the definition says that f is continuous at a if the y values of f are arbitrarily close to $f(a)$ when x gets close to a. Notice that continuity at a point with x value a requires that a be in the domain of f. Without an $f(a)$, this definition makes no sense.

An equivalent way to write definition 9.1 is: *Let $f(x)$ be a function and let a be in the domain of f. We say f is continuous at a if $\lim_{h \to 0} f(a+h) = f(a)$.* That these two expressions are the same can be seen by simply substituting $(a+h)$ for the x in the first statement of the the definition to get this equivalent statement of the definition.

We say *f is continuous* if it is continuous at every number a in its domain. If you think about this for awhile, you can see the direct implication of a function being continuous on its domain: over any open subset (c, d) of the domain, the graph of the function can be sketched without lifting the pencil from the paper. For example, if the domain of f is (or contains) the interval $(-3, 4)$ then continuity means that the graph of f can be drawn from left to right over the interval $(-3, 4)$ without lifting the pencil. This is more easily seen intuitively when using the first statement of the continuity definition.

The second statement of the continuity definition is useful in enabling us to see a relationship between continutiy at a point and differentiability at that point. This is discussed at the end of this section.

Since Definition 9.1 involves a limit as x approaches real number a, we are assuming that x can approach a from both the left $(x < a)$ and the right $(x > a)$. We can use the corresponding appropriate one-sided limit definition to define one-sided continuity. We say that f is *left-continuous at a* if $\lim_{x \to a^+} f(x) = f(a)$. We say that f is *right-continuous at a* if $\lim_{x \to a^-} f(x) = f(a)$. This enables us to discuss the continuity of a function over a closed interval. For example if f is continuous over closed interval $[-3, 4]$ we want to be able to sketch the graph of f from the point $(-3, f(-3))$ to the point $(4, f(4))$ without lifting our pencil from the paper. This can be done if we require: $\lim_{x \to -3^+} f(x) = f(-3)$ and $\lim_{x \to a} f(x) = f(a)$ for all a in $(-3, 4)$ and $\lim_{x \to 4^-} f(x) = f(4)$. Again, we don't wish to get too bogged down with continuity minutia, but we do want a way to make sure that our nice connected "interior" graph is connected to its endpoints. Do you see why we need left continuity for the right endpoint, and right continuity for the left endpoint?

Some Continuous Functions[25]

Polynomials are continuous on their domain, $\mathbb{R}$.

For $P(x) = c_n x^n + c_{n-1}x^{n-1} + \ldots + c_2x^2 + c_1x + c_0$,

$$\begin{aligned}\lim_{x\to a} P(x) &= \lim_{x\to a}[c_n(x)^n + c_{n-1}(x)^{n-1} + \ldots + c_2(x)^2 + c_1(x) + c_0] \\ &= c_n a^n + c_{n-1}a^{n-1} + \ldots + c_2a^2 + c_1a + c_0 \\ &= P(a)\end{aligned}$$

Rational Functions are continuous on their domain, (all values of x for which the denominator is not zero).

For $R(x) = \dfrac{P(x)}{Q(x)}$ with domain $\{x \in \mathbb{R} : Q(x) \neq 0\}$,

$\lim\limits_{x\to a} R(x) = \dfrac{\lim\limits_{x\to a} P(x)}{\lim\limits_{x\to a} Q(x)} = \dfrac{P(a)}{Q(a)} = R(a)$

(The first step is legitimate because a is in the domain of R)

Simple Exponential Functions, c^x ($c > 0$ and $c \neq 1$) are continuous on their domain, $\mathbb{R}$.

For $f(x) = c^x$, $\quad \lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} c^x = c^{\lim\limits_{x\to a}(x)} = c^a = f(a)$

Simple Logarithmic Functions, $\log_c x$ are continuous on their domains, $(0, \infty)$

The graph of $f(x) = \log_c x$ is a reflection over the line $y = x$ of the function $g(x) = c^x$. Since the exponential function is continuous, the logarithmic function must be continuous also.

Simple Root Functions, $\sqrt[n]{x}$ are continuous on their domains.

For n an odd number we give the same graph reflection argument that was used for logarithmic functions.

For n an even number we have a restricted domain, $[0, \infty)$. We can still use the same graph reflection argument. Notice that this continuity involves right-continuity at $x = 0$.

Example 9.1.

1. $f(x) = 4x^3 - 2x + 8$ is continuous on $\mathbb{R}$ because it is a polynomial.
2. $g(x) = \dfrac{x^2 - 9}{x + 3}$ is discontinuous only at $x = -3$ because g is a rational function and $x = -3$ is the only real number not in its domain.
3. $h(x) = \sqrt{x + 4}$ is continuous on $[-4, \infty)$. More specifically, it is right-continuous at $x = -4$ and it has two-sided continuity on $(-4, \infty)$.
4. $j(x) = \log_5 e^x$ is continuous on its domain, $\mathbb{R}$.

[25] For a review of these functions, see Sections 3 and 4.

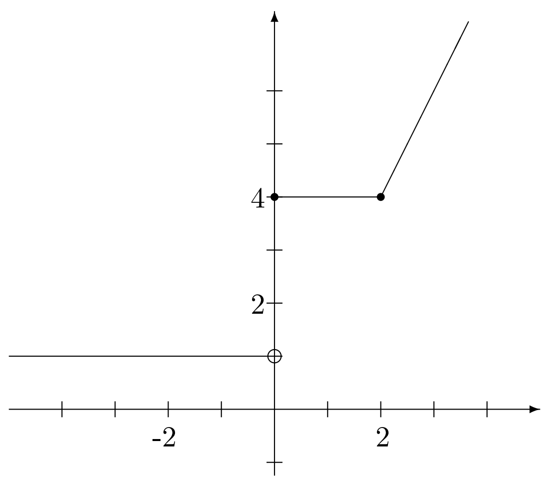
Piecewise Defined Functions

Example 9.2. At which values of x in the domain $\mathbb{R}$ is the function f continuous? Justify your answer. $f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 4 & \text{if } 0 \leq x \leq 2 \\ 2x & \text{if } x > 2 \end{cases}$

Answer: The function f is not continuous at $x = 0$ because $\lim_{x \to 0^-} f(x) = 1$ but $f(0)$=4. However, f is continuous at $x = 2$ because $\lim_{x \to 2^-} f(x) = 4$ AND $\lim_{x \to 2^+} f(x) = 2(2) = 4$ and $f(2) = 4$. Finally, f is continuous on $(-\infty, 0)$ and on $(0, 2)$ and on $(2, \infty)$ because on each of these intervals f is a polynomial.

When dealing with a piecewise defined function, you must check for continuity at each of the domain split points (in this case $x = 0$ and $x = 2$) because the function could behave differently on either side of each of these split points. This requires checking the left-hand limit, right-hand limit and function value at each of these points. If at a split point these three entites are equal, then there is continutiy at that point. If any one of the three is different, then there is discontinuity at that point.

Consider this from a graphing perspective. Piecewise defined functions have their domain divided into disjoint sets (pieces). When sketching a graph of a piecewise defined function, one sketches the graph of each piece over its respective domain piece. The function f from Example 9.2 is graphed below. Compare the graph with the limits found in the example.



Example 9.2

Note that while a continuous function has the geometric property that its graph can be sketched without lifting pencil from paper, this itself is not the definition of continuity. The definition is specific (see Definition 9.1). To claim continuity (or discontinuity) at a point, one must use the limit definition.

Example 9.3. Find all x in $\mathbb{R}$ where f is discontinuous. $f(x) = \begin{cases} \frac{1}{x-6} & \text{if } x \leq -2 \\ 2x+1 & \text{if } -2 < x < 0 \\ e^x & \text{if } x > 0 \end{cases}$

Answer: f is discontinuous at $x = -2$ because $\lim\limits_{x \to -2^+} f(x) = 2(-2) + 1 = -3$ is not equal to $f(-2) = \frac{1}{-8}$.
f is discontinuous at $x = 0$ because $f(0)$ does not exist (i.e., 0 is not in the domain of f).

Why do we care about continuity?

We have said that graphing functions is not just a matter of plotting some points and then playing dot-to-dot. However, with continuity, we can justify the connectedness of the graphs of these functions. The graphs of polynomials and simple exponential, logarithmic and root functions can in fact be sketched with one (continuous) curve. The graphs of rational functions must be disconnected at any values of x that are not in the domain of the function, but those are the *only* places where the graph is disconnected. For example $f(x) = \dfrac{x}{x-5}$ will be one continuous curve on the domain interval $(-\infty, 5)$ and another continuous curve on domain interval $(5, \infty)$. The only break is at $x = 5$, which is not in the domain of f. Continuity, or more accurately, *dis*continuity is useful for showing us where a function is *not* differentiable.

Theorem 9.1. *If $f'(a)$ exists and $f(x)$ makes sense for all x near a (including a itself) then f is continuous at a.*

Corollary to Theorem 9.1 *If f is not continuous at a then $f'(a)$ does not exist.*

Proof of the Theorem. $f(a+h) - f(a) = \dfrac{f(a+h) - f(a)}{h} \cdot h$ when $h \neq 0$. Take limits as $h \to 0$. The limit of a product is the product of the limits as long as the separate limits exist. But $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists. So $\lim\limits_{h \to 0}(f(a+h) - f(a)) = f'(a) \cdot 0 = 0$. Since $\lim\limits_{h \to 0}(f(a+h) - f(a)) = 0$ and $\lim\limits_{h \to 0}(f(a+h) - f(a)) = \left(\lim\limits_{h \to 0} f(a+h)\right) - f(a)$, we get: $\left(\lim\limits_{h \to 0} f(a+h)\right) - f(a) = 0$, or $\lim\limits_{h \to 0} f(a+h) = f(a)$. □

Proof of the Corollary. We are given that f is not continuous at a, so if $f'(a)$ exists the Theorem is contradicted. Thus $f'(a)$ cannot exist. □

Theorem 9.1 tells us that differentiability at a point guarantees continuity at that point. The converse is NOT true. A function may be continuous at a point, but not differentiable at that point. The classic example of this is the function $f(x) = |x|$. This function is continuous at $x = 0$, but is not differentiable there. The proof is left as an exercise.

Section 9 - Exercises (answers follow)

1. List all the numbers x for which the given function is *not* continuous.

   (a) $f(x) = x^4 - x^2$

   (b) $f(x) = \frac{x}{2x^2+1}$

   (c) $f(x) = \frac{x^2-4}{x-2}$

   (d) $f(x) = \dfrac{x^2 - 4x + 4}{x^2 + x - 6}$

   (e) $f(x) = 2^x$

   (f) $f(x) = \ln |x|$

2. Find all values of x where the function is not continuous. Justify your claims of discontinuity using the definition.

   (a) $f(x) = \begin{cases} x & \text{if } x \leq 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$

   (b) $f(x) = \begin{cases} 6 & \text{if } x < -1 \\ x^3 + 2 & \text{if } -1 \leq x \leq 3 \\ 8 & \text{if } x > 3 \end{cases}$

   (c) $f(x) = \begin{cases} 4 & \text{if } x < 0 \\ x & \text{if } 0 \leq x \leq 2 \\ x^2 - 2 & \text{if } x > 2 \end{cases}$

   (d) $f(x) = \begin{cases} -1 & \text{if } x \text{ is an integer} \\ 1 & \text{if } x \text{ is not an integer} \end{cases}$

   (e) $f(x) = \begin{cases} -1 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$

3. Find the values of the constant c so that the function f will be continuous for all x in $\mathbb{R}$.

   $f(x) = \begin{cases} cx - 3 & \text{if } x < 2 \\ 3 + x + x^2 & \text{if } x \geq 2 \end{cases}$

4. Find the values of constants a and b so that the function f will be continuous for all x in $\mathbb{R}$.

   $f(x) = \begin{cases} 3x & \text{if } x \leq 2 \\ ax + b & \text{if } 2 < x < 5 \\ -6x & \text{if } x \geq 5 \end{cases}$

5. A company charges \$1.20 per pound for a certain product on all orders not over 100 lb, and \$1 per pound for orders over 100 lb. Let $f(x)$ represent the cost for buying x lb of the product. Write a piecewise defined function to describe this situation. Find the cost of buying the following:

   (a) 60 lb. 200 lb. 100 lb.

   (b) Where is $f(x)$ not continuous?

6. The cost function for a certain commodity is defined by

$$C(x) = \begin{cases} 5x & \text{if } 0 < x < 10 \\ 4x & \text{if } 10 \le x < 30 \\ 3.5x & \text{if } 30 \le x < 60 \\ 3.3x & \text{if } x \ge 60 \end{cases}$$

   where x is the number of pounds sold and $C(x)$ is in dollars. Sketch the graph of the function C and determine the values of x for which the function C is not continuous.

7. Look at the income tax function (Example 2.2, Part 2 on page 9). At which values of I is this function discontinuous? Does this seem reasonable to you? Why or why not?

8. In several places in the text we claimed that if a function f is continuous and there is a function g whose graph is a reflection of the graph of f over the line $y = x$, then g must also be continuous. From an intuitive perspective, why is this statement reasonable?

9. Show that $f(x) = |x|$ is continuous at $x = 0$ but that $f'(0)$ does not exist.

Section 9 - Answers

1. (a) None (b) None (c) $x = 2$ (d) $x = -3$ and $x = 2$ (e) None (f) $x = 0$.

2. (a) Continuous everywhere

   (b) Discontinuous at $x = -1$ $\lim_{x\to -1^-} f(x) = 6$ $\lim_{x\to -1^+} f(x) = 1$ and
   Discontinuous at $x = 3$ $\lim_{x\to 3^-} f(x) = 29$ $\lim_{x\to 3^+} f(x) = 8$

   (c) Discontinuous at $x = 0$ $\lim_{x\to 0^-} f(x) = 4$ $\lim_{x\to 0^+} f(x) = 0$

   (d) Discontinuous at every integer. For any integer z, $f(z) = -1$ which is not equal to $\lim_{x\to z} f(x) = 1$

   (e) Discontinuous at all real numbers. For every real number, s, $\lim_{x\to s} f(x)$ does not exist.

3. $c = 6$

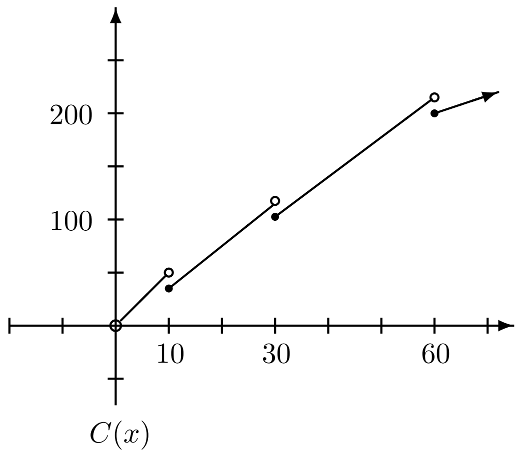
4. $a = -12$, $b = 30$

5. $f(x) = \begin{cases} 1.20x & \text{if } 0 \le x \le 100 \\ 1.00x & \text{if } x > 100 \end{cases}$

(a) \$72 \$200 \$120

(b) Discontinuous at $x = 100$

6. Discontinuous at $x = 10$, $x = 30$ and $x = 60$



7. Continuous everywhere. It would not be reasonable to have "jumps" in the tax owed for very small changes in income.

8. If you can draw f without lifting your pencil from the paper, then you should be able to draw g without lifting your pencil.

9. Hint: Rewrite $|x|$ as $\sqrt{x^2}$.

10 Calculating the Derivative I; Product and Quotient Rules

In using calculus you must be able to find $f'(x)$ for all sorts of given functions $f(x)$. When $f(x)$ is complicated it could be a chore to work out what $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ looks like. Fortunately that's rarely necessary. Complicated functions can usually be broken down into simple parts. When you know the derivatives of the simple parts, general rules will tell you how to combine these derivatives in order to find the derivative of the complicated function. The derivative rules are not "magic." They follow from the limit definition of derivative. Some proofs of these rules are shown and some are included for you to do in the exercises. Some rules are not proved in this text; you can take them on faith, or look them up elsewhere if you are curious.

Recall the definition of derivative function:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Rule 1. Let $f(x)$ be a *constant function* i.e., $f(x) = c$ for some fixed number c. Then $f'(x) = 0$.

Proof. $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{c-c}{h} = \lim_{h\to 0} 0 = 0$ □

Rule 2. Let $f(x) = x$, the *identity function.* Then $f'(x) = 1$.

The proof is left as an exercise.

Rule 3. If c is a (constant) number, $(cf)' = c(f')$.

Proof. $(cf)'(x) = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = \lim_{h\to 0}\frac{c\left[f(x+h)-f(x)\right]}{h} = c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
$= cf'(x)$. □

If we combine rules 2 and 3 we can get the derivative of $g(x) = 5x$: $g'(x) = 5\cdot 1 = 5$

Rule 4. $(f+g)' = f' + g'$. That is, *the derivative of the sum of two functions is the sum of their derivatives.*

The proof is left as an exercise.

By combining rules 1, 2, 3 and 4 we get the derivative of $h(x) = 7x+3$: $h'(x) = 7\cdot 1 + 0 = 7$.

It is not true that the derivative of a product of two functions is the product of their derivatives. The rule is more strange AND MUST BE MEMORIZED:

Rule 5. (Product Rule): $(fg)' = f'g + g'f$.

Proof.

$$\begin{aligned}
(fg)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} \cdot g(x) + \frac{g(x+h) - g(x)}{h} \cdot f(x+h) \right] \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h\to 0} g(x) + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \cdot \lim_{h\to 0} f(x+h) \\
&= f'(x)g(x) + g'(x)f(x)
\end{aligned}$$

□

Rule 6. (PowerRule): If $f(x) = x^n$, where n is any real number, then $f'(x) = nx^{n-1}$.

Notice that for $n = 1$, this rule is the same as Rule 2. The proofs for $n = 2$ and $n = 3$ are left as exercises. (Can you see a pattern when n is a positive integer?) The complete proof is postponed until Section 11 as it relies on the Chain Rule.

Example 10.1. Find the derivatives for the following functions.

(a) $f(x) = x^5$ Answer: $f'(x) = 5x^4$

(b) $g(x) = \sqrt[3]{x}$ Answer: Since $\sqrt[3]{x} = x^{\frac{1}{3}}$, we get $g'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

(c) $h(x) = \frac{1}{x^5}$ Answer: Since $\frac{1}{x^5} = x^{-5}$ we get $h'(x) = -5x^{-6}$

(d) $j(x) = 3x^7$ Answer: $j'(x) = 3 \cdot 7x^6 = 21x^6$

By these six rules we can differentiate any polynomial. If $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ then $f'(x) = na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_2x + a_1$.

Example 10.2. Find the derivatives for the following functions.

(a) $f(x) = x^3 - 4x^2 + x - 9$ Answer: $f'(x) = 3x^2 - 4 \cdot 2x + 1 = 3x^2 - 8x + 1$

(b) $g(x) = -5x^{100} + \frac{1}{2}x^{34} + 2x$ Answer: $-500x^{99} + 17x^{33} + 2$

Example 10.3. Use the Product Rule to find derivatives for the following functions. Do not simplify.

(a) $f(x) = (x^2 + 1)(3x^5 - 10x + 2)$
Answer: $f'(x) = (x^2 + 1)'(3x^5 - 10x + 2) + (3x^5 - 10x + 2)'(x^2 + 1)$
$= 2x(3x^5 - 10x + 2) + (15x^4 - 10)(x^2 + 1)$

(b) $g(x) = \left(\frac{1}{x^2} + 3\right)\left(\frac{2}{x^3} + x\right)$

Answer: Rewrite $g(x)$ as: $g(x) = \left(x^{-2} + 3\right)\left(2x^{-3} + x\right)$.

Then, $g'(x) = (-2x^{-3})(2x^{-3} + x) + (-6x^{-4} + 1)(x^{-2} + 3)$

How do we differentiate a rational function? More generally, how do we differentiate $f(x)/g(x)$ if we know the derivatives of $f(x)$ and $g(x)$?

Rule 7. (Quotient Rule): $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$. The proof is a bit complicated for this course. Like the Product Rule, (Rule 5) the Quotient Rule MUST BE MEMORIZED.

Example 10.4. Find the derivative of $f(x) = \frac{3x^2 + 2x + 1}{x^3 - 7}$.

$$\begin{aligned} f'(x) &= \frac{(3x^2 + 2x + 1)'(x^3 - 7) - (x^3 - 7)'(3x^2 + 2x + 1)}{(x^3 - 7)^2} \\ &= \frac{(6x + 2)(x^3 - 7) - (3x^2)(3x^2 + 2x + 1)}{(x^3 - 7)^2} \\ &= \frac{-3x^4 - 4x^3 - 3x^2 - 42x - 14}{(x^3 - 7)^2} \end{aligned}$$

Any rational function can be differentiated this way. When you become comfortable with this rule, you can leave out the first step.

In summary, take a look at our seven rules. The first four are pretty easy to remember. Everyone who has had even a little calculus memorizes Rule 6. The ones you must carefully commit to memory are the Product Rule and the Quotient Rule. You should know them so well that you can use them when you need them without hesitation. In particular note where the minus sign is in the Quotient Rule!

A special case of the Quotient Rule is the Reciprocal Rule. The proof is easy and is left as an exercise.

Rule 7a (Reciprocal Rule): $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$.

This rule is sometimes a convenient short-cut for finding derivatives of functions that are in the form $\frac{1}{g(x)}$. It isn't necessary that you memorize this formula, though, because it is really just an application of the Quotient Rule. In Section 11 we will see another short-cut way to find derivatives of functions of this form.

Example 10.5. Find the derivative of $f(x) = \frac{1}{3x^5 + 3x - 8}$.

Answer: $f'(x) = \frac{(3x^5 + 3x - 8)'}{(3x^5 + 3x - 8)^2} = \frac{15x^4 + 3}{(3x^5 + 3x + 8)^2}$

Two hints:

To find the derivative of a function like $f(x) = \dfrac{3}{x^2}$, you *could* use the quotient rule, but it is more efficient to think of f as $f(x) = 3x^{-2}$ and use the power rule. $f'(x) = -6x^{-3}$.

To find the derivative of a function like $g(x) = \dfrac{x^3}{7}$, you *could* use the quotient rule, but it is more efficient to think of g as $g(x) = \frac{1}{7}x^3$ and use the power rule. $g(x) = \frac{3}{7}x^2$.

Exponential Functions

Be very careful with the power rule. The rule applies to functions of the form $g(x) = x^n$. The variable is the base and the exponent is a constant. This is not the same as $f(x) = n^x$ where the variable is the exponent and the base is a constant. The derivative of f is NOT xn^{x-1}.

So, how do we handle exponential functions, those of the form $f(x) = a^x$? We will look at one exponential function now and do the others in Section 11.

The exponential function that rates our immediate attention is $f(x) = e^x$. This function is unique in that it is the only function which is its own derivative. That is, $f'(x) = e^x$! If you very carefully sketch a graph of $f(x) = e^x$ you can observe that for any point (x, y) on the curve, the y coordinate is the same as the slope of the line tangent to the curve at that point. The slope of the line tangent to f at the point $(0, 1)$ is 1. The slope of the line tangent to f at the point $(1, e)$ is e, etc. While other exponential functions with bases $a > 1$ have the same increasing, swooping shape as $f(x) = e^x$, they do not do so in a way that the steepness of the curve at each point corresponds exactly to the y-value.

Rule 8. (Derivative of e^x): If $f(x) = e^x$, then $f'(x) = e^x$.

The proof of this is beyond the scope of this course, but an explanation of the proof is given in Section 20. By that time we will have studied some of the terminology and concepts to make the outline of the proof understandable. But, for now, understand that it is true that $\lim\limits_{h\to 0}\left(\dfrac{e^{x+h} - e^x}{h}\right) = e^x$.

Example 10.6. Use the various rules learned in this section to find the derivatives.

$$
\begin{array}{ll}
f(x) = e^x + x^e & f'(x) = e^x + ex^{e-1} \\
f(x) = 3e^x - 5x - 7 & f'(x) = 3e^x - 5 \\
f(x) = e^x(4x^2 + x + e) & f'(x) = e^x(4x^2 + x + e) + (8x + 1)e^x \\
f(x) = \dfrac{4x + e^x}{7x - e^x} & f'(x) = \dfrac{(4 + e^x)(7x - e^x) - (7 - e^x)(4x + e^x)}{(7x - e^x)^2}
\end{array}
$$

Derivatives in Economics

In Section 3 we introduced the term "marginal." It was defined for linear functions to be the slope. The slope of a line gives us the change in the y value for each unit change in x. The slope of a line is constant.

Now that we are dealing with non-linear functions, we do not have a constant slope. The slope can vary from point to point. We use the derivative function to give us the slope of the function at any given point. This tells us the instantaneous change in the y value for each unit change in x.

So, how should we define “marginal?” It makes sense to use the derivative. The *marginal cost* is the derivative of the cost function, the *marginal profit* is the derivative of the profit function, etc. “marginal” means “rate of change of” or “derivative of”.

Example 10.7. The cost of manufacturing x items is $C(x) = 6 + 3x^2 + 9x^3$ dollars. The marginal cost is thus $C'(x) = 6x + 27x^2$ dollars per item.

Notice that using this definition (derivative) for “marginal,” we are consistent with what was said in Section 3. For a linear function, $f(x) = mx + b$, the slope and the derivative are the same (m).

Smoothing functions whose domains are discrete[26].

Think about the last example. In real life x would probably have to be a non-negative integer (since you don't usually manufacture $\frac{3}{2}$ items or $\sqrt{2}$ items). Yet taking the derivative $C'(x)$ suggests that $\lim_{h\to 0} \dfrac{C(x+h) - C(x)}{h}$ makes sense. What we are really doing here is to “smooth out” the function $C(n) = 6 + 3n^2 + 9n^3$ (n an integer ≥ 0). The function $C(x) = 6 + 3x^2 + 9x^3$ agrees with this when $x = n$, and is differentiable. This “smoothing out” is done all the time in economics, usually without comment. We will do it too.

[26] A set of isolated points in the line, such as the set of all rational numbers or the set of all non-negative integers is said to be *discrete*.

Section 10 - Exercises (answers follow)

1. Use the definition of derivative and the limit rules to prove: (a) Rule 2 (b) Rule 4

2. (a) Use the Product Rule (Rule 5) to prove Rule 6 for $n = 2$. Hint, note that $x^2 = x \cdot x$.

   (b) Use the Product Rule to prove Rule 6 for $n = 3$.

   (c) Use the Quotient Rule (Rule 7) to prove the Reciprocal Rule, Rule 7a.

3. Given $C(x) = 6x^2$ and $R(x) = 8x$. Find the marginal cost, marginal revenue, and marginal profit functions, and then find all the values of x for which the marginal profit is zero.

4. The demand function for a product is given by $p(q) = 0.1q^3 - 0.4q + 60$

   where q is the quantity sold (measured in thousands) and p is the unit price in dollars.

   (a) Find $p'(q)$.

   (b) What is the rate of change of the unit price when the quantity sold is 10,000 units ($q = 10$)? What is the unit price at that level of demand?

5. Assume that a demand equation is given by $q = 400 - 100p$. Find the marginal revenue for the following production levels (values of q). (*Hint*: Solve the demand equation for p.)
   (a) 60 units (b) 120 units (c) 400 units

6. Differentiate the given functions.
   (a) $f(x) = -1$ (b) $y = 6x^3 - 3x^2 + 2x + 5$ (c) $f(x) = 2x^{-6}$
   (d) $y = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{3x^3}$ (e) $f(r) = \frac{4}{3}\pi r^3$

7. Suppose $f(x) = 3x^4 + 2x$ and $g(x) = (5x - 1)$. Show that $(fg)' \neq f'g'$ by:

   (a) Find fg by multiplying the two polynomials.

   (b) Find $(fg)'$, the derivative of the result in part (a).

   (c) Find f' and g'.

   (d) Multiply f' by g'. Simplify.

   (e) Show that your answer to (b) does not equal the answer to (d).

8. Differentiate the given functions using the Product Rule:
   (a) $y = (x^2 + 1)(2x - 1)$ (b) $f(x) = x^4(x^3 - 2)$ (c) $f(x) = xe^x$

9. Differentiate the given functions using the Quotient Rule:
   (a) $y = \dfrac{3x^2 + 1}{x - 1}$ (b) $y = \dfrac{x + 14}{x - 1}$ (c) $g(x) = \dfrac{x^2 - 4x + 1}{x + 2}$ (d) $f(x) = \dfrac{e^x}{x}$

10. Find the equation of the line tangent to the graph of $f(x) = \dfrac{x^2 + 1}{x}$ at the point with x-coordinate $x = -1$.

11. For each of the following functions, find $f'(2)$: (a) $f(x) = \frac{1}{3}x^3 - 1$ (b) $f(x) = \dfrac{x}{x^2 + 2}$

12. Suppose f and g are both differentiable at $x = 3$. Also, suppose we know that $g(3) = 4$, $g'(3) = 5$, $f(3) = 7$, and $f'(3) = 6$. Find $h'(3)$ when $h(x) = f(x)g(x)$.

13. Suppose f and g are both differentiable at $x = 1$. Further, suppose we know that $f(1) = 1$, $f'(1) = 2$, $g(1) = \frac{1}{2}$, $g'(1) = -3$. Find:
(a) $(f+g)'(1)$ (b) $(f-g)'(1)$ (c) $(2f+3g)'(1)$ (d) $(fg)'(1)$ (e) $\left(\frac{f}{g}\right)'(1)$ (f) $\left(\frac{g}{f}\right)'(1)$

14. Suppose that f, g, h are all differentiable functions and $F(x) = f(x) \cdot g(x) \cdot h(x)$.
Show that $F'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$.

15. The demand function for a product is given by $p(q) = \dfrac{50}{0.01q^2 + 1}$ $(0 \le q \le 20)$ where q (measured in units of a thousand) is the quantity demanded per week and $p(q)$ is the unit price in dollars.

    (a) Find $p'(q)$.

    (b) Find $p'(5)$, $p'(10)$, and $p'(15)$.

16. More practice. Find the derivative for each function. Do not simplify.
(a) $f(x) = x^5 - 3x^3 + 1$ (b) $f(x) = \dfrac{x^{10}}{2} + \dfrac{x^5}{5} + 6$ (c) $f(x) = \dfrac{3}{x^2} - \dfrac{4}{x}$

(d) $f(x) = 3x^{-2} - 7x^{-1} + 6$ (e) $f(x) = x^2(3x^3 - 1)$ (f) $f(x) = (x^2 + 3x + 7)(2x - 9)$

(g) $f(x) = \dfrac{2x + 7}{3x - 1}$ (h) $f(x) = \dfrac{3x^2 + 7}{x^2 - 1}$ (i) $f(x) = \left(\dfrac{3x + 1}{x + 2}\right)(x + 7)$

(j) $f(x) = \dfrac{1}{x^2 + 3x - 7}$ (k) $f(x) = \dfrac{(x^2 + 3)(3x^3 + 2x)}{x^4 - 1}$ (l) $\left(\dfrac{x^2 + 2}{3x^3 - 9}\right)\left(\dfrac{2x^2 + 1}{x^4 + x}\right)$

(m) $f(x) = e^x x^2$ (n) $f(x) = \dfrac{e^x + 1}{e^x}$ (o) $f(x) = e^x + x^e + x + e$

Section 10 - Answers

1. (a) $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h) - x}{h} = \lim\limits_{h\to 0} \dfrac{h}{h} = \lim\limits_{h\to 0} 1 = 1$

   (b) $$(f+g)'(x) = \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} = \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$$
   $$= \lim_{h\to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} = \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right]$$
   $$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = f'(x) + g'(x)$$

2. (a) $(x^2)' = (x \cdot x)' = (x')x + (x')x = 1 \cdot x + 1 \cdot x = 2x$

   (b) $(x^3)' = (x \cdot x^2)' = (x^2)'x + x'(x^2) = 2x \cdot x + 1 \cdot x^2 = 2x^2 + x^2 = 3x^2$

   (c) $\left(\dfrac{1}{g}\right)' = \dfrac{1' \cdot g - g' \cdot 1}{g^2} = \dfrac{0 \cdot g - g'}{g^2} = \dfrac{g'}{g^2}$.

3. $C'(x) = 12x$ $R'(x) = 8$ $P'(x) = 8 - 12x$ $P'(x) = 0$ when $x = \frac{2}{3}$

4. (a) $p'(q) = .3q^2 - .4$ (b) \$29.60/thousand items \$156/thousand items

5. $R'(q) = -\frac{q}{50} + 4$ (a) 2.8 (b) 1.6 (c) -4

6. (a) $f'(x) = 0$ (b) $y' = 18x^2 - 6x + 2$ (c) $f'(x) = -12x^{-7}$
   (d) $y' = -\frac{1}{x^2} + \frac{2}{x^3} - \frac{1}{x^4}$ (e) $f'(r) = 4\pi r^2$ (surface area of sphere)

7. (a) $fg = 15x^5 - 3x^4 + 10x^2 - 2x$ (b) $(fg)' = 75x^4 - 12x^3 + 20x - 2$
   (c) $f' = 12x^3 + 2$ and $g' = 5$ (d) $f' \cdot g' = 60x^3 + 10$ (e) (b) $\neq$ (d).

8. (a) $y' = 2x(2x-1) + 2(x^2+1)$ (b) $f'(x) = 4x^3(x^3-2) + 3x^2 \cdot x^4$ (c) $f'(x) = e^x + xe^x$

9. $y' = \dfrac{6x(x-1) - (3x^2+1)}{(x-1)^2}$ $y' = \dfrac{(x-1) - (x+14)}{(x-1)^2}$ $g'(x) = \dfrac{(2x-4)(x+2) - (x^2-4x+1)}{(x+2)^2}$
   $f'(x) = \dfrac{e^x x - e^x}{x^2}$

10. $y = -2$

11. (a) 4 (b) $-\frac{1}{18}$

12. 59

13. (a) -1 (b) 5 (c) -5 (d) -2 (e) 16 (f) -4

14. Consider $F(x) = f \cdot g \cdot h$ as $F(x) = f \cdot (g \cdot h)$ and use the Product Rule.

15. (a) $\dfrac{-x}{(.01x^2+1)^2}$ (b) -3.2 -2.5 $-\dfrac{240}{169}$

16. (a) $5x^4 - 9x^2$ (b) $5x^9 + x^4$ (c) $\frac{-6}{x^3} + \frac{4}{x^2}$ (d) $-6x^{-3} + 7x^{-2}$
    (e) $2x(3x^3-1) + 9x^2 \cdot x^2$ (f) $(2x+3)(2x-9) + 2(x^2+3x+7)$ (g) $\dfrac{2(3x-1) - 3(2x+7)}{(3x-1)^2}$
    (h) $\dfrac{6x(x^2-1) - 2x(3x^2+7)}{(x^2-1)^2}$ (i) $\left[\dfrac{3(x+2) - 1(3x-1)}{(x+2)^2}\right](x+7) + 1\left(\dfrac{3x+1}{x+2}\right)$
    (j) $\dfrac{-(2x+3)}{(x^2+3x-7)^2}$ (k) $\dfrac{[2x(3x^3+2x) + (9x^2+2)(x^2+3)](x^4-1) - 4x^3(x^2+3)(3x^3+2x)}{(x^4-1)^2}$
    (l) $\left(\dfrac{2x(3x^3-9) - 9x^2(x^2+2)}{(3x^3-9)^2}\right)\left(\dfrac{2x^2+1}{x^4+x}\right) + \left(\dfrac{4x(x^4+x) - (4x^3+1)(2x^2+1)}{(x^4+x)^2}\right)\left(\dfrac{x^2+2}{3x^3-9}\right)$
    (m) $e^x x^2 + 2xe^x$ (n) $\dfrac{e^x e^x - e^x(e^x+1)}{(e^x)^2}$ (o) $e^x + ex^{e-1} + 1 + 0$

11 Calculating the Derivative II; Chain Rule

Suppose we have the function $F(x) = (x^4 - 3x)^2$ and we want to find $F'(x)$. One way to do this is to first rewrite $F(x)$ by multiplying: $F(x) = x^8 - 6x^5 + 9x^2$. Then finding $F'(x)$ is easy. $F'(x) = 8x^7 - 30x^4 + 18x$.

But suppose we make a "minor" change to the function. Suppose we are instead given $F(x) = (x^4 - 3x)^{200}$ and we want to find $F'(x)$. In this case, multiplying isn't very attractive. Fortunately, there is an alternative.

Notice that $F(x)$ is a composition[27] of the functions $f(x) = x^{200}$ and $g(x) = x^4 - 3x$. That is, $F(x) = f(g(x)) = (f \circ g)(x)$. There is a rule for finding the derivative of a composition of functions. It is called the Chain Rule. The proof of the chain rule is not given here, but can be found in any standard Calculus textbook.

Rule 9. (Chain Rule) $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$.

You must memorize the Chain Rule and learn how to use it.

Example 11.1. Use the Chain Rule to find $F'(x)$ for $F(x) = (x^4 - 3x)^{200}$.

Solution: We identified the composition above. When $f(x) = x^{200}$ and $g(x) = x^4 - 3x$ we have $F(x) = (f \circ g)(x)$. We see that $f'(x) = 200x^{199}$ and $g'(x) = 4x^3 - 3$. According to the Chain Rule, $F'(x) = f'(g(x)) \cdot g'(x)$. So, $F'(x) = 200(g(x))^{199} \cdot g'(x) = 200(x^4 - 3x)^{199}(4x^3 - 3)$.

Notice that the chain rule can be used on our initial problem, $F(x) = (x^4 - 3x)^2$. Here, $F(x) = (f \circ g)(x)$ when $f(x) = x^2$ and $g(x) = x^4 - 3x$. So, $F'(x) = 2(g(x)) \cdot g'(x)$ $= 2(x^4 - 3x)(4x^3 - 3) = 2(4x^7 - 3x^4 - 12x^4 + 9x) = 8x^7 - 30x^4 + 18x$. This is the same answer that we got above.

For the following examples, we will not specifically identify the f and g that make up the composition.

Example 11.2. $F(x) = \sqrt{3x^4 + x - 7}$. Find $F'(x)$. Do not simplify.

Solution: Note that $F(x) = (3x^4 + x - 7)^{\frac{1}{2}}$. So, $F'(x) = \frac{1}{2}(3x^4 + x - 7)^{-\frac{1}{2}}(12x^3 + 1)$.

The derivative of the following function could be found using the quotient rule or the reciprocal rule. It can also be found using the chain rule. You should verify that all three answers are the same.

Example 11.3. $F(x) = \dfrac{1}{5x^3 + x^2}$. Find $F'(x)$.

$F(x) = (5x^3 + x^2)^{-1}$, so $F'(x) = -(5x^3 + x^2)^{-2}(15x^2 + 2x)$.

In the following example, we need to use both a chain rule and a quotient rule to get the derivative.

Example 11.4. $G(x) = \left(\dfrac{2x^2 + 3x}{x - 5}\right)^3$. Find $G'(x)$. Do not simplify.

$$G'(x) = 3\left(\frac{2x^2 + 3x}{x - 5}\right)^2 \left(\frac{(4x + 3)(x - 5) - 1(2x^2 + 3x)}{(x - 5)^2}\right).$$

[27] To review composition of functions, see Section 2.

In the next example our function F is a composition of a composition. So we need to use the chain rule multiple times. Pay careful attention to the placement of parentheses in the derivative.

Example 11.5. $F(x) = [3x + (2x^2 - 7x)^4]^6$. Find $F'(x)$. Do not simplify.
$F'(x) = 6[3x + (2x^2 - 7x)^4]^5[3 + 4(2x^2 - 7x)^3(4x - 7)]$.

We can use the Chain Rule to complete the proof for the Power Rule: If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Proof. $f(x) = x^n = e^{n \ln x}$. By the Chain Rule, we get: $f'(x) = e^{n \ln x} \cdot n \cdot \frac{1}{x} = x^n \cdot n \cdot \frac{1}{x} = nx^{n-1}$. $\square$

The Chain Rule can help us find derivatives for other functions.

Rule 10. (Derivative of general exponential function) If $f(x) = a^x$, then $f'(x) = a^x(\ln a)$.

Proof. $f(x) = a^x = e^{x \ln a}$. By the Chain Rule, we get: $f'(x) = e^{x(\ln a)} \cdot (\ln a) = a^x(\ln a)$. $\square$

Rule 11. (Derivative of the natural logarithm function) If $f(x) = \ln x$ then $f'(x) = \dfrac{1}{x}$.

Proof. We know that $e^{\ln x} = x$. Taking the derivative of both sides, and using the Chain Rule, we get: $e^{\ln x} \cdot (\ln x)' = 1$. Dividing, we get: $(\ln x)' = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$. $\square$

Rule 12. (Derivative of general logarithm function) If $f(x) = \log_a x$, then $f'(x) = \dfrac{1}{x(\ln a)}$.
The proof is left as an exercise.

Example 11.6. Find derivatives for the following functions.

$f(x) = 3^x$ $\qquad f'(x) = 3^x \ln 3$

$g(x) = \log x$ $\qquad g'(x) = \dfrac{1}{x \ln 10}$

$h(x) = e^{x^2}$ $\qquad h'(x) = e^{x^2} 2x$

$j(x) = \ln(3x^2 + x - 5)$ $\qquad j'(x) = \dfrac{1}{3x^2 + x - 5}(6x + 1)$

$k(x) = \sqrt{x^2 + \ln(x^3 - 7x)}$ $\qquad k'(x) = \frac{1}{2}\left[x^2 + \ln(x^3 - 7x)\right]^{-\frac{1}{2}}\left[2x + \dfrac{1}{x^3 - 7x}(3x^2 - 7)\right]$

Example 11.7. Find the equation of the line tangent to the graph of $f(x) = \log_2(x + \frac{1}{x})$ at the point where $x = 1$.

Answer: For the equation of a line, we need a point, and a slope. The point is $(1, f(1)) = (1, 1)$.
The slope is $f'(1)$. $f'(x) = \dfrac{1}{(x + \frac{1}{x}) \cdot \ln 2} \cdot \left(1 - \frac{1}{x^2}\right)$. So, $f'(1) = 0$
The tangent line is the horizontal line, $y = 1$.

We can use the Chain Rule to complete the proof for the Power Rule: If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.

Proof. $f(x) = x^n = e^{n \ln x}$. By the Chain Rule, we get: $f'(x) = e^{n \ln x} \cdot n \cdot \frac{1}{x} = x^n \cdot n \cdot \frac{1}{x} = nx^{n-1}$. $\square$

Section 11 - Exercises (answers follow)

1. Find $f(g(x))$ and $g(f(x))$

   (a) $f(x) = \frac{x}{8} + 12 \qquad g(x) = 3x - 1$

   (b) $f(x) = 2x + 1 \qquad g(x) = -\frac{1}{x}$

2. Find the derivative of the given function. You do not have to simplify.

   (a) $f(t) = (2t + 1)^2$

   (b) $f(x) = (2x + 1)^{-4}$

   (c) $f(x) = \dfrac{1}{(4x^2 + 1)^7}$

   (d) $f(x) = (5 - 2x)^{10}$

   (e) $f(x) = \dfrac{1}{(4x + 1)^5}$

   (f) $f(x) = \left(\sqrt{3}x^2 + x - \sqrt{11}\right)^{-8}$

   (g) $f(x) = \sqrt{x^2 + 2x + 3}$

   (h) $f(x) = \dfrac{1}{\sqrt{x^2 + 1}}$

   (i) $f(x) = (3x^2 + 7)^2(5 - 3x)^3$

   (j) $f(x) = \left(\dfrac{x^2 + x}{1 - 2x}\right)^4$

   (k) $f(x) = \sqrt{x + \sqrt{x}}$

   (l) $f(x) = 6x(5x^4 - 1)^2$

   (m) $f(x) = \left(\dfrac{2x + 4}{3x - 1}\right)^2$

   (n) $f(x) = \dfrac{(x^2 + 2)^3}{(x^2 - 1)^5}$

   (o) $f(x) = \left(\dfrac{9x + 1}{1 - 12x}\right)^{20}$

3. Find the derivative of the following functions. You do not have to simplify.

   (a) $f(x) = e^{x+4}$

   (b) $f(x) = 3e^{4x}$

   (c) $y = -e^{x+1}$

   (d) $f(x) = 30 + 10e^{-0.01x}$

   (e) $f(x) = 5^{-x}$

   (f) $f(x) = xe^x$

   (g) $y = (x - 3)^2 e^{2x}$

(h) $f(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

(i) $f(x) = (2x + e^{-x^2})^2$

(j) $f(x) = \sqrt{2}^x + x^{\sqrt{2}}$

(k) $f(x) = e^{\sqrt{x}}$

(l) $f(x) = \sqrt{1 - 2e^x}$

(m) $f(x) = e^{-\frac{1}{x}}$

(n) $f(x) = \dfrac{e^{3x}}{1 + e^x}$

(o) $f(x) = xe^{-x^2}$

(p) $f(x) = \sqrt[3]{2x + e^{2x}}$

(q) $f(x) = (2x - 5e^x)^3$

(r) $f(x) = 3^{6x^2+2x+1}$

(s) $f(x) = e^{e^{e^x}}$

4. Find the equation of the line tangent to the graph of the function at the point specified.

   (a) $f(x) = \dfrac{1}{\sqrt{2x+7}}$ at the point $(1, \frac{1}{3})$

   (b) $g(t) = t(t^2 - 4t + 5)^4$ where $t = 1$

   (c) $f(x) = 4^x$ where $x = 3$

   (d) $f(x) = \ln(8 - 4x)$ at the point $(1, \ln 4)$.

   (e) $f(x) = x^2 e^{-x}$ where $x = 1$

5. $f(x) = x^x$. Find $f'(x)$. (Hint: Rewrite x^x into an equivalent expression with base e).

6. Each week q items are sold, where $q = -\dfrac{4(p+1)^2}{3} + 80$ and p is the price per item (in dollars). Express weekly revenue as a function of p, and then calculate $R'(4)$.

7. Suppose the cost in dollars of manufacturing q items is given by $C = 200q + 35,000$, and the demand equation is given by $q(p) = 1,500 - 1.5p$. The demand equation gives the number of items in demand as a function of the price p dollars charged per item.

   (a) Find an expression for the revenue $R(p)$;

   (b) Find an expression for the profit $P(p)$;

   (c) Find an expression for the marginal profit;

   (d) Determine the value of the marginal profit when the price is \$500.

8. Prove Rule 12: If $f(x) = \log_a x$, then $f'(x) = \dfrac{1}{x \ln a}$.

9. Given f and g both differentiable at $x = 5$ and $x = 7$ and given $f(5) = -3$, $f'(5) = 10$, $f(7) = 0$, $f'(7) = 20$, $g(5) = 7$, $g'(5) = \frac{1}{4}$, $g(7) = \frac{3}{5}$, $g'(7) = \frac{2}{3}$.

(a) Find $(f \circ g)'(5)$

(b) Can you find $(g \circ f)'(5)$? Explain.

10. Use the fact that $|x| = \sqrt{x^2}$ to show that $|x|' = \frac{x}{|x|}$.

11. Use the information in problem 10 to find the derivative of $f(x) = |x^2 - 3x|$

12. Given $(\sin x)' = \cos x$ and $(\tan x)' = \sec^2 x$, find the derivative for each of the following functions. You do not have to simplify.

    (a) $f(x) = \sin(x^2 + 3x)$

    (b) $g(x) = \sin(\tan x)$

    (c) $h(x) = \sin(\tan(6x))$

    (d) $j(x) = \tan^5 x$

    (e) $k(x) = \tan^5(\sin x)$

13. Find the derivative of the given function.

    (a) $y = \ln(8x)$

    (b) $g(x) = \ln(4x - 1)$

    (c) $y = \ln\sqrt{2x+1}$

    (d) $h(x) = \ln\left(\frac{9x}{4x-2}\right)$

    (e) $f(x) = \ln e^{28x}$

    (f) $f(x) = \dfrac{\ln x}{x+1}$

    (g) $y = \log_8 \sqrt{2x-3}$

    (h) $f(t) = \log_6(t + (1/t))$

    (i) $f(x) = \ln(x^3 - 2x + 3)$

    (j) $f(x) = \dfrac{\ln x}{x}$

    (k) $f(x) = x \ln x - x$

    (l) $f(x) = \ln(\ln x)$

    (m) $f(x) = \log_5 x$

    (n) $f(x) = \log(3^x)$

    (o) $f(x) = e^{(\ln x)^2}$

14. Prove that $\frac{d}{dx} \ln|x| = \frac{1}{x}$.

15. Assume that the total revenue received from the sale of x items is given by $R(x) = 15\ln(4x+1)$, and the cost function is given by $C(x) = 3x$. Find the marginal profit function.

16. Suppose the price per item for x units of a certain item is $p = 100 + \frac{50}{\ln x}, \quad x > 1$, where p is in dollars. Find the marginal revenue.

Section 11 - Answers

1. (a) $f(g(x)) = \dfrac{3x-1}{8} + 12 \qquad g(f(x)) = 3(\frac{x}{8} + 12) - 1$

   (b) $f(g(x)) = -\frac{2}{x} + 1 \qquad g(f(x)) = \dfrac{-1}{2x+1}$

2. (a) $2(2t+1)(2)$

   (b) $f'(x) = -4(2x+1)^{-5}(2)$

   (c) $-7(4x^2+1)^{-8}(8x)$

   (d) $10(5-2x)^9(-2)$

   (e) $-5(4x+1)^{-6}(4)$

   (f) $-8\left(\sqrt{3}x^2 + x - \sqrt{11}\right)^{-9}\left(2\sqrt{3}x + 1\right)$

   (g) $\frac{1}{2}\left(x^2+2x+3\right)^{-\frac{1}{2}}(2x+2)$

   (h) $-\frac{1}{2}\left(x^2+1\right)^{-\frac{3}{2}}(2x)$

   (i) $2(3x^2+7)(6x)(5-3x)^3 + 3(5-3x)^2(-3)(3x^2+7)^2$

   (j) $4\left(\dfrac{x^2+x}{1-2x}\right)^3\left(\dfrac{(2x+1)(1-2x) - -2(x^2+x)}{(1-2x)^2}\right)$

   (k) $\frac{1}{2}\left(x+\sqrt{x}\right)^{-\frac{1}{2}}\left(1+\frac{1}{2}x^{-\frac{1}{2}}\right)$

   (l) $f'(x) = 6(5x^4-1)^2 + 2(5x^4-1)20x^3 6x$

   (m) $2\left(\dfrac{2x+4}{3x-1}\right)\left(\dfrac{2(3x-1) - 3(2x+4)}{(3x-1)^2}\right)$

   (n) $f'(x) = \dfrac{3(x^2+2)^2 2x(x^2-1)^5 - 5(x^2-1)^4 2x(x^2+2)^3}{(x^2-1)^{10}}$

   (o) $20\left(\dfrac{9x+1}{1-12x}\right)^{19}\left(\dfrac{9(1-12x) - -12(9x+1)}{(1-12x)^2}\right)$

3. (a) $f'(t) = e^{x+4}$

   (b) $f'(x) = 12e^{4x}$

   (c) $y' = -e^{x+1}$

   (d) $f'(x) = -.1e^{-0.01x}$

   (e) $f'(x) = -5^{-x}\ln 5$

   (f) $f'(x) = e^x(x+1)$

   (g) $y' = 2(x-3)e^{2x} + e^{2x}\cdot 2(x-3)^2$

   (h) $f'(x) = \dfrac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2}$

   (i) $f'(x) = 4(2x + e^{-x^2})(1 - xe^{-x^2})$

   (j) $f'(x) = \sqrt{2}^x(\ln\sqrt{2}) + \sqrt{2}x^{\sqrt{2}-1}$

   (k) $f'(x) = e^{\sqrt{x}}\dfrac{1}{2\sqrt{x}}$

(l) $f'(x) = \frac{1}{2}(1-2e^x)^{-\frac{1}{2}}(-2e^x)$

(m) $f'(x) = e^{-\frac{1}{x}}\dfrac{1}{x^2}$

(n) $f'(x) = \dfrac{e^{3x}\cdot 3(1+e^x) - e^x \cdot e^{3x}}{(1+e^x)^2}$

(o) $f'(x) = e^{-x^2} + e^{-x^2}(-2x)x$

(p) $f'(x) = \frac{1}{3}\left(2x+e^{2x}\right)^{-\frac{2}{3}}\left(2+e^{2x}\cdot 2\right)$

(q) $f'(x) = 3(2x-5e^x)^2(2-5e^x)$

(r) $f'(x) = 3^{6x^2+2x+1}\ln 3(12x+2)$

(s) $f'(x) = e^{e^{e^x}} \cdot e^{e^x} \cdot e^x$

4. (a) $y - \frac{1}{3} = -\frac{1}{27}(x-1)$ (b) $y - 16 = -48(x-1)$ (c) $y - 64 = 64\ln 4(x-3)$
(d) $y - \ln 4 = -(x-1)$ (e) $y - \frac{1}{e} = \frac{1}{e}(x-1)$

5. $f'(x) = x^x(\ln x + 1)$

6. $R(p) = -\dfrac{4p(p+1)^2}{3} + 80p$ $R'(4) = -\frac{20}{3}$

7. (a) $R(p) = 1500p - 1.5p^2$ (b) $P(p) = -1.5p^2 + 1800p - 335,000$ (c) $P'(p) = -3p + 1,800$
(d) $P'(500) = 300$

8. Hint: Rewrite $\log_a x$ as $\frac{\ln x}{\ln a}$ and use the Chain Rule.

9. (a) 5 (b) No. You would need to have a value for $g'(-3)$.

10. $f(x) = |x| = \sqrt{x^2} = \left(x^2\right)^{\frac{1}{2}}$, So $f'(x) = \frac{1}{2}(x^2)^{-\frac{1}{2}}\cdot 2x = \dfrac{x}{(x^2)^{\frac{1}{2}}} = \dfrac{x}{|x|}$

11. $f'(x) = \dfrac{x^2-3x}{|x^2-3x|}\cdot(2x-3)$

12. (a) $\cos(x^2+3x)(2x+3)$ (b) $\cos(\tan x)\sec^2 x$ (c) $\cos(\tan(6x))\sec^2(6x)6$ (d) $5(\tan^4 x)\sec^2 x$
(e) $[5(\tan^4(\sin x)][\sec^2(\sin x)]\cos x$

13. (a) $y' = \frac{1}{x}$

(b) $g'(x) = \dfrac{4}{4x-1}$

(c) $y' = \dfrac{1}{\sqrt{2x+1}}\cdot\frac{1}{2}(2x+1)^{-\frac{1}{2}}(2)$

(d) $h'(x) = \dfrac{1}{\frac{9x}{4x-2}}\left(\dfrac{9(4x-2)-4\cdot 9x}{(4x-2)^2}\right)$

(e) $f'(x) = 28$

(f) $f'(x) = \dfrac{x+1-x\ln x}{x(x+1)^2}$

(g) $y' = \dfrac{1}{(2x-3)\ln 8}$

(h) $f'(t) = \dfrac{t^2 - 1}{\ln 6(t^2+1)t}$

(i) $f'(x) = \dfrac{3x^2 - 2}{x^3 - 2x + 3}$

(j) $f'(x) = \dfrac{\frac{1}{x} \cdot x - \ln x}{x^2}$

(k) $f'(x) = \ln x + \frac{1}{x} \cdot x - 1 = \ln x$

(l) $f'(x) = \frac{1}{\ln x} \frac{1}{x}$

(m) $f'(x) = \frac{1}{x \ln 5}$

(n) $f'(x) = \dfrac{1}{3^x \ln 10} \cdot 3^x \ln 3 = \dfrac{\ln 3}{\ln 10} = \log 3$
Note: If you simplify the function first, this derivative is trivial.

(o) $f'(x) = e^{(\ln x)^2}(2 \ln x) \cdot \frac{1}{x}$

14. Hint: Separate the problem into two cases: $x > 0$ and $x < 0$. OR Use the result from problem 10.

15. $P'(x) = \dfrac{60}{4x+1} - 3$

16. $R'(x) = 100 + \frac{50(\ln x - 1)}{(\ln x)^2}$

12 The d/dx Notation and Higher Order Derivatives

Consider the function $x^3 + 2x^2 + 3$.

If you call it $f(x)$, i.e., let $f(x) = x^3 + 2x^2 + 3$, then its derivative is written $f'(x) = 3x^2 + 4x$. If you call it y, i.e., let

$$y = x^3 + 2x^2 + 3$$

then its derivative can also be called $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = 3x^2 + 4x.$$

So we have two quite different notations for the derivative.[28] You must learn to be comfortable with both since the whole world uses both: in some contexts one is convenient, in other contexts the other is better.

$\dfrac{dy}{dx}$ looks like a fraction but it isn't a fraction. The notation reminds us that the derivative is a limit of $\dfrac{\Delta y}{\Delta x}$, but don't think of dy and dx as having separate meanings (at least not yet). Instead, think of $\dfrac{d}{dx}$ as an "operator" called "derivative of". This will enable us to not have to always write, "The derivative of" . To say "The derivative of $(x^2 + 5)$ is $2x$," we can simply write $\dfrac{d}{dx}(x^2 + 5) = 2x$. Indeed, you can think of $\dfrac{dy}{dx}$ as $\dfrac{d}{dx}(y)$, the derivative of y.

In words, one reads $\dfrac{d}{dx}$ as "d by dx".

The notation for evaluating f' at $x = a$ is $f'(a)$. The notation for evaluating $\dfrac{dy}{dx}$ at $x = a$ is $\left.\dfrac{dy}{dx}\right|_{x=a}$. Using the function $f(x) = x^3 + 2x^2 + 3$ above, $f'(1) = \left.\dfrac{dy}{dx}\right|_{x=1} = 7$.

The Chain Rule Again

The $\dfrac{dy}{dx}$ notation gives us another view of the chain rule. In Section 11 we learned the chain rule as: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$. If we call $y = f(g(x))$ and let $u = g(x)$, then $y = f(g(x)) = f(u)$. We now have y as a function of u and u as a function of x. Their derivatives are $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$ respectively.

This produces the chain rule using $\dfrac{dy}{dx}$ (Leibnitz) notation:

(12.1)
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 12.1. Find $\dfrac{dy}{dx}$ for $y = (2x^4 - 6x + 5)^3$

Soultion: Let $u = 2x^4 - 6x + 5$. So, $y = u^3$. This gives us derivatives: $\dfrac{du}{dx} = 8x^3 - 6$, and $\dfrac{dy}{du} = 3u^2$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2 \cdot (8x^3 - 6) = 3(2x^4 - 6x + 5)^2(8x^3 - 6)$$

[28] The f' notation is attributed to Joseph Lagrange (1736-1818) and the $\frac{dy}{dx}$ notation is attributed to Gottfried Leibnitz (1646-1716).

Here is a more complicated example:

Example 12.2. Find $\dfrac{dy}{dx}$ for $y = \ln[(2x+41)^{15}]$.

Answer: $y = \ln u$ where $u = (2x+41)^{15}$. So $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \dfrac{1}{u} \cdot \dfrac{du}{dx}$.

Now, $\dfrac{du}{dx} = \dfrac{d}{dx}(2x+41)^{15}$. This requires a chain rule: Let $v = 2x+41$.

Then $\dfrac{dv}{dx} = 2$. So, $\dfrac{du}{dx} = \dfrac{du}{dv} \cdot \dfrac{dv}{dx} = 15v^{14} \cdot 2$

Putting this together, we have $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dv} \cdot \dfrac{dv}{dx} = \dfrac{1}{u} \cdot 15v^{14} \cdot 2$.

When we substitute to rewrite in terms of x we get $\dfrac{dy}{dx} = \dfrac{1}{(2x+41)^{15}} \cdot 15(2x+41)^{14} \cdot 2$.

In this Example we found $\dfrac{dy}{dx}$ without multiplying out the original very complicated expression first.

This notation may seem more complicated, but we will see in Section 14 that it can be very useful. If you look at Equation 12.1 you can see that it looks like the "du"s on the right could just be canceled to leave you with $\dfrac{dy}{dx}$. A similar thing appears to happen in Example 12.2. In fact this sort of canceling does "work" from a notation standpoint. You can find equivalencies as though these derivatives were fractions. They are NOT fractions, they are rates of change, but they can be algebraically manipulated like fractions.

The following is a rather contrived illustration, but it might give you more insight into the chain rule and the $\dfrac{dy}{dx}$ notation. We will do more interesting applications in Section 14.

Example 12.3. Suppose we know that Albert always runs twice as fast as Barney. Suppose also that Barney always runs three times as fast as Chuck. Did you get that? Who is the fastest runner? (Albert) Who is the slowest? (Chuck). The question is: How much faster is Albert compared to Chuck? Stop a moment and think about this question. (insert *Final Jeopardy* music). You want to say that Albert runs six times as fast as Chuck. This is correct. How did you figure it out? You used the Chain Rule:

Consider Albert and Barney. As an expression of their relative changes in position over time we could say that $\dfrac{dA}{dB} = \dfrac{2}{1} = 2$. We could similarly use $\dfrac{dB}{dC} = 3$ to express the relative changes in the positions of Barney and Chuck. We want to find the relative change in the positions of Albert and Chuck $\left(\dfrac{dA}{dC}\right)$. Barney is the link in the chain. $\dfrac{dA}{dC} = \dfrac{dA}{dB} \cdot \dfrac{dB}{dC} = 2 \cdot 3 = 6$.

Higher Order Derivatives

Suppose that f is a differentiable function. Then its derivative f' is a function. Since f' is a function in its own right, it might have a derivative. We write the derivative of f' as $f''$and say "f double-prime." The derivative of f' is called the *second derivative* of f. Of course, f'' is a function, so it might have a derivative, f''', "f triple-prime." The function f''' is the second derivative of f' and the third derivative of f. This could go on and on, but we stop putting unweildly prime marks and write $f^{(4)}$, $f^{(5)}$,… $f^{(23)}$ …, for the fourth derivative, fifth derivative, …twenty-third derivative, …etc.

Example 12.4. Suppose $f(x) = 2x^4 + 3x^3 - x^2 + x + 10$. Then:

$$\begin{aligned}
f' &= 8x^3 + 9x^2 - 2x + 1 \\
f'' &= 24x^2 + 18x - 2 \\
f''' &= 48x + 18 \\
f^{(4)} &= 48 \\
f^{(5)} &= 0 \\
f^{(6)} &= 0 \\
f^{(7)} &= 0 \ldots \\
\ldots f^{(n)} &= 0 \quad \text{for all} \quad n \geq 5
\end{aligned}$$

Do not confuse $f^{(n)}$ with f^n. The first is the n^{th} derivative. The second is f multiplied times itself n times. Using f from Example 12.4, $f^{(4)} = 48$, but f^4 is the 16^{th}-degree polynomial $(2x^4 + 3x^3 - x^2 + x + 10)^4$.

The expressions for higher derivatives using Leibniz notation are gotten by applying the operator $\dfrac{d}{dx}$ multiple times.

$$\begin{aligned}
\frac{d}{dx}(y) &= \frac{dy}{dx} && \text{first derivative} \\
\frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d^2y}{dx^2} && \text{second derivative} \\
\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) &= \frac{d^3y}{dx^3} && \text{third derivative} \\
\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) &= \frac{d^ny}{dx^n} && n^{\text{th}}\text{derivative}
\end{aligned}$$

Again using Example 12.4, we would say $\left.\dfrac{d^3y}{dx^3}\right|_{x=2} = 48(2) + 18 = 114.$

We will have little use for derivatives higher than the second derivative, but they are worth a look if only to get more feel for functions and their derivatives.

As we continued to find higher and higher derivatives in Example 12.4 we eventually got to a point where all of the derivatives were zero. Will this always happen? Let's look at $f(x) = \dfrac{1}{x}$.

Example 12.5.

$$\begin{aligned}
f(x) &= \frac{1}{x} = x^{-1} \\
f'(x) &= -x^{-2} \\
f''(x) &= -(-2)x^{-3} = 2x^{-3} \\
f'''(x) &= -(-2)(-3)x^{-4} = -6x^{-4} \\
f^{(4)}(x) &= -(-2)(-3)(-4)x^{-5} = 24x^{-5} \\
f^{(5)}(x) &= -(-2)(-3)(-4)(-5)x^{-6} = -120x^{-6}
\end{aligned}$$

Do you see the pattern here? What would $f^{(27)}(x)$ be? $f^{(n)}(x)$?

Answer: $f^{(27)} = (-1)(-2)(-3)\ldots(-26)(-27)x^{-28}$. This can be written[29] as: $f^{(27)}(x) = -27!x^{-28}$.

Answer: $f^{(n)}(x) = (-1)^n n! x^{-(n+1)}$.

Above it was easier to see the pattern from the non-simplified expression of the derivatives. Simplifying the coefficients obscured from whence they came. If you are not looking for a pattern, but a "low" higher order derivative, it is often useful to use intermediate simlification.

Example 12.6. Find y''' for $y = \sqrt{x^2+1}$.

$$\begin{aligned} y &= (x^2+1)^{\frac{1}{2}} \\ y' &= \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot 2x \\ &= x(x^2+1)^{-\frac{1}{2}} \\ y'' &= (x^2+1)^{-\frac{1}{2}} - \frac{1}{2}(x^2+1)^{-\frac{3}{2}} \cdot 2x \cdot x \\ &= (x^2+1)^{-\frac{3}{2}}[(x^2+1) - x^2] \\ &= (x^2+1)^{-\frac{3}{2}} \\ y''' &= -\frac{3}{2}(x^2+1)^{-\frac{5}{2}} \cdot 2x \end{aligned}$$

Example 12.7. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $y = e^{x^2}$.

$$\begin{aligned} \frac{dy}{dx} &= e^{x^2} \cdot 2x \\ \frac{d^2y}{dx^2} &= e^{x^2} \cdot 2x \cdot 2x + 2e^{x^2} \end{aligned}$$

It's hard to grasp the idea that $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ measures the rate of change of the rate of change of y with respect to x, except that we see it in daily life.

Acceleration is the rate of change of velocity with respect to time, and velocity is the rate of change of position with respect to time, so acceleration is the rate of change of the rate of change of position with respect to time. In symbols: acceleration $a(t) = \dfrac{d}{dt}v(t) = \dfrac{d}{dt}\left(\dfrac{ds}{dt}\right) = \dfrac{d^2s}{dt^2}$.

Example 12.8. At time t seconds, a particle is $s(t) = 4t^3 - 2t^2 + 5t - 1$ feet from a given reference point. Find the velocity function $v(t)$ and the acceleration function $a(t)$. What is the acceleration of the particle at $t = 3$?
Answer: $v(t) = 12t^2 - 4t + 5 \qquad a(t) = 24t - 4 \qquad a(3) = 68\ \text{ ft/sec}^2$.

Real life examples of third and higher derivatives are harder to find, but in advanced math higher derivatives are useful.

[29]The "!" is a *factorial* symbol. For any positive integer n, $n!$ is defined to be the product of all of the integers between n and 1, inclusive. $n! = (n)(n-1)(n-2)\ldots(3)(2)(1)$.

Section 12 - Exercises (answers follow)

1. Find $\frac{dy}{du}$, $\frac{du}{dx}$, and $\frac{dy}{dx}$.
   (a) $y = u^2 + 1$ and $u = 3x - 2$
   (b) $y = u^{4/5}$ and $u = x^2 - x + 1$
   (c) $y = \frac{1}{u-1}$ and $u = x^3$
   (d) $y = \frac{1}{u}$ and $u = \sqrt{x} + 6$

2. Find f'' for each of the following. Then find $f''(0)$ and $f''(2)$.
   (a) $f(u) = 4u^2 - 2u + 1$
   (b) $f(t) = \sqrt{t+4}$
   (c) $f(x) = (2x^2 - 1)(3x^2)$

3. $f(x) = \dfrac{-x}{x^2+1}$. Show that $f''(x) = \dfrac{-2x(x^2-3)}{(x^2+1)^3}$.

4. Find $g'(x)$ and $g''(x)$ for each of the following.
   (a) $g(x) = \dfrac{\ln x}{4x}$
   (b) $g(x) = \ln(x^2 + 1)$.
   (c) $g(x) = -10^{3x^2-2}$

5. How many non-zero higher-order derivatives does the function $f(x) = x^{12}$ have?

6. Find $f^{(28)}(x)$ and $f^{(n)}(x)$ for $f(x) = \dfrac{1}{5x-2}$, for n a positive integer.

7. Find $f^{(28)}(x)$ and $f^{(n)}(x)$ for $f(x) = e^{-2x}$, for n a positive integer.

8. Show that $y = e^{2x} + e^{-3x}$ satisfies the equation $y'' + y' - 6y = 0$.

9. A ball is dropped from a 100 ft. high platform. The position of the ball (measured in feet from the ground) t seconds after it is dropped is given to be $s(t) = -16t^2 + 100$.
   (a) How high is the ball 2 seconds after it is dropped? How long will it take until the ball hits the ground?
   (b) Find the velocity function $v(t)$. What is the velocity of the ball 2 seconds after it is dropped? How *fast* is it dropping at that time?
   (c) Find the acceleration function. Considering your experience with falling objects, what do you think of the statement "When acceleration is negative, it means that the object is slowing down?"

Section 12 - Answers

1. (a) $\frac{dy}{du} = 2u \qquad \frac{du}{dx} = 3 \qquad \frac{dy}{dx} = 6(3x-2)$

(b) $\frac{dy}{du} = \frac{4}{5}u^{-\frac{1}{5}}$ $\quad \frac{du}{dx} = 2x - 1$ $\quad \frac{dy}{dx} = \frac{4}{5}\left(x^2 - x + 1\right)^{-\frac{1}{5}}(2x - 1)$

(c) $\frac{dy}{du} = -\frac{1}{(u-1)^2}$ $\quad \frac{du}{dx} = 3x^2$ $\quad \frac{dy}{dx} = -\frac{3x^2}{(x^3-1)^2}$

(d) $\frac{dy}{du} = -\frac{1}{u^2}$ $\quad \frac{du}{dx} = \frac{1}{2\sqrt{x}}$ $\quad \frac{dy}{dx} = \frac{-1}{(\sqrt{x}+6)^2 2\sqrt{x}}$

2. (a) $f''(u) = 8$ $\quad f''(0) = 8$ $\quad f''(2) = 8$

   (b) $f''(t) = -\frac{1}{4}(t + 4)^{-\frac{3}{2}}$ $\quad f''(0) = -\frac{1}{32}$ $\quad f''(2) = -\frac{1}{24\sqrt{6}}$

   (c) $f''(x) = 72x^2 - 6$ $\quad f''(0) = -6$ $\quad f''(2) = 282$

3. Hint: When simplified, $f'(x) = \dfrac{x^2 - 1}{(x^2 + 1)^2}$.

4. (a) $g'(x) = \frac{1-\ln x}{4x^2}$ $\quad g''(x) = \frac{-3+2\ln x}{4x^3}$

   (b) $g'(x) = \frac{2x}{x^2+1}$ $\quad g''(x) = \frac{2(1-x^2)}{(x^2+1)^2}$.

   (c) $g'(x) = -6x(10^{3x^2-2})\ln 10$ $\quad g''(x) = -6\ln 10(10^{3x^2-2})(1 + 6x^2\ln 10)$

5. 12

6. $f^{(28)}(x) = 28!(5x - 2)^{-28} \cdot 5^{28}$ $\quad f^{(n)}(x) = (-1)^n n!(5x - 2)^{-(n+1)} \cdot 5^n$

7. $f^{(28)}(x) = e^{-2x} \cdot (-2)^{28}$ $\quad f^{(n)}(x) = e^{-2x} \cdot (-2)^n$

8. $y = e^{2x}+e^{-3x}$, $y' = 2e^{2x}-3e^{-3x}$ and $y'' = 4e^{2x}+9e^{-3x}$. Substitute these values into $y''+y'-6y$ and you should get zero.

9. (a) $s(2) = 36$ ft. above the ground. $\quad s(t) = 0$ at $t = 2.5$ seconds.

   (b) $v(t) = -32t$ $\quad v(2) = -64$ ft./sec. $\quad$ Speed $= |v(t)|$, so speed is 64 ft./sec.

   (c) $a(t) = -32$ $\quad$ The statement must be false because experience tells us that an object increases speed as it drops, but here we have the acceleration negative for all times t.

13 Implicit Differentiation

When y is given explicitly in terms of x we have seen how to compute $\dfrac{dy}{dx}$. But sometimes y is only given "implicitly" in terms of x by means of some equation linking y and x. A simple example is

$$(*) \qquad x^2 + y^2 = 1.$$

Take $\dfrac{d}{dx}$ on both sides to get

$$2x + 2y\frac{dy}{dx} = 0.$$

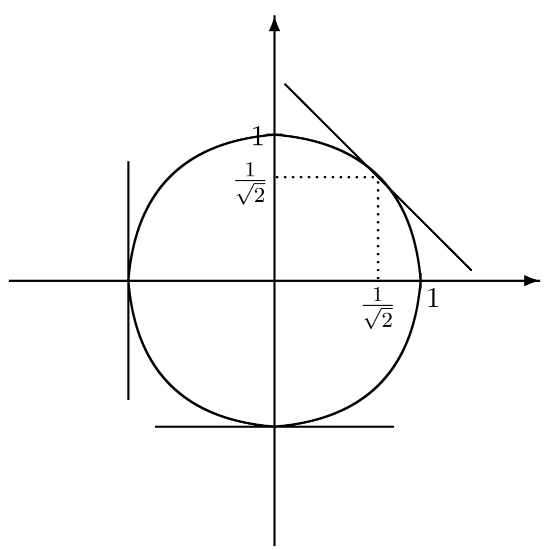
PAUSE: What are we doing here? We are pretending that y is a function of x and we are applying the Chain Rule to y^2: $\dfrac{d}{dx}(y^2) = \dfrac{d}{dy}(y^2) \cdot \dfrac{dy}{dx} = 2y \cdot \dfrac{dy}{dx}$.

Now solve $2x + 2y\dfrac{dy}{dx} = 0$ for $\dfrac{dy}{dx}$ to get

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The curve in the plane whose equation is $(*)$ is a circle with center $(0,0)$ and radius 1. A point on that circle is $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$. So $\dfrac{dy}{dx}\Big|_{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})} = -\dfrac{1}{\sqrt{2}}\Big/\dfrac{1}{\sqrt{2}} = -1$. Another point on that circle is $(0,1)$. So $\dfrac{dy}{dx}\Big|_{(0,1)} = -\dfrac{0}{1} = 0$. What we are finding here is that the slope of the tangent to the circle at $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ is -1 and the slope of the tangent to the circle at $(0,1)$ is 0.

Another point on the circle is $(-1,0)$. Here, the result $\dfrac{dy}{dx} = -\dfrac{x}{y}$ becomes $\dfrac{dy}{dx}\Big|_{(-1,0)} = -\dfrac{(-1)}{0}$ which has no meaning: in other words the tangent to the circle at $(-1,0)$ has no slope, so it must be vertical.



$x^2 + y^2 = 1$ and tangents

What we have done here is to find the slope of the tangent line by *implicit differentiation*. This process involves two (or three) steps:

1. Take $\frac{d}{dx}$ of both sides of the equation.
2. Solve for $\frac{dy}{dx}$.
3. Substitute in specific values of x and y (if the slope is desired at a specific point).

When taking the derivative of both sides we recognize y to represent a function of x and we remember to use all of the established derivative rules (chain rule, quotient rule, product rule, etc.).

Example 13.1. Find $\frac{dy}{dx}$ for $x^2y^5 + y^3 = 7x$

Solution:

$$\begin{aligned}
x^2y^5 + y^3 &= 7x \\
\frac{d}{dx}(x^2y^5 + y^3) &= \frac{d}{dx}(7x) \\
2xy^5 + 5y^4\frac{dy}{dx}x^2 + 3y^2\frac{dy}{dx} &= 7 \\
5y^4x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} &= 7 - 2xy^5 \\
\frac{dy}{dx}(5y^4x^2 + 3y^2) &= 7 - 2xy^5 \\
\frac{dy}{dx} &= \frac{7 - 2xy^5}{5y^4x^2 + 3y^2}
\end{aligned}$$

Example 13.2. Find $\frac{dy}{dx}$ for $e^{xy} = 5$

Solution:

$$\begin{aligned}
e^{xy} &= 5 \\
\frac{d}{dx}e^{xy} &= \frac{d}{dx}5 \\
e^{xy}\left(y + \frac{dy}{dx}x\right) &= 0 \\
y + x\frac{dy}{dx} &= 0 \\
x\frac{dy}{dx} &= -y \\
\frac{dy}{dx} &= -\frac{y}{x}
\end{aligned}$$

Example 13.3. Find the equation of the line tangent to the graph of $\dfrac{\sqrt{x}}{y} = 3 - y$ at the point $(4, 2)$.

Solution:

$$\begin{aligned}
\frac{\sqrt{x}}{y} &= 3 - y \\
\frac{d}{dx}\left(\frac{\sqrt{x}}{y}\right) &= \frac{d}{dx}(3 - y) \\
\frac{\frac{1}{2}x^{-\frac{1}{2}}y - \frac{dy}{dx}\sqrt{x}}{y^2} &= -\frac{dy}{dx} \\
\frac{y}{2\sqrt{x}} - \frac{dy}{dx}\sqrt{x} &= -\frac{dy}{dx}(y^2) \\
-\frac{dy}{dx}\sqrt{x} + \frac{dy}{dx}(y^2) &= -\frac{y}{2\sqrt{x}} \\
\frac{dy}{dx}(-\sqrt{x} + y^2) &= -\frac{y}{2\sqrt{x}} \\
\frac{dy}{dx} &= \frac{-\frac{y}{2\sqrt{x}}}{-\sqrt{x} + y^2} \\
\left.\frac{dy}{dx}\right|_{(4,2)} &= \frac{-\frac{2}{2\sqrt{4}}}{-\sqrt{4} + 2^2} = \frac{-\frac{2}{4}}{2} = -\frac{1}{4}
\end{aligned}$$

So, the equation of the tangent line is $y - 2 = -\frac{1}{4}(x - 4)$.

Example 13.4. Find $\dfrac{d^2y}{dx^2}$ for the equation $y^3 + x^2 - 7x + 1 = 0$.

To find the second derivative, we first find $\frac{dy}{dx}$.

$$\begin{aligned}
y^3 + x^2 - 7x + 1 &= 0 \\
\frac{d}{dx}(y^3 + x^2 - 7x + 1) &= \frac{d}{dx}0 \\
3y^2\frac{dy}{dx} + 2x - 7 &= 0 \\
\frac{dy}{dx} &= \frac{-2x + 7}{3y^2}
\end{aligned}$$

Now we differentiate again.

$$\begin{aligned}
\frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d}{dx}\left(\frac{-2x + 7}{3y^2}\right) \\
\frac{d^2y}{dx^2} &= \frac{-2 \cdot 3y^2 - 6y\frac{dy}{dx}(-2x + 7)}{9y^4}
\end{aligned}$$

It is good form to express our answer in terms of only x and y. So any $\frac{dy}{dx}$ is replaced with its equivalent expression in x and y. The unsimplified "final" answer is:

$$\frac{d^2y}{dx^2} = \frac{-2 \cdot 3y^2 - 6y\left(\frac{-2x+7}{3y^2}\right)(-2x + 7)}{9y^4}.$$

So what is the real meaning of this procedure? Let's go back to the first example $x^2 + y^2 = 1$. This circle is not the graph of a function, but rather the combined graphs of two functions $y = \pm\sqrt{1-x^2}$: that is, $y = \sqrt{1-x^2}$ is one function and $y = -\sqrt{1-x^2}$ is another function. With the exception of the points where the tangent is vertical, namely $(\pm 1, 0)$, a point on the circle is on one function graph or the other. Once you fix such a point, say (a, b), on one of the graphs, $\left.\dfrac{dy}{dx}\right|_{(a,b)}$ is the slope of the tangent to the graph at (a, b); and $\left.\dfrac{dy}{dx}\right|_{(a,b)}$ measures the rate of change of y with respect to x at (a, b). Implicit differentiation is a fast way of getting the same information. When x is near a, y is given by some function of x (this function depends on a) but one doesn't need to know the function explicitly in order to get $\dfrac{dy}{dx}$.

Section 13 - Exercises (answers follow)

1. Verify that the derivative found by implicit differentiation for the circle $x^2 + y^2 = 1$ (at the beginning of this section), is consistent with derivatives one would find when splitting the equation into the functions that describe the top half and bottom half of the circle.

2. Use implicit differentiation to find $\dfrac{dy}{dx}$ for each of the following equations.

   (a) $x^2 + 3y^2 = 6$

   (b) $9x - x^2y^2 = 2xy$

   (c) $3xy - \frac{y}{3} = \frac{2}{x}$

   (d) $y \ln x + 8 = x^{3/2}y^{7/2}$

   (e) $(x^2 + y^3)^5 = 2xy$

   (f) $3x^2 - 4y^3 + 3 = \sqrt{5x + y}$

   (g) $x^2y - xy^2 = 10.$

3. Find the equation of the line tangent to $y^2 - x^2 = 16$ at the point $(2, 2\sqrt{5})$.

4. Find the equation of the line tangent to $x^2 - xy + y^3 = 8$ at the point $(0, 2)$.

5. Find the equation of the line tangent to $y + x\sqrt{y} = 8$ at the point $(2, 4)$.

6. Find the slope of the line tangent to the curve $e^{xy} = x$ at $x = 3$.

7. Find the slope of the line tangent to $(x + y^2)^5 + 6 = -3x - y$ at the point $(-3, 2)$

8. For a certain product, cost C and revenue R (in dollars) are given as follows, where q is the number of units sold (in hundreds).

$$\text{Cost: } C^2 = q^2 + 100\sqrt{q} + 100$$
$$\text{Revenue: } 900(q - 4)^2 + R^2 = 25{,}500$$

   (a) Find and interpret the marginal cost dC/dq at $q = 5$.

   (b) Find and interpret the marginal revenue dR/dq at $q = 5$.

9. Find $\dfrac{d^2y}{dx^2}$ for $(y + 2)^3 = x^2$.

10. For $y^2 = x$, show that: (a) $\dfrac{d^2y}{dx^2} = -\dfrac{1}{4y^3}$ (b) $\dfrac{d^3y}{dx^3} = \dfrac{3}{8y^5}$.

Section 13 - Answers

1. When $y = \sqrt{1 - x^2}$, $\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}} = -\frac{x}{y}$
   When $y = -\sqrt{1 - x^2}$, $\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}} = -\frac{x}{y}$

2. (a) $\dfrac{dy}{dx} = \dfrac{-x}{3y}$

(b) $\dfrac{dy}{dx} = \dfrac{9 - 2xy^2 - 2y}{2x^2y + 2x}$

(c) $\dfrac{dy}{dx} = \dfrac{-(6 + 9x^2y)}{9x^3 - x^2}$

(d) $\dfrac{dy}{dx} = \dfrac{\frac{-y}{x} + \frac{3}{2}x^{\frac{1}{2}}y^{\frac{7}{2}}}{\ln x - \frac{7}{2}y^{\frac{5}{2}}x^{\frac{3}{2}}}$

(e) $\dfrac{dy}{dx} = \dfrac{2y - 10x(x^2 + y^3)^4}{15y^2(x^2 + y^3)^4 - 2x}$

(f) $\dfrac{dy}{dx} = \dfrac{-6x + \frac{5}{2}(5x + y)^{-1/2}}{-12y^2 - \frac{1}{2}(5x + y)^{-1/2}}$

(g) $\dfrac{dy}{dx} = \dfrac{-2xy + y^2}{x^2 - 2xy}$

3. $y - 2\sqrt{5} = \frac{1}{\sqrt{5}}(x - 2)$

4. $y = \frac{1}{6}x + 2$

5. $y - 4 = -\frac{4}{3}(x - 2)$

6. $m = \dfrac{1 - \ln 3}{9}$ Note: when $x = 3$, $y = \frac{\ln 3}{3}$

7. $m = -\frac{8}{21}$

8. (a) $\dfrac{dC}{dq} = \dfrac{q}{C} + \dfrac{25}{C\sqrt{q}}$ At $q = 5$, $\dfrac{dC}{dq} = \dfrac{1 + \sqrt{5}}{\sqrt{5 + 4\sqrt{5}}}$, so at the instant that 500 units are being sold, the cost is *increasing* at the rate of $\frac{1+\sqrt{5}}{\sqrt{5+4\sqrt{5}}} \approx .87$ dollars/100 items sold.

(b) $\dfrac{dR}{dq} = -\dfrac{900(q - 4)}{R}$ At $q = 5$, $\dfrac{dR}{dq} = -\dfrac{90}{\sqrt{246}}$, so at the instant that 500 units are being sold, the revenue is *decreasing* at the rate of $\frac{90}{\sqrt{246}} \approx 5.74$ dollars/100 items sold.

9. $\dfrac{d^2y}{dx^2} = \dfrac{6(y + 2)^2 - 12x(y + 2)\left(\frac{2x}{3(y+2)^2}\right)}{9(y + 2)^4}$

14 Related Rates

Suppose you drop a stone into a still pond. Circular ripples form in the water and they expand, maintaining their perfect circular shape until they meet an obstacle or die out. The center of the circles is the place where you dropped the stone.

We watch one ripple as it is " born" and expands. How does the area of that circle change compared to the change in the radius of that circle? We know that $A = \pi r^2$. So when the radius changes from 2 in. to 3 in. (an increase of one inch), the area changes from 4π in.^{2} to 9π in.^{2} (an increase of 5 in.^{2}). However, when the radius changes from 3 in. to 4 in. (an increase of one inch), the area changes from 9π in.^{2} to 16π in.^{2} (an increase of 7 in.^{2}). So, we know that the rate of change of the area compared to the radius does not change at a constant rate. If you draw a picture of concentric circles and look at the increasing areas of these circles, this should not be a surprise to you. An increase in radius from 10 in. to 11 in., or from 20 in. to 21 in. will have a larger and larger increase in area. The rate at which the area is increasing depends on what the radius is at that point.

We have calculus to express the instantaneous rate of change of the area of the circle compared to the radius. We will use Leibnitz notation here, and for the remainder of this section, because it is very descriptive. We can tell exactly the entities that are being compared:

$$\begin{aligned} A &= \pi r^2 \\ \frac{dA}{dr} &= 2\pi r \qquad \text{(remember, } \pi \text{ is a constant)} \end{aligned}$$

The derivative equation tells us what we already knew: The rate at which the area is increasing depends on the value of r.

Suppose we notice that the radius of the circle is increasing at the constant rate of 5 in./sec. Now we are introducing *time* into our discussion. We are saying how *fast* the radius is increasing. We can express this as $\frac{dr}{dt} = 5$ in./sec. Notice that the units of measure are consistent with the derivative: r (radius) on top corresponds to inches on top; t (time) on bottom corresponds to seconds on bottom.

A reasonable question now is "How fast is the area increasing?" Certainly the area is some function of time. So, what is $\frac{dA}{dt}$? We can use implicit differentiation. We take our area equation and differentiate both sides with respect to time (t):

$$\begin{aligned} A &= \pi r^2 \\ \frac{d}{dt}(A) &= \frac{d}{dt}(\pi r^2) \\ 1 \cdot \frac{dA}{dt} &= 2\pi r \cdot \frac{dr}{dt} \\ \frac{dA}{dt} &= 2\pi r(5) = 10\pi r \end{aligned}$$

Again, this result is not surprising in that it tells us that the speed at which the area is increasing depends on r. The larger the radius, the faster the area is increasing. At the instant that the radius is 2 in., the area is increasing at a rate of $20\pi \approx 62.8$ in.^{2}/sec. At the instatnt that the radius is 10 in., the area is increasing at the rate of $100\pi \approx 314.2$ in.^{2}/sec.

Related Rates problems

Section 14 is the study of related rates. In related rates problems there are two entities related to each other by an equation. Both entities depend on a basic variable which is usually, but not always, time. You are given some information about the entities and the rate of change (derivative) of one of them and you are to find the rate of change (derivative) of the other. In the introductory example, the two entities were area and radius. They are related by $A = \pi r^2$. Both entities depended on time, measured from when the stone first hit the pond. We were given $\frac{dr}{dt}$ and asked to find $\frac{dA}{dt}$.

Positive and Negative Rates

We know from Section 8 that velocity $v(t) = \dfrac{ds}{dt}$ can be positive or negative and that *speed* is the absolute value of velocity. In the very same way other rates of change dependent on time can be either positive or negative. When an entity is getting smaller as time goes on, the derivative is negative. When an entity is getting larger as time goes on, the derivative is positive. In either event, the absolute value of the derivative tells the *speed* at which the entity is shrinking or growing. In the example of the stone in the pond, the radius and area were both increasing as time went on. So $\dfrac{dr}{dt}$ and $\dfrac{dA}{dt}$ were both positive. So, in this case our use of the term "speed" was accurate. In the homework exercises you will come across a spherical snowball that is melting. Its volume is decreasing as time goes on. There you will want $\dfrac{dV}{dt}$ to be negative.

The Chain Rule – yet again

Recall from Section 12 that we expressed the chain rule as:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

We see the chain rule again when we look at the derivatives for the example of the stone in the pond. Our initial derivative claim in that example was:

$$\frac{dA}{dr} = 2\pi r$$

We later claimed:

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

When you substitute the first equation into the second, you get the chain rule form:

$$\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$$

General Stategy for Solving Word Problems

Below is a general guide for solving word problems. Every problem has its own variances and nuances, so these steps still require some creativity. In the examples that follow, try to relate the solutions to the six steps.

1. Read the problem twice.

   (a) Identify in words what you are trying to compute or find. Often, but not always, you can find this in a sentence that ends with a question mark.

   (b) Identify in words the facts that are given.

   Get clear in your mind which parts are (a) and which parts are (b).

2. Translate (a) and (b) into mathematical statements using mathematical symbols. A phrase about how something is changing or about the marginal such-and-such refers to a derivative. The phrases "at what rate?" or "how fast?" indicate a derivative with respect to time (dt). Often, identifying the units of measure for the entities involved can help you keep straight what is going on (cm/sec would indicate a derivative, whereas cm^3 would indicate a volume).

3. Determine a mathematical relationship that connects the given (b) with the unknown (a). Often, drawing a picture can help immensely.

4. Restate the mathematical relationship into an equation(s) and solve for the unknown. For related rates problems solving for the unknown will often require taking a derivative with respect to time (t).

5. Clearly identify your answer, including any units of measure.

6. Check your answer for reasonableness.

Example 14.1. The profit, in dollars, that a company makes on one q units of product is given by the equation $P(q) = 5q - 0.01q^2 - 1000$. The demand for the product is increasing at a rate of 10 units per week. How fast is the profit changing when the demand is at 100 units per week?

Solution: We want to know how fast the profit is changing when the demand is at 100 units per week. So, we are looking for $\left|\frac{dP}{dt}\right|$ when $q = 100$.

We are given the equation that expresses the relationship between P and q. We are given that the demand is increasing at a rate of ten units per week, so $\frac{dq}{dt} = 10$.

Differentiating $P(q)$ with respect to t we get: $1 \cdot \frac{dP}{dt} = 5\frac{dq}{dt} - 0.02q\frac{dq}{dt}$.

Substituting our knowns and solving: $\frac{dP}{dt} = 5(10) - .02(100)(10) = 30$.

So, when the demand is at 100 units per week, the profit is increasing at \$30/week.

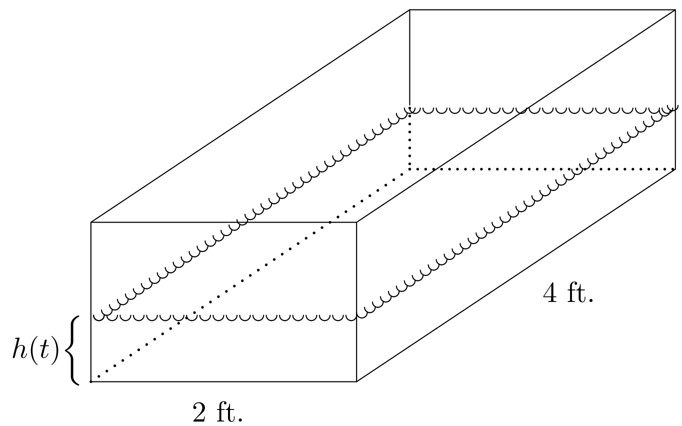
Example 14.2.

Water is pouring into a rectangular tank at the rate of $\frac{1}{2}$ cubic foot per minute. The tank is 4 feet long and 2 feet wide. How fast is the water level rising?

Solution: We want to find how fast the water level in the tank is rising. We know that the tank is rectangular and has a base of 4 ft. by 2 ft. We know that the volume of water in the tank is increasing at a rate of $\frac{1}{2}$ ft^3/min.

It helps to draw a picture (see below). We want to find $\frac{dh}{dt}$ where h is the height of the waterline, measured in feet from the bottom of the tank. We know that $\frac{dV}{dt} = \frac{1}{2}$ ft^3/min.

The volume of the water is $V = 4 \cdot 2 \cdot h = 8h$.
Differentiating with respect to t we get: $\frac{dV}{dt} = 8\frac{dh}{dt}$.
Substituting our known value for $\frac{dV}{dt}$ we have: $\frac{1}{2} = 8\frac{dh}{dt}$. So, $\frac{dh}{dt} = \frac{1}{16}$.
The water is rising at the rate of $\frac{1}{16}$ ft./min.



Example 14.2

Notice that when solving these problems, the information is not substituted into the relationship equation until after the derivative has been taken. In addition, sometimes the information to be substituted in the derivative equation is not specifically given in the problem. That information needs to be calculated from other values and often requires use of the "when" statement. This occurs in examples 14.4 and 14.5.

Example 14.3. A street light is at the top of a 4-meter-tall pole. A dog, 0.8 meters in height, is running away from the pole along a straight path. If the dog is running at a speed of 6.4 m/sec., how fast is the dog's shadow lengthening when the dog is 20 meters from the pole?

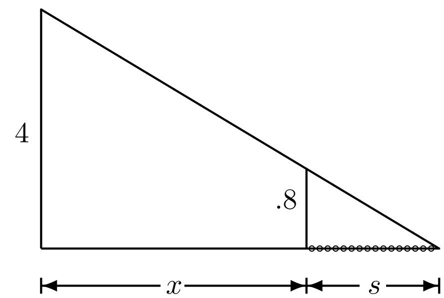
Solution: It is helpful to draw a picture (see below). We assume a level ground and sketch a horizontal line. We then sketch two vertical lines, perpendicular to the ground line. One vertical line represents the 4 meter pole and a shorter vertical line represents the 0.8 meter tall dog. The slanted line that extends from the top of the pole to the ground represents the path of light that creates the dog's shadow. We label the shaded part of the ground that is the shadow s. We know that the dog is running away from the pole at a rate of 6.4 m/sec. We label the part of the ground that is between the dog and the pole x. So, we are given that $\frac{dx}{dt} = 6.4$. We want to find $\frac{ds}{dt}$ when $x = 20$.

Our two triangles are similar (they have the same angles but not the same side lengths). From high school geometry we recall that the sides of similar triangles are proportionate. We want to find a proportion that relates x and s. There are a few to choose from, but probably the easiest to work with is: $\dfrac{x+s}{4} = \dfrac{s}{0.8}$. We simplify this equation to $x = 4s$.

Differentiating the simplified equation with respect to t: $\frac{dx}{dt} = 4\frac{ds}{dt}$.

We know that $\frac{dx}{dt} = 6.4$, so $\frac{ds}{dt} = 1.6$.

The shadow is increasing in length at a rate of 1.6 meters/sec. (The "when $x = 20$" is irrelevant to this calculation).



Example 14.3

Example 14.4. It is estimated that when the circulation of a certain newspaper is x thousand, the annual advertising revenue received by the newpaper will be $R(x) = \frac{1}{2}x^2 + 3x + 160$ thousand dollars. The circulation of the paper is currently 10,000 ($x = 10$) and is increasing at a rate of 2,000 per year. At what rate will the advertising revenue be changing with respect to time, three years from now?

Solution: We are looking for the rate that the revenue will be changing three years from now. So, we are looking for $\frac{dR}{dt}$ when $t = 3$.

We know that the circulation is increasing at 2,000 papers per year and that the current circulation is 10,000. This tells us that $\frac{dx}{dt} = 2$ and that $x = 10$ when $t = 0$.

We are given the relationship between R and x, so we differentiate with respect to t: $\frac{dR}{dt} = x\frac{dx}{dt} + 3\frac{dx}{dt}$.

When $t = 3$, $x = 10 + 3(2) = 16$. So, $\frac{dR}{dt} = 16(2) + 3(2) = 38$

Three years from now revenue will be increasing at a rate of \$38,000 per year.

Example 14.5.

The big bad wolf decides that he wants pork for supper, so he starts heading for the straw house of a little pig. The wolf sneaks due north at a rate of 96 meters/minute. When the wolf is 320 meters away from the straw house, the little pig spies the wolf and decides to run to the safety of his brother's brick house. Unseen by the wolf, the little pig heads due east, running at a rate of 80 meters/minute.

(a) How close to the straw house is the wolf two minutes after the pig begins his escape?

(b) How fast is the distance between the pig and the wolf changing two minutes after the pig begins his escape?

Solution: For this adventure, it is best to draw a picture (see below). The wolf is running north *towards* the straw house and the pig is running east *away from* the straw house. So, we can draw a right angle. The straw house is at the corner of the angle, the wolf is at the end of one leg of the angle, and the pig is at the end of the other leg of the angle. We let w represent the distance from the straw house to the wolf and let p represent the distance from the straw house to the pig. Both w and p are functions of time, t.

With this picture, we know that $\frac{dw}{dt} = -96$ m/min. (negative because the distance between the wolf and straw house gets smaller as time goes on). We know that $\frac{dp}{dt} = 80$ m/min. We can chose $t = 0$ to be any of several instances during these events, but since both questions refer to "two minutes after the pig begins his escape," we will choose $t = 0$ to be the time when the pig first spots the wolf and begins running away. So, when $t = 0$, we have $w = 320$ and $p = 0$.

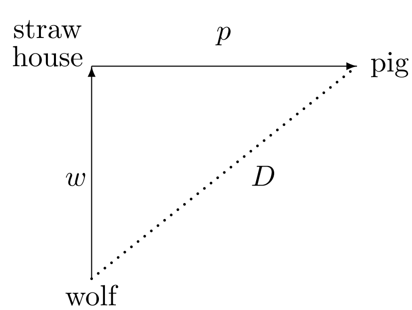
For question (a) we want to know w when $t = 2$. Since distance=(rate x time), we know that in the time between $t = 0$ and $t = 2$, the wolf has traveled 96 x 2 = 192 meters closer to the straw house. So, two minutes after the pig begins to escape, the wolf is $320 - 192 = 128$ m. from the straw house.

For question (b) we are looking for the rate of change, with respect to time, of the distance between the wolf and the pig at the time $t = 2$. We edit our sketch by drawing a dotted line to connect the ends of the two legs of our right triangle. So the distance between the wolf and the pig is represented by the hypotenuse of the right triangle. We will label the distance D. So, question (b) is asking for $\frac{dD}{dt}$ when $t = 2$.

The relationship between the unknown, D and the knowns w and p is expressed by the Pythagorean Theorem: $D^2 = w^2 + p^2$. Differentiating both sides with respect to t: $2D\frac{dD}{dt} = 2w\frac{dw}{dt} + 2p\frac{dp}{dt}$.

We now want to substitute in known values and solve for $\frac{dD}{dt}$. First we need to make some calculations. From part (a) we know that $w = 128$ when $t = 2$. Using distance=(rate x time) again, we can get that $p = 80$ x $2 = 160$ when $t = 2$. Now, using the Pythagorean Theorem, we get that $D = \sqrt{128^2 + 160^2}$ meters when $t = 2$.

Now we can answer question (b). $\frac{dD}{dt} = \dfrac{128 \cdot -96 + 160 \cdot 80}{\sqrt{128^2 + 160^2}} \approx 2.5$ m/sec.



Example 14.5

Section 14 - Exercises (answers follow)

1. Assume x and y are functions of t and $xy - x + 2y^3 = -70$. Evaluate dy/dt when $\frac{dx}{dt} = -5, x = 2$, and $y = -3$.

2. Suppose q thousand watches can be sold at p dollars per watch where $p + q^2 = 144$. How fast is the demand changing when $q = 9$, $p = 63$, and the price per watch is increasing at the rate of \$2/week?

3. A genteel giant is relaxing with a cup of hot tea. He notices that the sugar cube in his tea is melting in such a way that the shape of the cube is constantly maintained. The volume of the cube is decreasing at a constant rate of 36 cm^3/minute. (a) How fast is a side of the cube decreasing when the side is exactly 2 cm. long? (b) How fast is the surface area of the sugar cube decreasing when the side is exactly 2 cm. long?

4. Given the revenue and cost functions $R = 50x - .4x^2$ and $C = 5x + 15$, where x is the daily production (and sales), find the following when 40 units are produced daily and the rate of change of production is 10 units per day.

   (a) The rate of change of revenue with respect to time

   (b) The rate of change of cost with respect to time

   (c) The rate of change of profit with respect to time

5. The base of a 50-foot ladder is being pulled away from a wall at a rate of 10 feet per second. How fast is the top of the ladder sliding down the wall at the instant when the base of the ladder is 30 feet from the wall?

6. An oil tanker ruptures and oil spills, spreading in a circular pattern. If the radius of the circle of oil increases at the constant rate of 1.5 ft./sec., how fast is the area of the spreading oil increasing (a) when the radius is 30 ft.? (b) 2 minutes after the rupture?

7. Percy the ant leaves his nest in search of chips. He travels due north at a rate of 2 cm/sec. When Percy is 20 cm. from the nest, his friend Clive leaves the nest to hunt for salsa. Clive travels due east at a rate of 3 cm/sec. How fast is the distance between the two ants changing 10 seconds after *Clive* leaves the nest?

8. A person 6 feet tall walks away from a street light at the rate of 5 feet per second. If the light is 18 feet above ground level how fast is the tip of the person's shadow moving?

9. A hospital estimates that $N(p) = p^2 + 5p + 900$ people will seek treatment in the emergency room when the population of the community is p thousand people. The population is now 20,000 (that is, $p = 20$) and is growing at 1,200 per year. At what rate is the number of people seeking emergency room treatment rising?

10. The formula for $N(p)$ in the last question suggests that when the population is 4000 then 936 people will seek help in the emergency room. Common sense suggests that this is rather high.

When the population is 40,000 it predicts 2700 visitors to the emergency room which seems more reasonable. When the population is 0 it predicts 900 emergency room cases which is ridiculous. What conclusion do you draw about this mathematical model?

11. A spherical snowball is melting so that its volume is decreasing at 1 cubic centimer per minute. At what rate is the diameter decreasing when the diameter is 10 centimeters? Note: If the radius of a sphere is r then the diameter is $2r$ and the volume is $\frac{4}{3}\pi r^3$.

12. Two products are competing. The sales of product A are related to the sales of product B according to the following formula: $3\sqrt{A}+5\sqrt{B}=55$. When 64 units of product B are being sold, the sales are increasing at a rate of 4 units/day. At what rate are the sales of product A changing?

13. A resort community estimates that when the summer tourist population is x thousand, the income from parking meters will be $f(x)=\frac{1}{5}x^2+3x+12$ thousand dollars. Currently, the summer tourist population is 14,000 ($x=14$) and is increasing at 2000 per summer. At what rate will the meter income be changing, three summers from now?

14. A triangle has base length 30 cm. and altitude height 20 cm. Suddenly the dimensions begin to change. The base length increases at a rate of 5 cm/sec, and the altitude decreases at a rate of 2 cm/sec. At what rate is the area of the triangle changing three seconds after the triangle begins to transform? Is the area increasing or decreasing at this time?

15. A camper is frying bacon for breakfast. A bear, 7 miles due east of the campsite smells the bacon and decides to investigate. At exactly 8:00 a.m., the bear begins walking toward the campsite at a rate of 6 miles/hr. Meanwhile, the camper finishes his breakfast and decides to go for a hike. He leaves the campsite at exactly 8:30 a.m. and heads due north at a rate of 4 miles/hr. How fast is the distance between the bear and the camper changing at 9:00 a.m.?

16. A conical pile of sand was dumped at a construction site a few weeks ago. Originally it was 2 meters high and the radius of the base was 3 meters, so its volume was 6π cubic meters. Since then the pile has been slowly spreading and flattening. The pile remains conical in shape and the volume stays constant, but the height decreases at a rate of 0.1 meter/day. At what rate is the radius increasing after 10 days? Note: The volume of a cone with height h and radius r is $V=\frac{1}{3}\pi r^2 h$.

17. And now,...a“novel” word problem (long, but not difficult or particularly time-consuming):
Chapter 1 – The Accident

Once upon a time there was a BU student who was double-majoring in mathematics and music. His name was Alan. Alan also had another major interest. Her name was Joanna.

One evening, in a attempt to gain Joanna's affection, Alan decided to serenade her. As he strummed his guitar beneath Joanna's dorm window, singing his latest creation, *“My Love has no Limit …Yours is Undefined,”* he noticed that Joanna's window sill was exactly 71 feet above the ground.

Hearing the music, Joanna rushed to her window and flung it open, upsetting a potted geranium which fell from the sill with an initial velocity of $-\frac{1}{3}$ ft./sec. Exactly two seconds later it struck Alan on the head, sending him to his knees.

"Alan!" shrieked Joanna, "Say something!"

"The position above the ground of an object in vertical motion can be described by the function $s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$ where v_0 is the initial velocity and s_0 is the initial height," replied Alan.

"Oh, gee!" exclaimed Joanna desperately.

"G? oh, g is gravity. You can assume that $g = 32$," mumbled Alan before slipping into unconsciousness.

(a) Write the exact function that gives the geranium's position above the ground at time t (fill in the values for v_0, s_0, and g).

(b) Write the equation that gives the velocity of the geranium at time t.

(c) What was the velocity of the geranium when it struck Alan?

(d) How tall is Alan?

Chapter 2 – The Rescue

Attracted by the noise, three students, Wally, Nancy and Phoebe rushed to Alan.

"We need to get help," said Wally. I'll run due west to that blue emergency phone." He took off, running at a constant rate of 20 ft./sec.

"But there's a closer phone due north," said Nancy. So, exactly five seconds after Wally began to run, Nancy headed due north running at a constant rate of 15 ft./sec.

"Hmmm...," said Phoebe to no one in particular, "I guess I'll just call for help on my cell phone." She began to dial....

(a) How fast was the distance between Wally and Nancy changing exactly ten seconds after Nancy started to run? You should solve for the desired variable, so that your answer is a constant, but you do not need to simplify your answer or we'll never get Alan to the hospital in time.

Not to be outdone by Wally and Nancy, Joanna also ran ... down the stairs, out the door and then directly to the limp body of Alan. Her heart was beating wildly $\left(\frac{d\heartsuit}{dt} \approx 120 \text{ beats/min}\right)$. She waited anxiously as the ambulance drove towards them. She was frustrated as it stopped $\left(\frac{dA}{dt} = 0 \text{ mi/hr}\right)$ at each crosswalk to let people go by. Finally, (it seemed like $t \to \infty$), Alan was rushed to the hospital, with Joanna at his side, clutching the guitar.

Chapter 3 – Chez L'Hospital

Unable to sit still in the hospital waiting room, Joanna slowly paced along the corridor (in a perfectly straight line of course). Her position, measured in yards relative to a fixed point (the water cooler), at time t minutes would be described by Alan (if he were conscious) as $s(t) = t^3 - 9t^2 + 15t$ for $0 \leq t \leq 6$.

(a) How far is Joanna from the water cooler when she begins pacing? When she stops pacing?

(b) What is Joanna's velocity at time $t = 2$?

(c) What is Joanna's acceleration at time $t = 2$?

(d) At what time does Joanna reverse direction during her pacing?

(e) What is the total distance that Joanna walks during the six minutes?

Chapter 4 – The Concussion and Conclusion

A man in a white lab coat approached Joanna.

"Are you a friend of our head case?" he asked. "Good," he said when she nodded. "We could use your help. The patient keeps saying strange things. Can you come with me?"

Unsure what to expect, Joanna followed him. Alan was propped up in bed with a large bandage and ice pack on his head. He was pleased to see her, and the guitar.

"Joanna, I am so relieved that you are here. I am really OK but the doctors don't seem convinced."

"Does your head hurt a lot?" she cooed.

"Oh, it's much better now. I have a bump that is the shape of a perfect hemisphere. Apparently for the first six seconds after I was hit by the...what WAS that?"

"A geranium."

"Right. Well, as I was saying, for the first six seconds I could feel the radius of the bump increasing at a constant rate of $\frac{1}{2}$ cm/sec. Then I must have passed out. I was trying to tell these doctors what equation to use to determine the rate at which the volume of the bump was growing at that time but I wasn't sure to what degree of accuracy they wanted – because, of course, π is involved, and...and then they sent for you."

"It's alright, doctors," smiled Joanna. "I think Alan is back to normal. Thank you very much."

~ *The End* ~

(a) What is the equation that Alan was trying to tell the doctors (i.e., give the function that expresses the rate of change of the volume of the bump over time)?

(b) At what rate was the radius of the bump growing when Alan passed out?

(c) At what rate was the volume of the bump growing when Alan passed out?

(d) The bump stopped swelling when it reached a volume of $\frac{128\pi}{3}$ cm^3. How long was the swelling process?

Section 14 - Answers

1. $-\frac{5}{14}$ (Don't forget to use the product rule when differentiating.)
2. $\frac{dq}{dt} = -\frac{1}{9}$, so *decreasing* at rate of approximately 111 watches/wk.
3. (a) decreasing at rate of 3 cm./min. (b) decreasing at rate of 72 cm^2/min.
4. (a) 180 \$/day (b) 50 \$/day (c) 130 \$/day
5. 7.5 ft/sec
6. (a) 90π ft^2/sec. (b)540π ft^2/sec.
7. $\frac{17}{5}$ cm/sec
8. 7.5 ft/sec
9. Increasing at a rate of 54 patients per year.
10. It needs a domain restriction. Perhaps allow only populations greater than the current 40,000 people.
11. $\frac{1}{50\pi}$ cm/min
12. $-\frac{25}{6}$ units/day
13. increasing at \$22,000/summer
14. -10 cm^2/sec. Area is decreasing
15. $2/\sqrt{5}$ mi/hr.
16. $\frac{3\sqrt{2}}{20}$ meters/day
17. Chapter 1: (a) $s(t) = -16t^2 - \frac{1}{3}t + 71$ (b) $v(t) = -32t - \frac{1}{3}$ (c) $-64\frac{1}{3}$ ft/sec (d) $6\frac{1}{3}$ ft.= 6 ft., 4 in.
    Chapter 2: (a) $\dfrac{300(20) + 150(15)}{\sqrt{300^2 + 150^2}}$ ft/sec
    Chapter 3: (a) Joanna is *at* the water cooler when she begins pacing and is 18 yards away when she stops pacing. (b) -9 yds/min. (c) -6 yd/min^2. (d) $t = 1$ and $t = 5$ (e) 46 yds. (Find the distance walked in each direction separately and then add them.)
    Chapter 4: (a) $\frac{dV}{dt} = 2\pi r^2 \frac{dr}{dt}$ (b) $\frac{1}{2}$ cm/sec (given) (c) 9π cm^3/sec (d) 8 secs

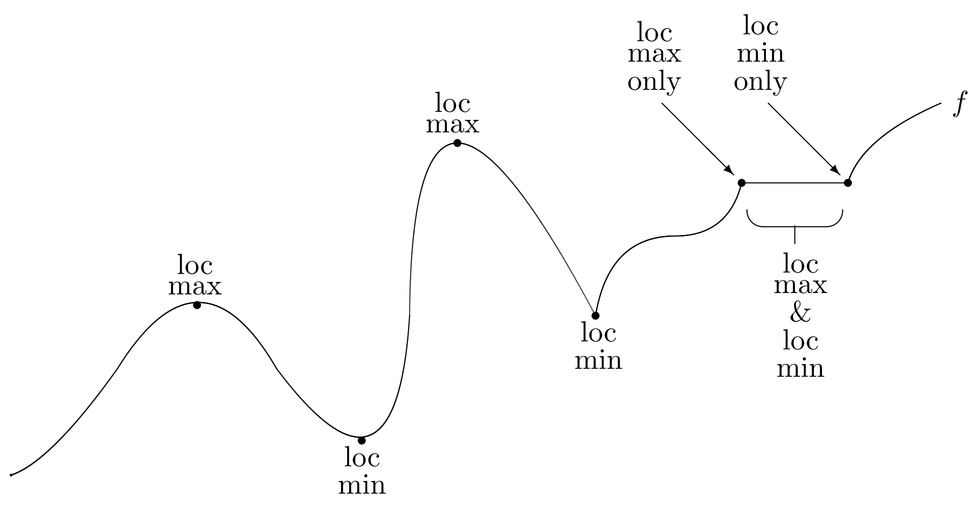
15 Local Maxima and Minima

Suppose f is a function whose domain and range are some subsets of $\mathbb{R}$. We say f has a *local maximum*[30] at $x = a$ if there is a number $\epsilon > 0$ such that $f(x) \leq f(a)$ whenever the distance in $\mathbb{R}$ from x to a is less than ϵ, i.e. when x lies in the open interval $(a - \epsilon, a + \epsilon)$. Similarly, f has a *local minimum* [31] at $x = b$ if there is a number $\epsilon > 0$ such that $f(x) \geq f(b)$ whenever the distance in $\mathbb{R}$ from x to b is less than ϵ.

To say that $f(x)$ has a local maximum at a, is to say that its y-value, $f(a)$, is greater than or equal to the y-values for all points x "near" a. But "further away" from a there might well be points x with y-values greater than $f(a)$. Since we are only concerned with x values "close" to a, the word "local" is appropriate. A good analogy is a range of mountains. You may be on top of one of the peaks but there are probably higher peaks in the range: you are at a place where the "height-above-sea-level" function has a local maximum. Your friend may be in a crater, but there may be deeper craters in the area. Your friend is at a place where the "height-above-sea-level" function has a local minimum.

Just as a mountain range may have more than one peak or valley, a function may have more than one local maximum or local minimum. The plural for "maximum" is "maxima" and the plural for "minimum" is "minima." Sometimes we will wish to speak of these extremes together without having to say "maximum or minimum." We use the term "local extremum" (or the plural "local extrema") to indicate that we are talking about either a local maximum or a local minimum.

Notice that the definition of local extrema is not a strict inequality. For local maximum we have $f(x)$ is less than OR EQUAL TO $f(a)$. So, if the mountain top on which you are standing is in fact a plateau, you are still at the site of a local maxium. If your friend is standing on a level valley floor, he is still at the site of a local minimum.



Local Extrema for Function f

[30]Some texts say *relative maximum*; We will use the words "local" and "relative" interchangeably.
[31]or *relative minimum*

Question. How do we find the local extrema (that is, the local maxima and/or local minima) of some function f?

The first part of the answer is to reduce the list of possible places to a short list.

Theorem 15.1. *If f is differentiable at the point a in its domain and if f has a local extremum at a then $f'(a) = 0$.*

Proof. We are given a function f that is differentiable at $x = a$ and has a local maximum at $x = a$. Suppose $f'(a) \neq 0$. Then either $f'(a) > 0$ or $f'(a) < 0$. If $f'(a) > 0$ then $\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} > 0$. Choosing h sufficiently small and positive this says $f(a+h) > f(a)$. This conclusion contradicts the fact that f has a local maximum at a. On the other hand, if $f'(a) < 0$ then $\frac{f(a+h) - f(a)}{h} < 0$ when h is sufficiently near 0. Taking h to be sufficiently near 0 and negative gives (multiplying across the inequality[32] by the negative number h) $f(a+h) - f(a) > 0$, i.e. $f(a+h) > f(a)$. Again, this is a contradiction to our given. Thus our "suppose" at the beginning of this proof leads to a contradiction. A similar argument can be made for the case where f has a local minimum at a. The assumption that "$f'(a) \neq 0$" leads to a contradiction. So it must be true that if f has a local maximum or a local minimum at a then $f'(a) = 0$. $\square$

Example 15.1. Let $f(x) = 6x^3 - 3x^2 - 12x - 4$ with domain the open interval $(0, 5)$. Then

$$\begin{aligned} f'(x) &= 18x^2 - 6x - 12 \\ &= 6(3x^2 - x - 2) \\ &= 6(3x + 2)(x - 1). \end{aligned}$$

So $f'(x)$ can only be 0 when $x = 1$ or $-\frac{2}{3}$. Now $-\frac{2}{3}$ is not in $(0, 5)$. So the only possible local maximum or local minimum of f on $(0, 5)$ is at $x = 1$.

This example leaves two questions unanswered:

(i) Even though $f'(1) = 0$, might it be the case that f has neither a local maximum nor a local minimum at $x = 1$?

(ii) Even if some knowing person tells you that f has either a local maximum or a local minimum at $x = 1$, how would you tell which?

These questions are answered in Section 17.

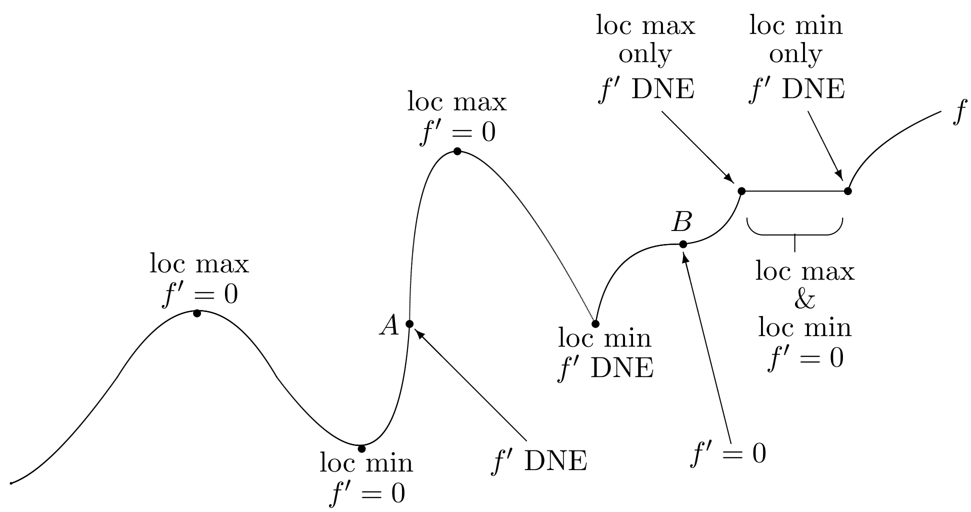
An equivalent, and perhaps more easily applied, statement of Theorem 15.1 is:
If a function f has a local extremum at $x = a$, then either $f'(a) = 0$ or $f'(a)$ does not exist.

Warning: You must be careful with Theorem 15.1. It does NOT say "If $f'(a) = 0$ or $f'(a)$ does not exist, then f has a local extremum at $x = a$. " This theorem is often misapplied by students

[32] Recall: if $a < b$ then $-a > -b$, so multiplying across an inequality by a negative number changes $<$ into $>$ and $>$ into $<$. Similarly for $\leq$ and $\geq$.

who do not pay careful attention to which part is the "if" statement of the theorem and which part is the "then" statement. The classic example that shows that Theorem 15.1 doesn't work in the opposite direction is $g(x) = x^3$ with $a = 0$. Here, $g'(x) = 3x^2$ so $g'(0) = 0$, yet g does not have a local maximum or a local minimum at 0. The graph of $g(x) = x^3$ near $x = 0$ behaves like the graph of f (below) near point B.

Below is a copy of the local extrema graph shown earlier (page 125). In keeping with Theorem 15.1, wherever f has a local extremum, $f' = 0$ or f' does not exist. Notice also that there are two points, marked A and B, where there is no local extremum.



Local Extrema and Critical Points for Function f

If a is in the domain of f, and if $f'(a) = 0$ or $f'(a)$ does not exist, we say that a is a *critical point* of f, and the number $f(a)$ is a *critical value* of f. Theorem 15.1 says that if f has a local extremum at a then a is a critical point of f. So, to find local extrema, we need to look only at the critical points of f. These are our only candidates for the locations of local extrema.

Example 15.2. Find all of the critical points for $f(x) = x^3 + 3x^2 - 24x$.
Answer: $f'(x) = 3x^2 + 6x - 24 = 3(x^2 + 2x - 8) = 3(x + 2)(x - 4)$

$f'(x) = 0$ at $x = -2$ and $x = 4$

$f'(x)$ is defined on $\mathbb{R}$ (there are no places where $f'(x)$ does not exist, but we had to check).

So, critical points are $x = -2$ and $x = 4$.

Example 15.3. Find all of the crical points for $f(x) = \sqrt[3]{x^2 - x}$.
Answer: $f'(x) = \frac{1}{3}(x^2 - x)^{-\frac{2}{3}}(2x - 1) = \dfrac{2x - 1}{3\sqrt[3]{(x^2 - x)^2}}$

$f'(x) = 0$ at $x = \frac{1}{2}$

$f'(x)$ D.N.E. at $x = 0$ and $x = 1$

So, critical points are $x = \frac{1}{2}$, $x = 0$ and $x = 1$

Example 15.4. Find all of the critical points for $f(x) = x^{\frac{1}{3}} - x^{-\frac{2}{3}}$
Answer: $f'(x) = \frac{1}{3}x^{-\frac{2}{3}} + \frac{2}{3}x^{-\frac{5}{3}} = \frac{1}{3}x^{-\frac{5}{3}}(x+2)$

$f'(x) = 0$ at $x = -2$

$f'(x)$ D.N.E. at $x = 0$, but since $x = 0$ is not in the domain of f, it is not a critical point.

So, the only critical point is $x = -2$

Example 15.5. Find all of the critical points for $f(x) = 2^x \cdot x$
Answer: $f'(x) = 2^x(\ln 2)x + 1 \cdot 2^x = 2^x(x \ln 2 + 1)$

$f'(x) = 0$ at $x = -\frac{1}{\ln 2}$

$f'(x)$ is defined on $\mathbb{R}$.

So, the only critical point is $x = -\frac{1}{\ln 2}$

Section 15 - Exercises (answers follow)

1. Sketch the graph of a function that has the given domain and extrema.

   (a) Domain $[-2, 4)$ local maximum only at $x = 0$; local minimum only at $x = 3$

   (b) Domain $\mathbb{R}$. f has three local maxima and two local minima.

   (c) Domain $(2, 6)$; Each x-value in the domain is BOTH a local max and a local min.

   (d) Domain $[1, 5]$; no local extrema

2. For functions (a) through (f), sketch the graph. From your graph, decide at which values of x the function has local maxima or local minima.

   (a) $f(x) = -x^2$

   (b) $f(x) = |x|$

   (c) $f(x) = x^3$

   (d) $f(x) = \frac{1}{x}$ where $x \geq 2$

   (e)
   $$f(x) = \begin{cases} 1 & x \text{ is an integer} \\ 0 & x \text{ is not an integer} \end{cases}$$

   (f)
   $$f(x) = \begin{cases} x & x \text{ is an integer} \\ 0 & x \text{ is not an integer} \end{cases}$$

3. The following functions are impossible to sketch reasonably because between any two rational numbers there is an irrational number (and vice-versa). "Mentally" sketch these graphs and determine where each function has local extrema.

   (a)
   $$f(x) = \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$$

   (b)
   $$f(x) = \begin{cases} x & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}$$

4. For each function, find its critical points and corresponding critical values.

   (a) $f(x) = x^2 - 10x - 8$

   (b) $f(x) = \frac{1}{4}x^4 - 6x + 2$

   (c) $f(x) = x^3 - x^2 - 1$

   (d) $g(t) = 15t^4 - 15t^2 - 90$

   (e) $f(x) = \sqrt{x}$

(f) $f(x) = \dfrac{x^2 - 2x + 1}{x - 3}$

(g) $f(x) = \dfrac{x + 1}{x^2 + x + 1}$

(h) $f(x) = x + 3x^{\frac{2}{3}}$

(i) $f(x) = 4 + (x - 1)^{5/3}$

(j) $f(v) = v^2 - 4\sqrt{v}$

(k) $f(x) = x^2\sqrt{x - 4}$

(l) $f(x) = x \ln x$

Section 15 - Answers

1. (a) Graphs will vary.
   (b) Graphs will vary.
   (c) Graph is horizontal line on domain.
   (d) Graphs will vary.

2. (a) local max at $x = 0$ no local min
   (b) local min at $x = 0$ no local max
   (c) no local extrema
   (d) no local extrema
   (e) local max at every integer local max and local min at every non-integer
   (f) local max and min at every non-integer and at $x = 0$; local max at every positive integer; local min at every negative integer

3. (a) local max at every rational number local min at every irrational number
   (b) local max at every negative irrational number local min at every positive irrational number

4. (a) $(5, -33)$
   (b) $\left(\sqrt[3]{6}, \frac{-9\sqrt[3]{6}}{2} + 2\right)$
   (c) $(\frac{2}{3}, -\frac{31}{27})$ and $(0, -1)$
   (d) $(0, -90)$, $\left(\frac{1}{\sqrt{2}}, \frac{-375}{4}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{-375}{4}\right)$
   (e) $(0, 0)$
   (f) $(1, 0)$ and $(5, 8)$
   (g) $(0, 1)$ and $\left(-2, -\frac{1}{3}\right)$
   (h) $(-8, 4)$ and $(0, 0)$
   (i) $(1, 4)$
   (j) $(1, -3)$ and $(0, 0)$
   (k) $(4, 0)$ (Note: $x = 0$ and $x = \frac{16}{5}$ are not in the domain.)
   (l) $\left(\frac{1}{e}, -\frac{1}{e}\right)$

16 Some Useful Theorems: IVT, EVT, Rolle's Thm, MVT

In this section we look at four theorems. Each one is important to the study of calculus. The proofs for some of the theorems are given. You are not expected to reproduce the proofs, but you should read and be able to follow them. It is important that you understand the meaning of the theorems. As we study calculus, the important implications of these theorems should become apparent.

As with any theorem, these theorems have certain hypotheses (the "if" statements) that must be met before the conclusions (the "then" statements) are valid. Do not gloss over the hypotheses. Many times you hear,"Two negatives make a positive," or "$a^2+b^2=c^2$." These last two statements are not true in general. " *If the operation is multiplication*, two negatives make a positive" or "*If c represents the length of the hypotenuse of a right triangle and a and b represent the lengths of the other two sides, then* $a^2+b^2=c^2$" are true. The hypotheses state specific conditions under which a theorem is true.

A good way to help you understand these theorems is for you to draw graphs of functions that meet the criteria of the hypotheses and then verify that your drawing indeed satisfies the conclusions of the theorem. These graphs are not proofs but they can give you some insight as to what is being claimed by the theorems. All four of the theorems require in their hypotheses that f be a continuous function. When you are sketching graphs, recall that a continuous function can be drawn without lifting the pencil. Some theorems require in their hypotheses that f be differentiable. Recall that a differentiable function must be continuous and that it doesn't have any sharp corners or cusps; it is nice and smooth. You certainly don't have to have an algebraic expression for the graphs you draw. Any graph you draw that satisfies the vertical line test qualifies as a function.

Theorem 16.1. ***Intermediate Value Theorem***

Suppose that f is a continuous function defined on the closed interval $[a,b]$ and that $f(a) \neq f(b)$. Then for every number N between $f(a)$ and $f(b)$, there must be some number c in (a,b) where $f(c)=N$.

We will not prove the Intermediate Value Theorem, but if you draw a graph that meets the criteria, you can easily see that it must be true. Start by plotting the two points $(a,f(a))$ and $(b,f(b))$ on your graph and then connect them with a continuous line of any desired curvature. Make sure that $f(a)$ is not equal to $f(b)$. You have met the criteria for the theorem. The conclusion says that for every y value between $f(a)$ and $f(b)$, those y values must be represented by a point on your graph. Does yours do that? Do you see why the condition of continuity is essential?

A key use for this theorem is to enable us to establish that a particular function must have roots (x-intercepts). We wish to say that if a function is continuous and that if some of its y values are negative and some are positive, then it must have a y value of zero somewhere.

Example 16.1. Prove that the graph of the function $f(x)=x^3+x+7$ crosses the x-axis at least once.

Proof. The function f is a polynomial, so it is continuous on $\mathbb{R}$ and therefore continuous on the closed interval $[-2,1]$. We calculate that $f(-2)=-3$ and $f(1)=9$. Zero is a number between -3

and 9. So, by the Intermediate Value Theorem, there must be some x value in the interval $(-2, 1)$ whose y value is 0. Thus the graph of f must cross the x-axis. $\square$

In Example 16.1 we somewhat arbitrarily chose the closed interval $[-2, 1]$. Any interval $[a, b]$ would do as long as $f(a)$ and $f(b)$ had opposite signs.

Example 16.2. Suppose f is a continuous function and its only roots are a and b. Prove that either $f(x) > 0$ for all x in (a, b), or else $f(x) < 0$ for all x in (a, b).

Proof. We do this proof by contradiction. Suppose there is some c in (a, b) where $f(c) > 0$, and some d in (a, b) where $f(d) < 0$. Since f is continous on $[a, b]$, it must be continuous on the subset $[c, d]$ (or $[d, c]$ of d happens to be less than c). We know that zero is between $f(d)$ and $f(c)$. So, by the Intermediate Value Theorem, there must be some x-value between c and d whose y-value is zero. But that would mean that f has a root between c and d. This is impossible because the only roots of f were given to be a and b, neither of which is between c and d. $\square$

The result from Example 16.2 will be very important to us. It essentially says that if a function is continuous, then for all of the x values between two consecutive roots, their corresponding y values will be either all positive or all negative. A sketch or two should convince you of this. Indeed its truth then becomes obvious. But its significance should not be underestimated.

Theorem 16.2. ***Extreme Value Theorem***

Suppose that f is a continuous function defined on the closed interval $[a, b]$. Then there must be some numbers M and m in $[a, b]$ such that $f(m) \leq f(x) \leq f(M)$ for all x in $[a, b]$.

The "English translation" of the conclusion to this theorem is that f *must* have both a maximum and minimum y-value at some x values in $[a, b]$. Sketch a few graphs to convince yourself that this is true. We don't prove this theorem here.

There will be times when we will be wanting to find a maximum (or minimum) y value for a function. Not all functions have these extrema. The Extreme Value Theorem tells us that if we are dealing with a function that meets the conditions of the hypotheses, then we will be guaranteed that the extrema do exist.

This theorem is also essential for the proof of the Mean Value Theorem, further below.

Example 16.3. Prove: The function $f(x) = \dfrac{x^2 - 1}{x + 2}$, defined on the interval $[0, 3]$ has a largest y value and a smallest y value.

Proof. f is a rational function with domain $(-\infty, -2) \cup (-2, \infty)$. Rational functions are continuous on their domain, so f is certainly continuous on $[0, 3]$. By the Extreme Value Theorem, f must have a maximum value and a minimum value. $\square$

Notice that the Extreme Value Theorem doesn't tell us what the extreme y values are, or what the corresponding x values are. The theorem tells us only that they do exist. Indeed all of the four theorems in this section are what we call "existence" theorems. Their conclusions guarantee us the existence of certain entities, but don't tell us what they are or specifically where they are located.

Theorem 16.3. ***Rolle's Theorem***[33]

Suppose that f is a continuous function defined on the closed interval $[a, b]$ and that $f(a) = f(b)$. Further, suppose that f is differentiable on the open interval (a, b). Then there must be some number c in the open interval (a, b) where $f'(c) = 0$.

Proof. If $f(x) = f(a) = f(b)$ for all x in (a, b) then f is constant. So $f'(x) = 0$ for all x in (a, b). If $f(x)$ is not constant then, by the Extreme Value Theorem (using the same notation as in the theorem as stated above), either $f(m) < f(a) = f(b)$ or $f(M) > f(a) = F(b)$ or both. If $f(M) > f(a) = f(b)$ then M lies in the open interval (a, b). Now, since f is given to be differentiable on (a, b), Theorem 15.1 guarantees that $f'(M) = 0$. Similarly, if $f(m) < f(a) = f(b)$ then m lies in the open interval (a, b), and Theorem 15.1 gives us $f'(m) = 0$. Thus all cases are covered. $\square$

Example 16.4. Verify that $f(x) = x^3 - x$ defined on $[-1, 1]$ meets the criteria for Rolle's Theorem and find the c referred to in the conclusion of the theorem.
Answer: f is continuous on $[-1, 1]$ because it is a polynomial. f is differentiable on $(-1, 1)$ because it is a polynomial. $f(-1) = (-1)^3 - (-1) = 0$ and $f(1) = (1)^3 - (1) = 0$. So, the criteria for Rolle's Theorem are met. According to the theorem, there must exist some x value, c, in $(-1, 1)$ where $f'(c) = 0$.

$f'(x) = 3x^2 - 1$. $f'(c) = 0 = 3c^2 - 1 \Longrightarrow c = \pm\sqrt{\frac{1}{3}}$. So, we found two values of c. Both are in the interval $(-1, 1)$.

Theorem 16.4. ***Mean Value Theorem***

Suppose that f is a continuous function defined on the closed interval $[a, b]$. Further, suppose that f is differentiable on the open interval (a, b). Then there must be some number c in the open interval (a, b) where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Proof. Given the function f that meets the hypotheses above, we can define a function g that includes f. We let $g(x) = f(x) - f(a) - \dfrac{f(b) - f(a)}{b - a}(x - a)$. Notice that $g(a) = 0$ and $g(b) = 0$. Also, g is continuous on $[a, b]$ because g is the sum of two functions, f and polynomial $-f(a) - \dfrac{f(b) - f(a)}{b - a}(x - a)$, that are each continuous on $[a, b]$. For the same reason, g is differentiable on (a, b). So, g meets the requirements for Rolle's Theorem. We can therefore conclude that there must exist some c in (a, b) where $g'(c) = 0$. But $g'(c)$ is simply $f'(c) - 0 - \dfrac{f(b) - f(a)}{b - a}$. So, this c that must exist is one where $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. $\square$

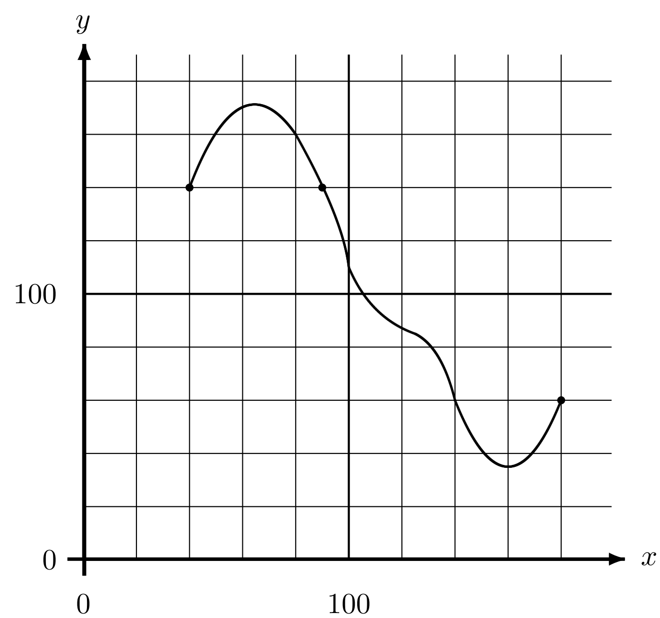
The Mean Value Theorem may look like just a lot of symbols, but look more closely at the conclusion equation $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. The right hand side is simply the slope of the line segment that goes from $(a, f(a))$ to $(b, f(b))$. That slope is the *average* rate of change of f over the interval $[a, b]$. What is $f'(c)$? This is the *instantaneous* rate of change at the point $(c, f(c))$. So, what the MVT is saying is that, if a function is continuous over a closed interval and differentiable

[33] "Rolle" rhymes with "foal,. not "folly."

on the interior of that interval, then there is some place inside that interval where the instantaneous rate of change is equal to the average rate of change over the entire interval.

Rolle's Theorem is really just a special case of the Mean Value Theorem. In Rolle's Theorem $f(a) = f(b)$, so the Mean Value Theorem conclusion equation $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ simply becomes $f'(c) = 0$.

Example 16.5. Consider the function below. Without doing any calculations, (a) estimate the values of c that satisfy the conclusion of the Mean Value Theorem for the interval $[40, 180]$, and (b) estimate the values of c that satisfy the conclusion of Rolle's Theorem for the interval $[40, 90]$.



(a) Answer: Sketch a line segment from the point $(40, 140)$ to the point $(180, 60)$. Then find any places on the graph where it looks like the tangent lines would be parallel to this line segment (using a straight-edge can help, but the objective here is thoretical understanding, not absolute precision). It looks like there are three values of c: $\approx 70, 117,$ and 137.

(b) Answer: The line segment between $(40, 160)$ and $(90, 160)$ is horizontal because those points have the same y value (as required by Rolle's Theorem). There is only one value c in interval $(40, 90)$ where the tangent line has slope 0. $c \approx 64$.

Example 16.6. Verify that $f(x) = \dfrac{x+4}{x}$ defined on $[1, 4]$ satisfies the hypotheses of the Mean Value Theorem. Then find all values of c described by the conclusion of the theorem.
Answer: $f'(x) = \dfrac{x - (x+4)}{x^2} = \dfrac{-4}{x^2}$. $f'(x)$ exists for every x in $\mathbb{R}$ except $x = 0$. Since f is continuous wherever it is differentiable, we conclude that f must be continuous on $[1, 4]$ and differentiable on $(1, 4)$. So, according to the Mean Value Theorem, there must be some x value, c, in $(1, 4)$ where $f'(c) = \dfrac{f(4) - f(1)}{4 - 1} = \dfrac{2 - 5}{3} = -1$.

$f'(c) = \frac{-4}{c^2} = -1 \Longrightarrow c = \pm 2$. But since -2 is not in the interval $(1,4)$, the only value referred to by the MVT is $c = 2$.

Are you feeling overwhelmed?

Again, you don't need to memorize the proofs of the theorems. It is sufficient that you have an idea of what they are saying and how to use them.

The names of these theorems are similar, but they are descriptive, so they can be kept straight. The *Intermediate* Value Theorem refers to the existence of a point on the graph whose y value is between (*intermediate* to) the y values $f(a)$ and $f(b)$. The *Extreme* Value Theorem refers to the existence of maximum and minimum (*extreme*) y values on an interval. The *Mean* [34] Value Theorem refers to an average (*mean*) rate of change.

Section 16 - Exercises (answers follow)

1. Tell which one of the four theorems studied in this section is represented by each of the following scenarios.
   (a) An elevator travels from the second floor of a building to the tenth floor without stopping. It must pass by the fifth floor.
   (b) Jim and Julie are on a roller-coaster. Julie screams when their car is at the highest point of the ride. Jim mocks her by screaming when the car is at the lowest point of the ride. Both Jim and Julie will be able to scream at least once during the ride.
   (c) Sasha drove her car from Owego to Binghamton. The ten-mile trip took twenty minutes. At some time during the trip, the car's speedometer must have read exactly 30 miles/hr.
   (d) On Timmy's second birthday he was exactly 3 feet tall. On Tim's twenty-second birthday, he refused to be called Timmy any longer because he was then exactly 6 feet tall. At some time between Tim's second and twenty-second birthdays he was exactly 4 ft., $6\frac{1}{2}$ inches tall.
2. What is the difference in hypotheses (the "if" statements) between Rolle's Theorem and the Mean Value Theorem?
3. Sketch a graph that shows that the Intermediate Value Theorem is not true if the hypothesis "f is continuous" is left out.
4. Sketch a graph that shows that the Extreme Value Theorem is not true if the function f is defined on an interval that is not closed.
5. Sketch a graph that shows that Rolle's Theorem is not true if the hypothesis "f is differentiable on (a,b)" is left out.

[34] "Mean" refers to the mathematical synonym for "average"; it is not an indication of the temperment of the theorem.

6. Carefully (reasonably accurately), sketch a graph of $f(x) = \sqrt{x}$ on the interval $[0, 4]$.
   (a) Verify that f meets the requirements of the Mean Value Theorem.
   (b) Use your graph to visually estimate any value(s) c referred to by the MVT. (Draw a line segment from $(0, 0)$ to $(4, 2)$ and look for a place on the graph of f where the tangent line is parallel to the line segment.)
   (c) Check your visual estimate by algebraically solving the MVT conclusion equation for c.
7. Given $f(x) = x^2 - 3x + 1$
   (a) Use the Intermediate Value Theorem to prove that f must have a root in the interval $(2, 5)$.
   (b) Use Rolle's Theorem to prove that f cannot have more than one root in the interval $(2, 5)$. Hint: Assume that there are two roots and arrive at a contradiction.
8. Use the Intermediate Value Theorem to prove that $f(x) = -x^3 + 2x + 5$ has a positive root.
9. Use the Mean Value Theorem to prove that if $f'(x) = 0$ for all x in $[3, 7]$, then $f(3) = f(7)$.
10. Verify the hypotheses of the Mean Value Theorem for each function below. Then find any value(s) "c" referred to by the theorem.
    (a) $f(x) = \sqrt{x+1}$ on $[3, 8]$
    (b) $f(x) = \dfrac{x-1}{x+1}$ on $[0, 3]$
    (c) $f(x) = \dfrac{x^2 - 2x - 3}{x+4}$ on $[-1, 3]$
    (d) $f(x) = x^{\frac{3}{4}} - 2x^{\frac{1}{4}}$ on $[0, 4]$ Hint: It will be easier to evaluate $f(4)$ if you first factor f.
11. Which of the functions in Problem 10 is an example of Rolle's Theorem?
12. For each of the following functions, there is no c in (a, b) where $f'(c) = \frac{f(b)-f(a)}{b-a}$. Explain why this is not a contradiction to the Mean Value Theorem.
    (a) $f(x) = \dfrac{3}{x-2}$ with domain $[1, 3]$
    (b) $f(x) = |x|$ with domain $[-2, 4]$

Section 16 - Answers

1. (a) Intermediate Value Theorem
   (b) Extreme Value Theorem
   (c) Mean Value Theorem
   (d) Intermediate Value Theorem
2. Rolle's Theorem requires that $f(a)$=$f(b)$.

3., 4., 5. Graphs will vary.

6c. $c = 1$

7a. Hint: Follow the pattern of Example 16.1

7b. You already had one hint, but here's another: If a and b are roots, then $f(a) = 0$ and $f(b) = 0$.

8. Hint: Since an interval was not given, you need to come up with your own interval. You need to choose your interval so that your root is positive.

9. Oh, c'mon! You shouldn't need a hint for this one.

10. (a) $c = \frac{21}{4}$ (b) $c = 1$ (note: -3 is not in the required interval, so it is not a number referred to by the MVT). (c) $c = -4 + \sqrt{21}$ (note: $-4 - \sqrt{21}$ isn't in the required interval) (d) $c = \frac{4}{9}$

11. (c) and (d) Note: In both of these problems it was true that $f(a)$ and $f(b) = 0$. That is not required of Rolle's Theorem. All that is required is that $f(a) = f(b)$.

12. (a) f is not defined at $x = 2$, so f is not continuous on $[1, 3]$.
(b) f is not differentiable at $x = 0$, so f is not differentiable on $(-2, 4)$.

17 Increasing and Decreasing; First Derivative Test

Calculus helps us to discover the shape of the graph of any differentiable function f. One question we can easily answer about the graph is: For what values of x is $f(x)$ increasing and for what values of x is $f(x)$ decreasing?

When we say "increasing" or "decreasing" we are talking about viewing the graph from left to right. The graph is increasing when the curve is going up; i.e., when the y values are getting larger as we read from left to right. We can state this mathematically: A function f is increasing on (domain) interval I if for any points a and b in I, when $a < b$ then $f(a) < f(b)$.

Similarly, a function f is decreasing on interval I if for any points a and b in I, when $a < b$ we have $f(a) > f(b)$. When reading from left to right, the y values of the function are decreasing.

Theorem 17.1. *If I is an open interval throughout which $f'(x) > 0$ then f is increasing on I. If $f'(x) < 0$ throughout I then f is decreasing on I.*

The proof for Theorem 17.1 is by contradiction. It follows directly from the Mean Value Theorem.

Proof. Suppose $f'(x) > 0$ on I but f is not increasing on I. Then there would be two numbers, a and b, in I, such that $a < b$ and $f(a) \geq f(b)$. So $\dfrac{f(b) - f(a)}{b - a} \leq 0$.

Since f is differentiable on I, it is both continuous and differentiable on the closed interval $[a, b]$, a proper subset of I. We now have met the requirements for the Mean Value Theorem and invoke it to claim that there must be some c in (a, b) where $f'(c) = \dfrac{f(b) - f(a)}{b - a} \leq 0$. But, $f'(c) \leq 0$ contradicts the given fact that $f'(x) > 0$ for all $x \in I$.

A similar argument works in the decreasing case, and is left as an exercise. □

It is reasonable to ask if the converse of Theorem 17.1 is true. In other words, if f is differentiable on the open interval I and is increasing on I, is it true that $f'(x) > 0$ for all $x \in I$? The answer is NO. Look at $I = (-1, 1)$ and $f(x) = x^3$. Then f is increasing on I, but $f'(0) = 0$. However a modified converse is true:

Theorem 17.2. *If I is an open interval on which f is increasing, then $f'(x) \geq 0$ for all $x \in I$.*

Proof. The proof of this is also by contradiction. Suppose $f'(a) < 0$ for some $a \in I$. Then, using an argument similar to that used in the proof of Theorem 15.1, there exists a positive number h such that $f(a + h) < f(a)$. This contradicts the fact that f is increasing on I. □

Solving Inequalities

There are a variety of ways in which students are taught to solve inequalities. One way that is especially useful here, involves the result that follows Example 16.2. That result claimed that if a function is continuous, then for all of the x values between two consecutive roots, their corresponding y values will either be all positive or all negative.

With this in mind, then, we only need to find all of the roots of a function and all of its points of discontinuity, divide the domain of the function by these points, and then check each resulting interval to see if the function is positive or negative there.

If the function that we are investigating is a derivative function, f', then finding the roots and the points where the function is discontinuous is the same as finding the critical points of f. We then divide the domian of f into intervals, using the critical points as interval breakers. Finally, we check each interval to see if the y values of f' are positive or negative. This tells us where f is increasing or decreasing.

Example 17.1. On what intervals is the function $f(x) = \frac{3}{2}x^2 - 7x + 2$ increasing? decreasing?
Answer: $f'(x) = 3x - 7$. $f'(x)$ exists everywhere. $f'(x) = 0$ at $x = \frac{7}{3}$. So, the only critical point for f is $x = \frac{7}{3}$. We divide the domain of f, which is $\mathbb{R}$, into two pieces, split by the critical point: $(-\infty, \frac{7}{3})$ and $(\frac{7}{3}, \infty)$. Now we check each interval to see if f' is positive or negative there. $x = 0$ is in the interval $(-\infty, \frac{7}{3})$. $f'(0) = -7 < 0$. $x = 100$ is in the interval $(\frac{7}{3}, \infty)$. $f'(100) > 0$. So, f is decreasing on the interval $(-\infty, \frac{7}{3})$ and increasing on interval $(\frac{7}{3}, \infty)$.

In the previous example, we used $x = 0$ as a test value for the interval $(-\infty, \frac{7}{3})$. We could have used any x value in the interval and gotten the same result. Again, this is due to the result that follows Example 16.2 and the fact that f' is continuous on $\mathbb{R}$.

You can save time and calculations if you choose your test values cleverly. We are only concerned about the sign (+ or $-$), of f' at the test point; we are not interested in the actual numeric value. So, choose test points that will make the determination of sign easy.

Example 17.2. On what intervals is $f(x) = \dfrac{1}{x}$ increasing? decreasing?
Answer: $f'(x) = -\frac{1}{x^2}$. $f'(x)$ exists for all x in the domain of f. $f'(x)$ is never zero. So, f has no critical points. The domain of f is $(-\infty, 0) \cup (0, \infty)$. So, we test each of these intervals in f'. We can see tht $f'(x)$ is always negative, so f is decreasing on each interval. f is decreasing on $(-\infty, 0) \cup (0, \infty)$.

In the previous example, notice that do not say that f is decreasing on its domain. This would imply that $f(-2) > f(5)$. (See graph of f on page 20 to see that this is not true).

Example 17.3. On what intervals is $f(x) = \dfrac{x^2 - 2x + 1}{x - 3}$ increasing? decreasing?
Answer: $f'(x) = \dfrac{(2x-2)(x-3) - (x^2 - 2x + 1)}{(x-3)^2} = \dfrac{x^2 - 6x + 5}{(x-3)^2} = \dfrac{(x-1)(x-5)}{(x-3)^2}$. The critical points are $x = 1$ and $x = 5$. The domain of f is $(-\infty, 3) \cup (3, \infty)$. We divide the domain into intervals, using the critical points as break points. This gives us $(-\infty, 1)$, $(1, 3)$, $(3, 5)$ and $(5, \infty)$. Testing, we get $f'(0) > 0$, $f'(2) < 0$, $f'(4) < 0$ and $f'(10) > 0$. Thus f is increasing on $(-\infty, 1) \cup (5, \infty)$ and decreasing on $(1, 3) \cup (3, 5)$.

In the previous example, the derivative looks imposing. Again, we only care about the sign. So, it is helpful to use the factored form of the derivative for the testing. Notice that the denominator is always positive because it is squared. The numerator, then, is the important part. At $x = 4$ for instance we see that the first factor is positive and the second factor is negative. Thus their product is negative. We don't care that $f'(4) = -3$; we only care that it is negative.

Example 17.4. On what intervals is the function $f(x) = xe^x$ increasing? decreasing?
Answer: $f'(x) = e^x + xe^x = e^x(1 + x)$. Since e^x is always positive, the only critical point is $x = -1$.

The domain of f is $\mathbb{R}$, so we are interested in the intervals $(-\infty,-1)$ and $(-1,\infty)$. Since the factor e^x is always positive, it is easy to see that $f' < 0$ on $(-\infty,-1)$ and $f' > 0$ on $(-1,\infty)$. Thus f is decreasing on $(-\infty,-1)$ and increasing on $(-1,\infty)$.

We can now answer the unresolved questions (i) and (ii) in Section 15.

We know from Theorem 15.1 that if f has a local extremum, then it will occur at a critical point. We have now seen that if f changes from increasing to decreasing, or from decreasing to increasing, it will do so at a critical point, or at a gap in $\mathbb{R}$ where the function is undefined.

In the case where f changes from increasing to decreasing at a critical point, there is a local maximum at that point. In the case where f changes from decreasing to increasing at a critical point, then there is a local minimum at that point. If f does not change direction at a critical point, then there is no local extremum at that critical point. Since we determine increasing and decreasing by using the first derivative of a function, we ultimately have a way of using the derivative to determine whether or not a critical point is the location of a local extremum. This method is called the First Derivative Test.

First Derivative Test: If a is a critical point of f and the sign of f' changes from positive to negative at a, then f has a local maximum at a. If the sign of f' changes from negative to positive at a, then f has a local minimum at a.

The First Derivative Test is valid for critical points where the derivative is zero as well as critical points where the derivative is undefined. Be reminded, however, that a critical point must be a point in the domain of the function.

Lets look back at the previous examples in this section and determine where these functions have local extrema.

Revisit Example 17.1: The only critical point for $f(x) = \frac{3}{2}x^2 - 7x + 2$ was $x = \frac{7}{3}$. Since $f'(0) < 0$ and $f'(100) > 0$, f' changed from negative to positive at $\frac{7}{3}$. So, f has a local minimum at $x = \frac{7}{3}$

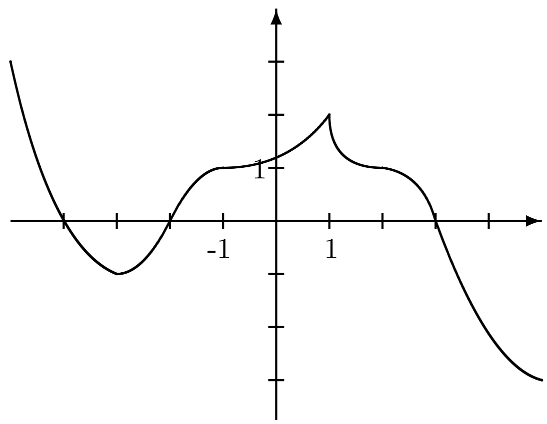
Revisit Example 17.2: $f(x) = \frac{1}{x}$ had no critical points, so it has no local extrema.

Revisit Example 17.3: The function $f(x) = \dfrac{x^2 - 2x + 1}{x - 3}$ had two critical points, $x = 1$ and $x = 5$. We are careful that our domain interval has a break at $x = 3$, but this is not a critical point because it is not in the domain of f. We found $f'(0) > 0$ and $f'(2) < 0$ (a change from positive to negative) so f has a local maximum at $x = 1$. We found $f'(4) < 0$ and $f'(10) > 0$, (a change from negative to positive) so f has a local minimum at $x = 5$.

Revisit Example 17.4: The function $f(x) = xe^x$ had one critical point, $x = -1$. f' changed from negative to positive at $x = -1$ so f has a local minimum at $x = -1$.

Section 17 - Exercises (answers follow)

1. Consider the graph of f below. Find the critical points of f. On what domain intervals is f' positive? negative?



$y = f(x)$

2. For each function find the critical points, the intervals where each function increases/decreases, and identify all local extrema.

   (a) $y = 2 + 24x - 8x^2$

   (b) $f(x) = x^4 - 5x^3 + 100$

   (c) $f(x) = \frac{2}{3}x^3 - x^2 - 24x - 10$

   (d) $f(x) = (x^2 - 1)^8$

   (e) $y = -2x + 4$

   (f) $f(x) = x + \frac{3}{x}$

   (g) $f(x) = \dfrac{x-2}{x-1}$

   (h) $f(x) = 1 + x^{1/5}$

   (i) $y = x^{5/3} - x^{8/3}$

   (j) $f(x) = 3x^4 - 8x^3 - 90x^2 + 70$

   (k) $f(x) = x - \ln x$

   (l) $f(x) = \dfrac{\ln x}{x}$

   (m) $f(x) = x - e^x$

3. Suppose the total cost $C(x)$ (in dollars) to manufacture a quantity x of some chemical (in hundreds of liters) is given by $C(x) = 2x^3 + 3x^2 + 6x + 24$. Where is $C(x)$ increasing? Where is $C(x)$ decreasing?

4. Suppose f is the quadratic function $f(x) = ax^2 + bx + c. \quad a \neq 0$

(a) Show that $x = -\frac{b}{2a}$ is the only critical point of f.

(b) Use the First Derivative Test to show that f has a local maximum at $x = -\frac{b}{2a}$ if $a < 0$ and a local minimum at $x = -\frac{b}{2a}$ if $a > 0$.

5. Prove the case of "decreasing" for Theorem 17.1.

Section 17 Answers

1. Critical points at $x = -3, x = -1, x = 1, x = 2$ $\quad f' > 0$ on $(-3, -1) \cup (-1, 1)$ $\quad f' < 0$ on $(-\infty, -3) \cup (1, 2) \cup (2, \infty)$

2. (a) c.pts: $x = \frac{3}{2}$; increasing $(-\infty, \frac{3}{2})$; decreasing $(\frac{3}{2}, \infty)$; loc max at $x = \frac{3}{2}$

   (b) c.pts: $x = 0, x = \frac{15}{4}$; increasing $(\frac{15}{4}, \infty)$; decreasing $(-\infty, \frac{15}{4})$; loc min at $x = \frac{15}{4}$.

   (c) c pts: $x = 4, x = -3$; increasing $(-\infty, -3) \cup (4, \infty)$; decreasing $(-3, 4)$; loc max at $x = -3$, loc min at $x = 4$

   (d) c.pts: $x = 0, x = -1, x = 1$; increasing $(-1, 0) \cup (1, \infty)$; decreasing $(-\infty, -1) \cup (0, 1)$ loc max at $x = 0$, loc mins at $x = \pm 1$

   (e) c pts: none; increasing nowhere; decreasing $(-\infty, \infty)$; no local extrema

   (f) c pts: $x = \sqrt{3}, x = -\sqrt{3}$; increasing $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$; decreasing $(-\sqrt{3}, 0) \cup (0, \sqrt{3})$; loc max at $x = -\sqrt{3}$, loc min at $x = \sqrt{3}$

   (g) c.pts: none; increasing $(-\infty, 1) \cup (1, \infty)$; decreasing nowhere; no local extrema

   (h) c.pts: $x = 0$; increasing $(-\infty, 0) \cup (0, \infty)$; decreasing nowhere; no local extrema

   (i) c.pts: $x = 0, , x = \frac{5}{8}$; increasing $(-\infty, \frac{5}{8})$; decreasing $(\frac{5}{8}, \infty)$; loc max at $x = \frac{5}{8}$

   (j) c.pts: $x = -3, x = 0, x = 5$; increasing $(-3, 0) \cup (5, \infty)$; decreasing $(-\infty, -3) \cup (0, 5)$ loc. max at $x = 0$, loc mins at $x = -3, x = 5$

   (k) c.pts: $x = 1$; decreasing $(0, 1)$; increasing $(1, \infty)$; loc. min at $x = 1$

   (l) c.pts: $x = e$; increasing on $(0, e)$; decreasing on (e, ∞); loc. max at $x = e$

   (m) c.pts: $x = 0$; increasing on $(-\infty, 0)$; decreasing on $(0, \infty)$; loc. max at $x = 0$

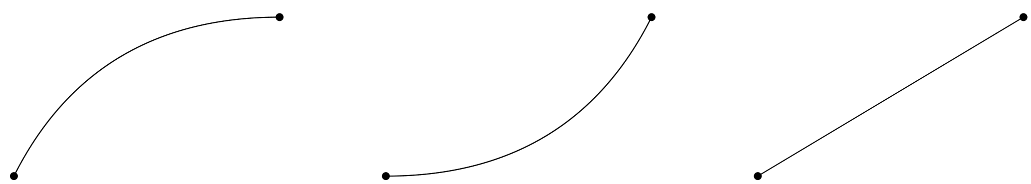
3. increasing $(-\infty, \infty)$; decreasing nowhere

18 Concavity and the Second Derivative Test

In Section 17 we saw that the derivative can tell us when a function f is increasing and when it is decreasing. The graph of f is sloping upward (the function is increasing) when $f'(x) > 0$ and downward (the function is decreasing) when $f'(x) < 0$. Now we want to refine our graphs.

Increasing functions

Suppose that f is continuous on interval $[a, b]$ and that $f'(x) > 0$ for all x on interval (a, b). This means that f is strictly increasing on this interval. There are three basic ways that the graph of f can increase from the point $(a, f(a))$ to the point $(b, f(b))$: the graph of f could increase quickly and then taper off, or the graph could increase slowly and then become steeper, or the graph could increase at a constant rate. These three possibilities are shown in the graphs below.



Why do we care which shape our graph has? If the function f describes our profit over the last year, it would be good news that $f'(x)$ is always positive. Our profits are ever increasing! But it would be more informative to know the *way* in which the profits are increasing. Which of the graphs above would you prefer to take to your shareholders' meeting?

Look carefully at the first graph. Sketch in some short tangent lines to indicate the slopes at several points. Reading from left to right, look at these slopes. Although the slopes are all positive, the slope values are decreasing. The tangent lines start out fairly steep but get flatter and flatter as you read to the right. The *slope values* are decreasing. We have a function that tells us the slope values at any point. It is the derivative function f'. Since the slope values are decreasing, and f' is the function that gives us slope values, we must conclude that f' is decreasing. AND NOW, (what we've been waiting for!) since f' is a decreasing function, we can conclude that *its* derivative function, f'', must be negative. So, a function whose graph has curvature like the first graph will have a positive first derivative and a negative second derivative.

What can we say about the second graph? If we draw tangent lines here and look at the slope values from left to right we can see that the slope values are increasing. So, a function whose graph has curvature like the second graph will have a positive first derivative and a positive second derivative.

The third graph has a constant slope. For this graph, $f'(x) = c$ for some positive constant c. So $f''(x) = 0$. A function whose graph is an increasing straight line will have a positive first derivative and a second derivative of zero.

Concavity

We use the term *concavity* to describe the curvature of a graph.

If f is a function such that $f''(x) > 0$ for all x in some open interval (a, b), we say that f is concave up on (a, b).

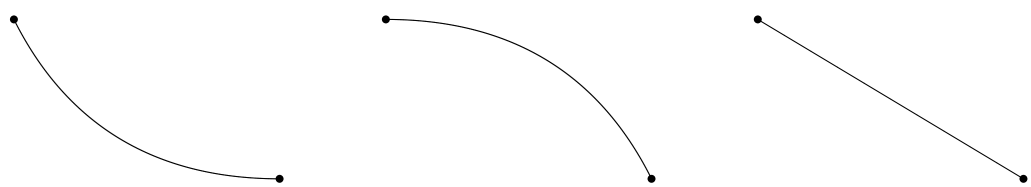
If f is a function such that $f''(x) < 0$ for all x in some open interval (a, b), we say that f is concave down on (a, b).

The first graph above is concave up, the second is concave down and third has no concavity.

When you are asked to find the "concavity" of a function this means you are to find where it is concave up and where it is concave down.

Decreasing functions

We can have a discussion of decreasing functions very similar to that of increasing functions. Here the graphs would look like:



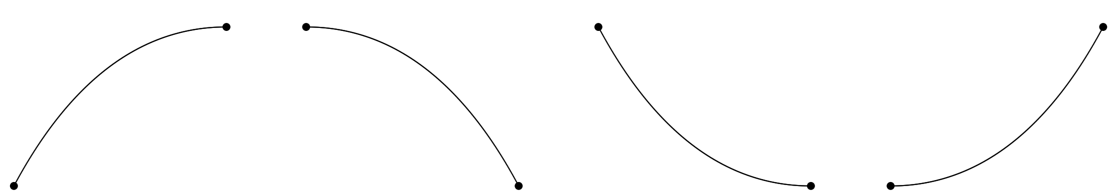
Keep in mind now that all of the derivatives, f', are negative. So, as we go from left to right of the first graph, we will have slopes like $-3, -1, -\frac{1}{2}, -\frac{1}{10} \ldots$. These numbers are *increasing* (getting less and less negative). So f' is increasing, which means $f'' > 0$. By our definition, this curve is concave up.

The second graph is concave down because $f''(x) < 0$.

The third graph has no concavity. The second derivative is zero.

Concavity summary

We can summarize our results with the following pictures:



$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
$f'' < 0$	$f'' < 0$	$f'' > 0$	$f'' > 0$
concave down		*concave up*	

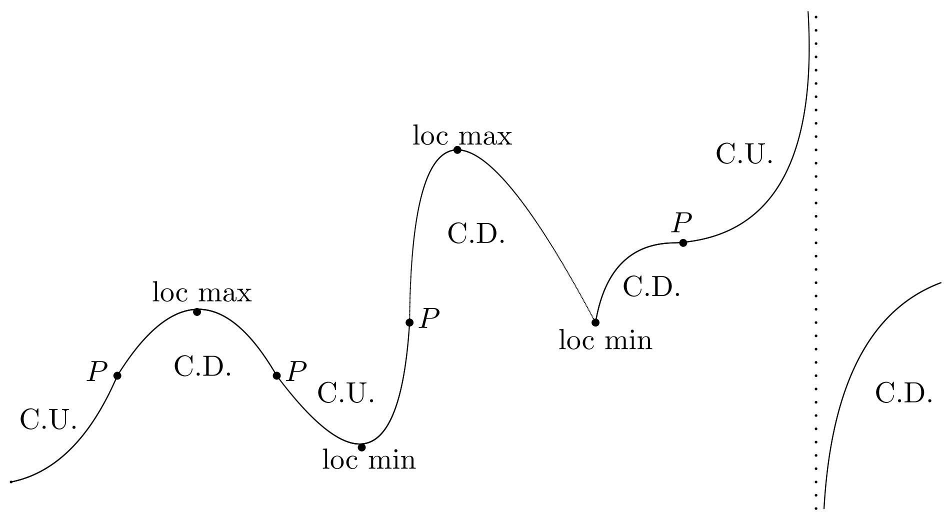
Notice that if you put the two concave down pieces together you get a "frown." If you put the two concave up pieces together you get a "smile." One way to remember the sign of the second derivative is to think that something negative will make you frown but something positive will make you smile. Sometimes students like to use: “Frown $\leftrightarrow$ Down” and “Cup $\leftrightarrow$ Up” to remember the concavity labels. Of course the best thing to do is to simply understand what is going on here: The second derivative is the rate of change of the slopes (the first derivative).

Points of Inflection

We say that f has a point of inflection at a if the concavity changes from up to down or from down to up at the point $(a, f(a))$.

A point of inflection can only occur at values of x where $f''(x) = 0$ or $f''(x)$ does not exist. Why?

Look at the Concavity Illustration below. The points of inflection are marked with a P. Notice that concavity changes at each point of inflection. By definition, a point of inflection must be a point on the graph. Although concavity is different on either side of the vertical asymptote, there is no point of inflection there.



Concavity Illustration

Example 18.1. Find intervals of concavity and points of inflection for $f(x) = \frac{1}{5}x^5 - x^4 - 5x$.

Answer: $f'(x) = x^4 - 4x^3 - 5 \qquad f''(x) = 4x^3 - 12x^2 = 4x^2(x-3)$.

$f''(0) = 0$ and $f''(3) = 0$. $\qquad f''(x)$ is defined on $\mathbb{R}$.

$f''(x) < 0$ when $x < 0$ and when $0 < x < 3$, so f is concave down on $(-\infty, 0) \cup (0, 3)$.

$f''(x) > 0$ when $x > 3$, so f is concave up on $(3, \infty)$.

f changes concavity only at $x = 3$ so the only point of inflection is at $x = 3$.

Example 18.2. Find intervals of concavity and points of inflection for $f(x) = \frac{1}{2}x^2 + \frac{9}{5}x^{\frac{5}{3}} + x - 7$.

Answer: $f'(x) = x + 3x^{\frac{2}{3}} + 1. \qquad f''(x) = 1 + 2x^{-\frac{1}{3}} = 1 + \frac{2}{\sqrt[3]{x}}$

$f''(-8) = 0 \qquad f''(0)$ D.N.E.

$f''(x) > 0$ when $x < -8$ or when $x > 0$, so f is concave up on $(-\infty, -8) \cup (0, \infty)$

$f''(x) < 0$ when $-8 < x < 0$, so f is concave up on $(-8, 0)$.

Concavity changes at $x = -8$ and at $x = 0$, so there are points of inflection at these x values.

Example 18.3. Find intervals of concavity and points of inflection for $f(x) = xe^x$.

Answer: $f'(x) = e^x + e^x x$. $\quad f''(x) = e^x + e^x x + e^x = e^x(2 + x)$

$f''(-2) = 0$. $\quad f''(x)$ exists for all x in $\mathbb{R}$.

$f''(x) < 0$ when $x < -2$, so f is concave down on $(-\infty, -2)$.

$f''(x) > 0$ when $x > -2$, so f is concave up on $(-2, \infty)$.

Concavity changes at $x = -2$, so there is a point of inflection at $x = -2$.

The Second Derivative Test

Besides using the second derivative to refine the curvature of the graph, the second derivative is useful in another way. Recall from Section 15 that the critical points of a function (the values of x where $f'(x) = 0$ or $f'(x)$ D.N.E) are the places where f might have a local extremum. For those critical points where $f'(x) = 0$ we can use the second derivative to test for local extrema.

The Second Derivative Test:

If $f'(a) = 0$ and the graph is concave down at a then there must be a local maximum at a.

If $f'(a) = 0$ and the graph is concave up at a then there must be a local minimum at a.

Hence: if $f'(a) = 0$ and $f''(a) < 0$ then f has a local maximum at a and

if $f'(a) = 0$ and $f''(a) > 0$ then f has a local minimum at a.

Look again at the Concavity Illustration above. At which values of x is $f'(x) = 0$ (i.e.; find the places where the tangent line is horizontal)? Now look at the concavity at those two points. Does the Second Derivative Test make sense?

Note that second derivatives do not give you the whole story on local extrema or points of inflection. It can happen that $f'(a) = 0$ and $f''(a) = 0$. In this case we cannot make a definite conclusion. For example consider $g(x) = x^3$ and $h(x) = x^4$ with $a = 0$. For both of these fuctions, the first and second derivatives are zero at $a = 0$. There is a point of inflection at $(0, 0)$ on the graph of g, but there is a local minimum at $(0, 0)$ on the graph of h.

Certainly the Second Derivative Test cannot be used for critical points where $f'(x)$ D.N.E. because if $f'(a)$ is undefined, then $f''(a)$ is not defined either. For these critical points you can use the First Derivative Test.

Example 18.4. Find the local extrema for $f(x) = x^3 + 2x^2 + x + 6$.

Answer: $f'(x) = 3x^2 + 4x + 1 = (3x + 1)(x + 1)$ so $f'(-\frac{1}{3}) = 0$ and $f'(-1) = 0$.

$f'(x)$ is defined everywhere. So our only critical points are $x = -\frac{1}{3}$ and $x = -1$.

$f''(x) = 6x + 4$. We test the critical points in f'':

$f''(-\frac{1}{3}) = -2 + 4 = 2 > 0$. So there is a local minimum at $x = -\frac{1}{3}$

$f''(-1) = -6 + 4 = -2 < 0$ So there is a local maximum at $x = -1$.,

Example 18.5. Find the local extrema for $f(x) = x - \ln x$

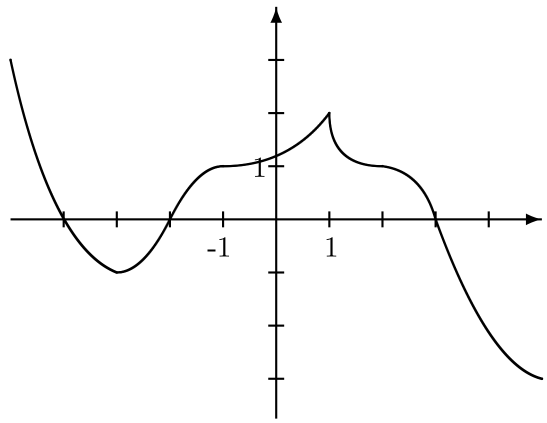
Answer: $f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$. So, $f'(1) = 0$.

Since the domain of f is $(0, \infty)$ the only critical point is $x = 1$.

$f''(x) = \frac{1}{x^2}$. $\quad f''(1) = 1 > 0$. So there is a local minimum at $x = 1$.

Section 18 - Exercises (answers follow)

1. Consider the graph of f below. On what intervals is f'' positive? negative?



$y = f(x)$

2. Find the intervals of concavity and any points of inflection.

   (a) $f(x) = \frac{1}{3}x^3 - 4x + 6$

   (b) $f(x) = -x^3 + \frac{9}{2}x^2 - 12x + 4$

   (c) $f(x) = (x-2)^3$

   (d) $f(x) = x(x+5)$

   (e) $f(x) = x + \frac{3}{x}$

   (f) $f(x) = \ln x + x$

   (g) $f(x) = x^{7/3} + x^{4/3}$

   (h) $f(x) = \sqrt{4x^2+3}$

   (i) $f(x) = \dfrac{3x}{x-2}$

   (j) $f(x) = \dfrac{2}{x^2+2x+2}$

   (k) $h(x) = (x-1)^{2/3}$

   (l) $g(x) = \frac{3}{10}x^5 - x^4 + x^3 + \frac{2}{10}x - 3$

3. Find any critical points for f and then use the Second Derivative Test to decide whether the critical points lead to local maxima or local minima.

   (a) $f(x) = x^2 - 4x + 8$

   (b) $f(x) = -x^3 + 4x$

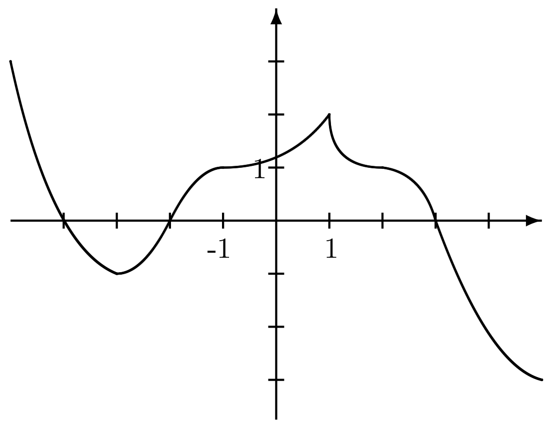
   (c) $f(x) = 2x^3 - 4x^2 + 2$

   (d) $f(x) = x^5 + x^4 + x^3$

   (e) $f(x) = xe^x$

(f) $f(x) = \frac{\ln x}{x}$

4. Suppose the graph below is the *derivative* graph, g'. On what intervals is the *function* g increasing? decreasing? concave up? concave down?



$y = g'(x)$

Section 18 Answers

1. $f'' > 0$ on $(-\infty, -2) \cup (-1, 1) \cup (1, 2) \cup (3, \infty)$ $\quad f'' < 0$ on $(-2, -1) \cup (2, 3)$

2. (a) CU $(0, \infty)$; CD $(-\infty, 0)$; POI $(0, 6)$
   (b) CU $(-\infty, \frac{3}{2})$; CD $(\frac{3}{2}, \infty)$; POI $(\frac{3}{2}, -\frac{29}{4})$
   (c) CU $(2, \infty)$; CD $(-\infty, 2)$; POI $(2, 0)$
   (d) CU $(-\infty, \infty)$; No POI
   (e) CU $(0, \infty)$; CD $(-\infty, 0)$; No POI
   (f) CD $(0, \infty)$; No POI
   (g) CU $(-\frac{1}{7}, 0) \cup (0, \infty)$, CD $(-\infty, -\frac{1}{7})$; POI $(-\frac{1}{7}, \frac{6}{7^{\frac{7}{3}}})$
   (h) CU $(-\infty, \infty)$; No POI
   (i) CU $(2, \infty)$; CD $(-\infty, 2)$ No POI
   (j) CU $(-\infty, -1-\frac{1}{\sqrt{3}}) \cup (-1+\frac{1}{\sqrt{3}}, \infty)$; CD $(-1-\frac{1}{\sqrt{3}}, -1+\frac{1}{\sqrt{3}})$; POI $(-1 + \frac{1}{\sqrt{3}}, \frac{3}{2}), (-1 - \frac{1}{\sqrt{3}}, \frac{3}{2})$
   (k) CD $(-\infty, 1) \cup (1, \infty)$; No POI
   (l) CU $(0, 1) \cup (1, \infty)$; CD $(-\infty, 0)$; POI $(0, -3)$

3. (a) Local min at $x = 2$
   (b) Local max at $x = \frac{2}{\sqrt{3}}$; Local min at $x = -\frac{2}{\sqrt{3}}$
   (c) Local max at $x = 0$; Local min at $x = \frac{4}{3}$
   (d) No Local extrema
   (e) Local min at $x = -1$

(f) Local max at $x = e$

4. g is increasing on $(-\infty, -4) \cup (-2, 3)$ decreasing on $(-4, -2) \cup (3, \infty)$ CU on $(-3, 1)$ CD on $(-\infty, -3) \cup (1, \infty)$

19 Graphs with No Asymptotes

Here are temporary instructions on how to sketch the graph of $y = f(x)$. They ignore "asymptotes" which will be discussed in Sections 20 and 21, but it is beneficial at this stage to apply the last few sections to the important problem of sketching graphs.

To sketch the graph of $y = f(x)$:

1. Determine the domain of f and mark it on the x-axis. Sometimes a specific domain will be given. If not, you must figure out the natural domain. Pay attention to division by zero, taking the square root (or other even root) of a negative number, and taking the logarithm of a non-positive number. Values of x which would make you do these things are not in the domain. When you sketch your graph, make sure that you do not include any points whose x values are not in the domain.

2. Find where the graph crosses the y-axis (the y-itercept, if there is one) by plugging in $x = 0$ to get $f(0)$. If convenient, find where the graph crosses the x-axis (the x-intercepts – there might be one, or more than one, or there might be none) by solving the equation $f(x) = 0$. This equation may be hard to solve; in that case don't bother with finding the x-intercepts.

3. Find $f'(x)$. Find the critical points of f (those values of x in the domain of f where $f'(x) = 0$ or $f'(x)$ is undefined). Determine where f is increasing ($f'(x) > 0$) and where it is decreasing ($f'(x) < 0$). This will tell you where any local maxima and minima are.

4. Find $f''(x)$. Find the values of x in the domain for which $f''(x) = 0$ or $f''(x)$ does not exist. Determine where f is concave up ($f''(x) > 0$) and where it is concave down ($f''(x) < 0$). This will tell you where the points of inflection are.

5. Plot the intercepts, the critical points, and the points of inflection.

6. Join them up with a smooth curve. Make sure that the concavity is clear.

7. Check your graph for inconsistencies against the increasing/decreasing information found in step 3. Recheck to be sure that your graph has the correct domain.

Using this process will give a good sketch of the graph of f.

Example 19.1. Sketch the graph of $f(x) = x^4 - 4x^3 + 10$ on the interval $[-1, 4]$.

Answer: The domain $[-1, 4]$ is given. $f(0) = 10$ is the y-intercept. The x-intercepts are not easily found.

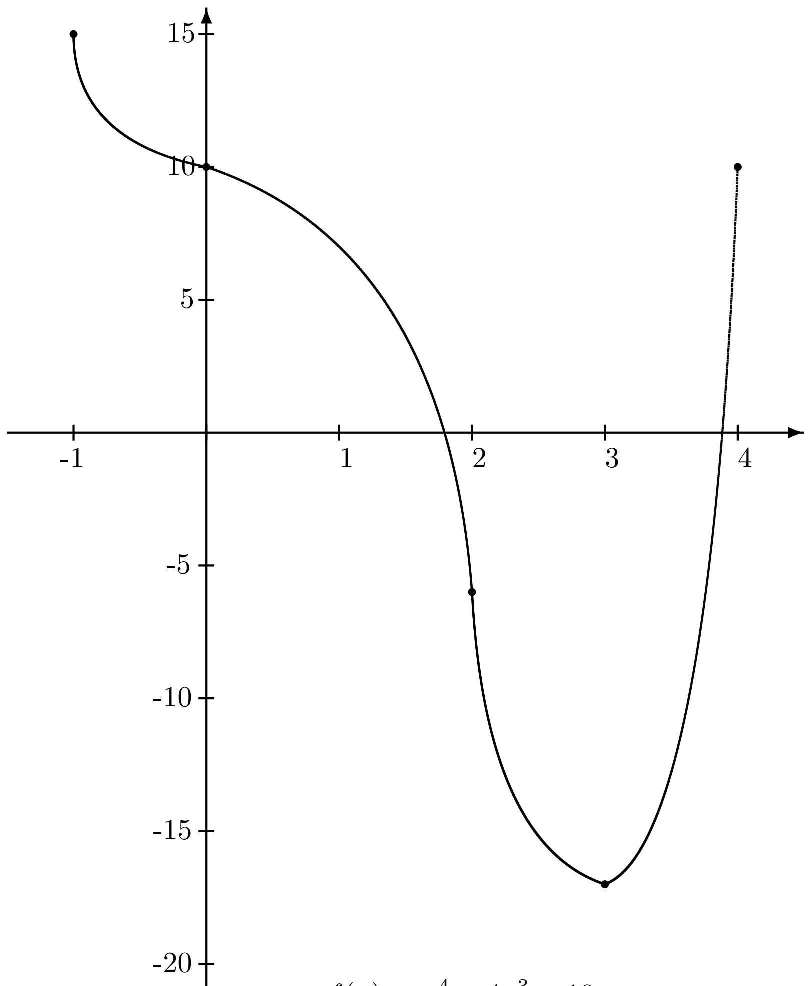
$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$. $f'(x) = 0$ at $x = 0$ and at $x = 3$. f' is defined everywhere on $[-1, 4]$. $f'(x) < 0$ when $-1 \leq x < 0$ and when $0 < x < 3$. So f is decreasing on $[-1, 3)$.[35] $f'(x) > 0$ when $3 < x \leq 4$, so f is increasing on $(3, 4]$.

$f''(x) = 12x^2 - 24x = 12x(x - 2)$. $f''(x) = 0$ at $x = 0$ and at $x = 2$. $f''(x)$ is defined everywhere on $[-1, 4]$. $f''(x) > 0$ when $-1 \leq x < 0$ and when $2 < x \leq 4$, so f is concave up on $[-1, 0) \cup (3, 4]$.

[35] We can combine these two adjacent intervals because polynomial f is continuous over its domain.

$f''(x) < 0$ when $0 < x < 2$ so f is concave down on $(0, 2)$. There are points of inflection at $x = 0$ and $x = 2$.

We need to plot the points $(-1, 15)$, $(0, 10)$, $(2, -6)$, $(3, -17)$ and $(4, 10)$. We complete the sketch by connecting these points with appropriate concavity.



$f(x) = x^4 - 4x^3 + 10$

Section 19 - Exercises (answers follow)

1. Sketch the graph of a function that has all of the following properties:
   (a) $f'(x) > 0$ when $x < 2$
   (b) $f'(x) < 0$ when $x > 2$
   (c) $f''(x) > 0$ when $x < 2$ and when $x > 2$

2. Sketch the graph of the function using the instructions given in Section 19.
   (a) $f(x) = -2x^3 - 9x^2 + 108x - 10$
   (b) $f(x) = x^4 - 6$
   (c) $h(x) = \frac{3}{2}x^4 - 2x^3 - 6x^2 + 8$
   (d) $f(x) = x^5 - 2x^3$
   (e) $f(x) = \frac{1}{2}x - \sqrt{x}$
   (f) $f(x) = \sqrt{x^2 - 1}$
   (g) $f(x) = 3x + x^{\frac{2}{3}}$
   (h) $f(x) = x^{\frac{3}{4}}(x - 2)$
   (i) $f(x) = \sqrt{x + 7}$
   (j) $f(x) = \sqrt[3]{x^3 - 3x}$

Section 19 - Answers

1. Graphs will vary in specifics. Your graph should be increasing and concave up when $x < 2$. Your graph should be decreasing and concave up when $x > 2$. Since f' exists for all values except at $x = 2$, f must be continuous everywhere except *possibly* at $x = 2$.

2. Answers for these problems include derivatives and other information so that you can easily find any errors that you might have made in trying to construct graphs from your derivatives and subsequent conclusions. All graphs are printed following the "data boxes" below.
   (a) $f(x) = -2x^3 - 9x^2 + 108x - 10$

Domain: $(-\infty, \infty)$	y-intercept: -10	x-intercept: Too hard. (Don't bother with it)
$f'(x) = -6x^2 - 18x + 108$ Incr. $(-6, 3)$ Loc. min. at $x = -6$	$f' = 0$ at $x = -6$ and $x = 3$ Decr. $(-\infty, -6) \cup (3, \infty)$ Loc. max at $x = 3$	f' DNE nowhere
$f''(x) = -12x - 18$ Conc. up $(-\infty, -\frac{3}{2})$	$f'' = 0$ at $x = -\frac{3}{2}$ Conc. down $(-\frac{3}{2}, \infty)$	f'' DNE nowhere P.O.I. at $x = -\frac{3}{2}$

(b) $f(x) = x^4 - 6$

Domain: $(-\infty, \infty)$	y-intercept: -6	x-intercept: $x = \pm\sqrt[4]{6} \approx \pm 1.57$
$f'(x) = 4x^3$	$f' = 0$ at $x = 0$	f' DNE: nowhere
Incr. $(0, \infty)$	Decr. $(-\infty, 0)$	
Loc. min. at $x = 0$	Loc. max.: none	
$f''(x) = 12x^2$	$f'' = 0$ at $x = 0$	f'' DNE: nowhere
Conc. up $(-\infty, \infty)$	Conc. down nowhere	P.O.I.: none

(c) $h(x) = \frac{3}{2}x^4 - 2x^3 - 6x^2 + 8$

Domain: $(-\infty, \infty)$	y-intercept: 8	x-intercept: Too hard. Don't bother with it.
$h'(x) = 6x^3 - 6x^2 - 12x$	$h' = 0$ at $x = -1, 0$ and 2	h' DNE: nowhere
Incr. $(-1, 0) \cup (2, \infty)$	Decr. $(-\infty, -1) \cup (0, 2)$	
Loc. min. at $x = -1$ and $x = 2$	Loc. max at $x = 0$	
$h''(x) = 18x^2 - 12x - 12$	$h'' = 0$ at $x = \frac{1 \pm \sqrt{7}}{3} \approx 1.2$, $x = -\frac{1}{2}$	h'' DNE: nowhere
C. up $\left(-\infty, \frac{1-\sqrt{7}}{3}\right) \cup \left(\frac{1+\sqrt{7}}{3}, \infty\right)$	C. down $\left(\frac{1-\sqrt{7}}{3}, \frac{1+\sqrt{7}}{3}\right)$	P.O.I.s at $x = \frac{1 \pm \sqrt{7}}{3}$

(d) $f(x) = x^5 - 2x^3$

Domain: $(-\infty, \infty)$	y-intercept: 0	x-intercepts: $0, \pm\sqrt{2}$
$f'(x) = 5x^4 - 6x^2$	$f' = 0$ at $x = 0$ and $x = \pm\sqrt{\frac{6}{5}}$	f' DNE: nowhere
Incr. $\left(-\infty, -\sqrt{\frac{6}{5}}\right) \cup \left(\sqrt{\frac{6}{5}}, \infty\right)$	Decr. $\left(-\sqrt{\frac{6}{5}}, \sqrt{\frac{6}{5}}\right)$	
Loc. min. at $x = \sqrt{\frac{6}{5}} \approx .77$	Loc. max at $x = -\sqrt{\frac{6}{5}} \approx -.77$	
$f''(x) = 20x^3 - 12x$	$f'' = 0$ at $x = \pm\sqrt{\frac{3}{5}}$ and 0	f'' DNE: nowhere
Conc. up $\left(-\sqrt{\frac{3}{5}}, 0\right) \cup \left(\sqrt{\frac{3}{5}}, \infty\right)$	Conc. down $\left(-\infty, -\sqrt{\frac{3}{5}}\right) \cup \left(0, \sqrt{\frac{3}{5}}\right)$	P.O.I. at $x = 0, \pm\sqrt{\frac{3}{5}}$

(e) $f(x) = \frac{1}{2}x - \sqrt{x}$

Domain: $[0, \infty)$	y-intercept: 0	x-intercepts: 0 and 4
$f'(x) = \frac{1}{2} - \frac{1}{2}x^{-\frac{1}{2}} = \frac{\sqrt{x}-1}{2\sqrt{x}}$	$f' = 0$ at $x = 1$	f' DNE at $x = 0$
Incr. $(1, \infty)$	Decr. $(0, 1)$	
Loc. min. at $x = 1$	Loc. max.: none	
$f''(x) = \frac{1}{4}x^{-\frac{3}{2}} = \frac{1}{4\sqrt{x^3}}$	$f'' = 0$ nowhere	f'' DNE at $x = 0$
Conc. up $(0, \infty)$	Conc. down nowhere	P.O.I.: none

(f) $f(x)=\sqrt{x^2-1}$

Domain: $(-\infty,-1]\cup[1,\infty)$	y-intercept: none	x-intercepts: ± 1
$f'(x)=\dfrac{x}{\sqrt{x^2-1}}$	$f'=0$ nowhere in domain	f' DNE at $x=\pm 1$
Incr. $(1,\infty)$	Decr. $(-\infty,-1)$	
Loc. min.: none	Loc. max.: none	
$f''(x)=\dfrac{-1}{\sqrt{(x^2-1)^3}}$	$f''=0$ nowhere	f'' DNE at ± 1
Conc. up nowhere	Conc. down $(-\infty,-1)\cup(1,\infty)$	P.O.I.: none

(g) $f(x)=3x+x^{\frac{2}{3}}$

Domain: $(-\infty,\infty)$	y-intercept: 0	x-intercept: 0 and $-\frac{1}{27}\approx -.037$
$f'(x)=3+\frac{2}{3}x^{-\frac{1}{3}}=3+\dfrac{2}{3\sqrt[3]{x}}$	$f'=0$ at $x=-\dfrac{8}{9^3}\approx -.011$	f' DNE at $x=0$
Incr. $\left(-\infty,-\frac{8}{9^3}\right)\cup(0,\infty))$	Decr. $\left(-\frac{8}{9^3},0\right)$	
Loc. min. at $x=0$	Loc. max at $x=-\frac{8}{9^3}\approx -.01$	
$f''(x)=-\frac{2}{9}x^{-\frac{4}{3}}=\dfrac{-2}{9\sqrt[3]{x^4}}$	$f''=0$ nowhere	f'' DNE at $x=0$
Conc.: up nowhere	Conc. down $(-\infty,\infty)$	P.O.I.: none

(h) $f(x)=x^{\frac{3}{4}}(x-2)$

Domain: $[0,\infty)$	y-intercept: 0	x-intercepts: 0 and 2
$f'(x)=\dfrac{7x-6}{4\sqrt[4]{x}}$	$f'=0$ at $x=\frac{6}{7}\approx .86$	f' DNE at $x=0$
Incr. $(\frac{6}{7},\infty)$	Decr. $(0,\frac{6}{7})$	
Loc. min. at $x=\frac{6}{7}$	Loc. max.: none	
$f''(x)=\frac{21}{16}x^{-\frac{1}{4}}+\frac{6}{16}x^{-\frac{5}{4}}$	$f''=0$ nowhere	f'' DNE at $x=0$
Conc. up $(0,\infty)$	Conc. down: nowhere	P.O.I.: none

(i) $f(x)=\sqrt{x+7}$

Domain: $[-7,\infty)$	y-intercept: $\sqrt{7}$	x-intercept: -7
$f'(x)=\dfrac{1}{2\sqrt{x+7}}$	$f'=0$ nowhere	f' DNE at $x=-7$
Incr. $(-7,\infty)$	Decr. nowhere	
Loc. min.: none	Loc. max.: none	
$f''(x)=\dfrac{-1}{\sqrt{(x+7)^3}}$	$f''=0$ nowhere	f'' DNE at $x=-7$
Conc. up nowhere	Conc. down $(-7,\infty)$	P.O.I.: none

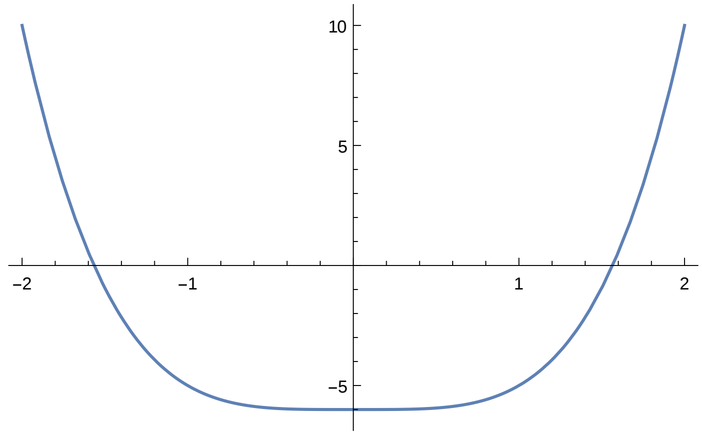
(j) $f(x) = \sqrt[3]{x^3 - 3x}$

Domain: $(-\infty, \infty)$	y-intercept: 0	x-intercepts: 0 and $\pm\sqrt{3}$
$f'(x) = \dfrac{x^2 - 1}{\sqrt[3]{(x^3 - 3x)^2}}$	$f' = 0$ at $x = \pm 1$	f' DNE at $x = 0$, $x = \pm\sqrt{3}$
Incr. $(-\infty, 1) \cup (1, \infty)$	Decr. $(-1, 1)$	
Loc. min. at $x = 1$	Loc. max. at $x = -1$	
$f''(x) = \dfrac{-2(x^2 + 1)}{\sqrt[3]{(x^3 - 3x)^5}}$	$f'' = 0$ nowhere	f'' DNE at $x = 0$, $x = \pm\sqrt{3}$
C. up $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$	C. down $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$	P.O.I. at $x = 0$, $x = \pm\sqrt{3}$

Graphs for most of the functions in Section 19, Exercise 2 can be found on the next few pages. They are done using Mathematica, a program you might want to explore.

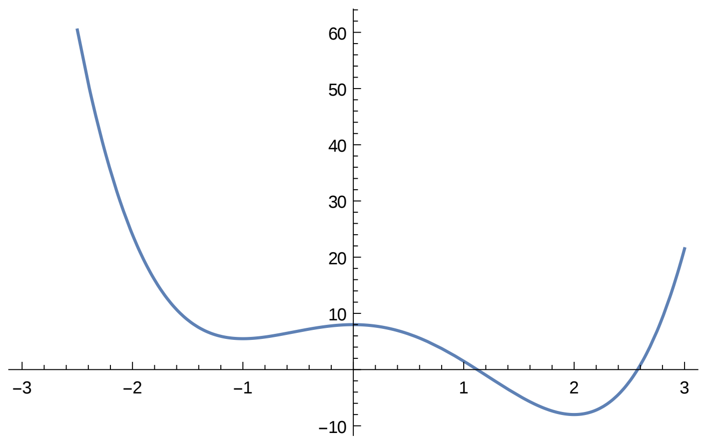
In[2]:= `Plot[x^4 - 6, {x, -2, 2}]`

Out[2]=



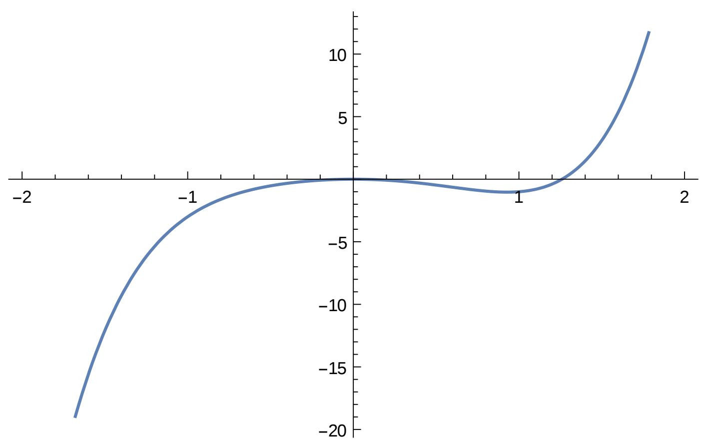
In[5]:= `Plot[{3 / 2} x^4 - 2 x^3 - 6 x^2 + 8, {x, -3, 3}]`

Out[5]=

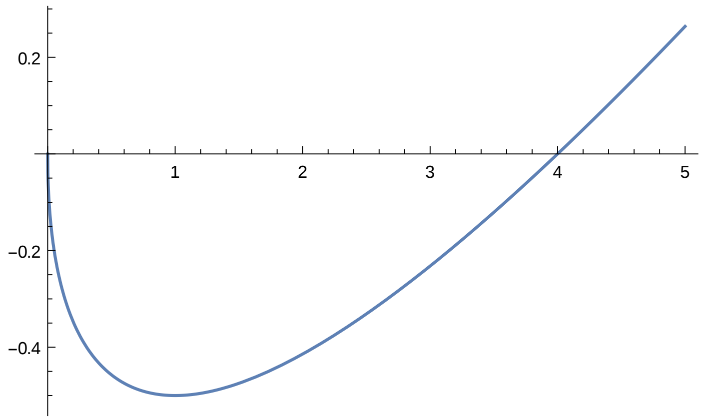


In[7]:= `Plot[x^5 - 2 x^2, {x, -2, 2}]`

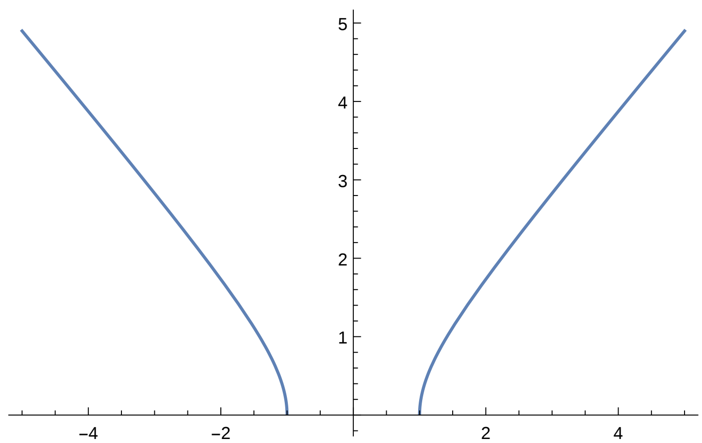
Out[7]=



In[9]:= `Plot[{1 / 2} x - x^{1 / 2}, {x, 0, 5}]`

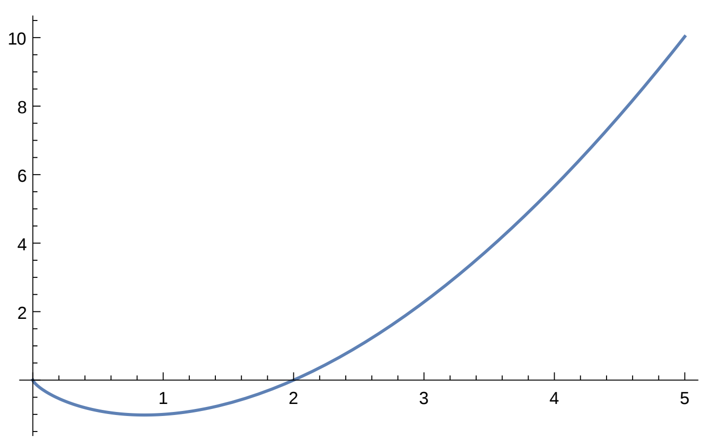
Out[9]=


In[12]:= `Plot[{x^2 - 1}^{1 / 2}, {x, -5, 5}]`

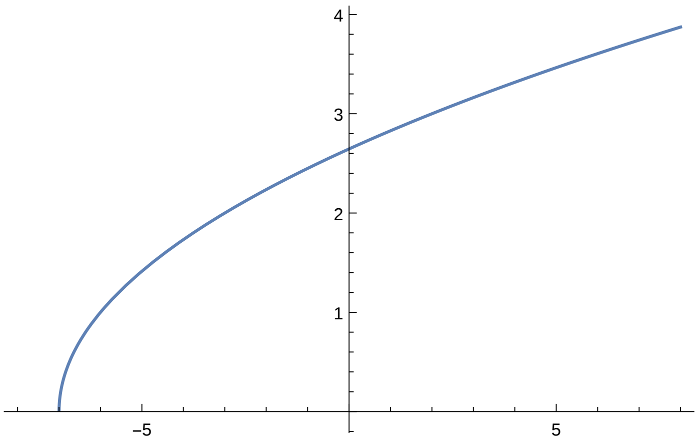
Out[12]=


In[32]:=

In[25]:= `Plot[x^{3 / 4} * (x - 2), {x, 0, 5}]`

Out[25]=


In[26]:=
```
Plot[(x + 7)^{1 / 2}, {x, -8, 8}]
```

Out[26]=


In[33]:=

20 Infinite Limits and Asymptotes

Vertical Asymptotes

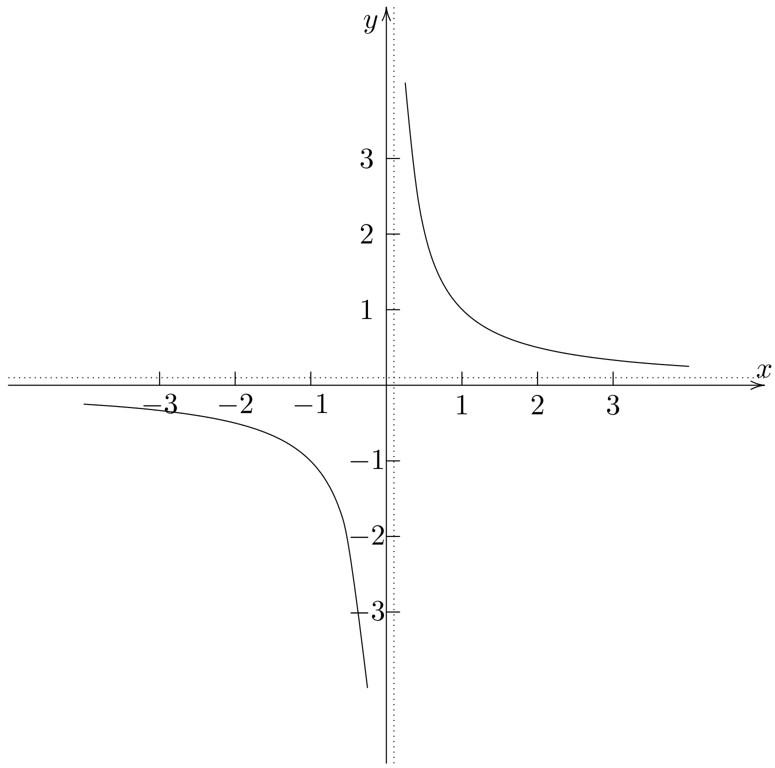
Consider the function $f(x) = \dfrac{1}{x}$. The domain of f does not include zero. In Section 6 we looked at how to evaluate $\lim\limits_{x\to 0} \dfrac{1}{x}$. Let's refresh our memories:

As with any limit of the form $\lim\limits_{x\to a} \dfrac{P(x)}{Q(x)}$, the first thing one should do is "plug in" the "a" value and see if there is any difficulty. Getting a zero in the denominator is cause for concern. If we get a zero in the numerator also, we then try to algebraically manipulate the function to get rid of (usually by "canceling" out) the offending denominator. Either this succeeds or we get an equivalent, reduced rational function that still yields a zero in the denominator, but not in the numerator. We then found that if $\lim\limits_{x\to a} \dfrac{P(x)}{Q(x)} \rightsquigarrow \dfrac{c}{0}$, (where c is a constant) then this limit does not lead to a real number. The limit is unbounded ($\pm\infty$). We needed to check the limit from both the left and the right side of a to determine the direction of the unboundedness. We represented the "answer" to each one-sided limit as ∞ or $-\infty$ as appropriate.

When we look at $\lim\limits_{x\to 0} \dfrac{1}{x}$ we see that the limit is unbounded. $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$ because the numerator is positive and the denominator is negative when $x < 0$. Also, $\lim\limits_{x\to 0^+} \dfrac{1}{x} = \infty$. Is it coming back to you? If not, take some time and review these limits in Section 6.

What we did not discuss in Section 6 is the significance of these one-sided limits to the graph of f. We are simply saying that as the x values get very close, closer and closer to 0 from the right, the corresponding y values are getting larger and larger and larger. The graph is going up, up and up. We indicate this on a graph with a vertical asymptote (line) $x = 0$. The graph approaches the asymptote as it shoots up. On the left side, where $x < 0$, the graph approaches the asymptote as it goes down.

An *asymptote* is simply a straight line to which a graph becomes arbitrarily close. The asymptote is not a part of the graph; it is more like scaffolding or a boundary frame. Asymptotes are generally drawn with dotted lines to indicate that they are not part of the graph. When $\lim\limits_{x\to a^\pm} f(x) = \pm\infty$ there is a vertical asymptote with equation $x = a$. For the graph of $f(x) = \frac{1}{x}$ on page 20 the vertical asymptote is not visible because it coincides with the y-axis. Below is a reminder of the graph, with a vertical asymptote included, just slightly offset for visibility.



$y = 1/x$

Horizontal Asymptotes

You probably noticed on the previous graph that there are two asymptotes drawn, one vertical (the line $x = 0$) and one horizontal (the line $y = 0$).

Again, an asymptote is just a line to which the graph becomes arbitrarily close. In this case the function has points such as $(2, \frac{1}{2})$, $(5, \frac{1}{5})$, $(100, \frac{1}{100})$, $(1,000,000,000, \frac{1}{1,000,000,000})$. You can see that the y values do get arbitrarily close[36] to $y = 0$ as the x values get larger and larger. We express this with a limit: $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$. A very similar thing is happening on the negative side of the graph of f. We write $\lim\limits_{x\to-\infty} \dfrac{1}{x} = 0$.

These limits are different from any of the ones that we have studied so far. Look at the format:

$$\lim_{x\to\infty} f(x) = c \quad \text{for some constant } c$$

With this limit we have $x \to \infty$; we do not have $x \to a$. All of our previous limits described the value of the function as x got very close to a constant (finite, real) value. This limit describes the behavior of the function as x gets very large. It describes the graph on the right "tail." With this limit we are saying that the y values on the positive side of the graph eventually become arbitrarily close to a constant (finite, real) value c. To sketch this, we draw a horizontal asymptote to the right. The equation of the asymptote is $y = c$.

[36]To review "arbitrarily close,. see page 60.

Everything that has been said about $\lim_{x\to\infty} f(x) = c$ can be said about $\lim_{x\to-\infty} f(x) = c$ with the appropriate changes. This latter limit describes the behavior of the graph on the left "tail."

It is important to understand that although we often draw horizontal asymptotes left-to-right across the entire graph, they have no meaning except on the extreme ends, or "tails," of the graph. Students sometimes are confused by this and are afraid to have their graphs "cross" a horizontal asymptote. Graphs can intersect or cross a horizontal asymptote many times, even infinitely many times. The asymptote is only meaningful as a guide to the graph on the extreme ends of the graph, when $x \to \infty$ or $x \to -\infty$.

One reason that we sometimes draw horizontal asymptotes across an entire graph is that for many of the functions we see, the same horizontal asymptote exists on both the right and the left sides of the graph. This is true for our example $f(x) = \frac{1}{x}$. However, it is not true in general. For example, $g(x) = 2^x$ has a horizontal asymptote to the left, but not to the right. Look at the graph of $g(x) = 2^x$ on page 31. We would write $\lim_{x\to-\infty} 2^x = 0$.

Look again at the graph of $f(x) = 2^x$. How would we describe the behavior of the function as x approaches infinity? As x gets larger and larger, $y = 2^x$ gets larger and larger. The function is increasing and is unbounded.

We write $\lim_{x\to\infty} f(x) = \infty$ to mean that the function values are *unbounded* as x gets larger and larger. It is not sufficient to simply say that the function values increase. The numbers 1, 1.1, 1.11, 1.111, ..etc. are increasing, but we wouldn't say that they are "headed to infinity."

What do you think we mean by each of the following:

$\lim_{x\to\infty} f(x) = -\infty \qquad \lim_{x\to-\infty} f(x) = \infty \qquad \lim_{x\to-\infty} f(x) = -\infty$?

Calculating Limits as $x \to \infty$

We now look at how to calculate these limits. It is clear that $\lim_{x\to\infty} \dfrac{1}{x} = 0$. The denominator is getting unboundedly large as x approaches infinity, so the value of the rational function $\frac{1}{x}$ is getting arbitrarily close to zero. We can make a stronger statement, however. It is true that for any positive rational number r, the function $y = x^r$ gets unboundedly large as x approaches infinity. Therefore, the reciprocal $\frac{1}{x^r}$ gets arbitrarily close to zero. We can make an even stronger statement by observing that multiplying $\frac{1}{x^r}$ by any real constant c does not change its unboundedness, and therefore does not change its reciprocal getting arbitrarily close to zero. We summarize this in the following theorem, which is offered without proof.

Theorem 20.1. *For any positive rational number r and any real constant c, $\lim_{x\to\infty} \dfrac{c}{x^r} = 0$. Also, for all values of r that make sense in the domain, $\lim_{x\to-\infty} \dfrac{c}{x^r} = 0$.*

We need to make the domain stipulation in the case of x approaching negative infinity. For example, if $r = \frac{1}{2}$ then $x^r = \sqrt{x}$. This makes no sense for negative values of x.

We can use Theorem 20.1 to handle more complex functions.

Example 20.1. Evaluate $\lim_{x\to\infty} \frac{3x+1}{5x}$.

Solution: $\lim_{x\to\infty} \frac{3x+1}{5x} = \lim_{x\to\infty} \frac{3+\frac{1}{x}}{5} = \frac{3+0}{5} = \frac{3}{5}$.

In Example 20.1 we divided the numerator and denominator by x and then applied Theorem 20.1 to the only x term remaining [37]. In general, it is helpful to divide by x raised to the highest power in the denominator. This guarantees that there will be a non-zero constant in the denominator when the limit is taken.

We need to remember the significance of this limit. In Example 20.1 we found a limit of $\frac{3}{5}$. This means that the function has a horizontal asymptote, $y = \frac{3}{5}$ on its extreme positive end. It is also true that the same asymptote is used on the extreme left of the graph because $\lim_{x\to-\infty} \frac{3x+1}{5x}$ works exactly the same way.

Example 20.2. Evaluate $\lim_{x\to-\infty} \frac{6x^5+2x^2+x+1}{-2x^5-x^2+1}$.

Solution: $\lim_{x\to-\infty} \frac{6x^5+2x^2+x+1}{-2x^5-x^3+10} = \lim_{x\to-\infty} \frac{6+\frac{2}{x^3}+\frac{1}{x^4}+\frac{1}{x^5}}{-2-\frac{1}{x^2}+\frac{10}{x^5}} = \frac{6+0+0+0}{-2+0+0} = -3$.

Example 20.3. Evaluate $\lim_{x\to\infty} \frac{2x^3+x-4}{x^4+x^3+x}$.

Solution: $\lim_{x\to\infty} \frac{2x^3+x-4}{x^4+x^3+x} = \lim_{x\to\infty} \frac{\frac{2}{x}+\frac{1}{x^3}-\frac{4}{x^4}}{1+\frac{1}{x^2}+\frac{1}{x}} = \frac{0+0+0}{1+0+0} = 0$.

Example 20.4. Evaluate $\lim_{x\to\infty} \frac{2x^7+x^2-12}{3x^4-x^3-5x^2+1}$.

Solution: $\lim_{x\to\infty} \frac{2x^7+x^2-12}{3x^4-x^3-5x^2+1} = \lim_{x\to\infty} \frac{\frac{2x^7}{x^4}+\frac{x^2}{x^4}-\frac{12}{x^4}}{\frac{3x^4}{x^4}-\frac{x^3}{x^4}-\frac{5x^2}{x^4}+\frac{1}{x^4}} = \lim_{x\to\infty} \frac{2x^3+\frac{1}{x^2}-\frac{12}{x^4}}{3-\frac{1}{x}-\frac{5}{x^2}+\frac{1}{x^4}}$.

Now we have a situation where the denominator goes to 3 (as x approaches infinity) but the numerator gets unboundedly large. When the numerator is unbounded but the denominator is not, the fraction value is unbounded. We look at the numerator. Since x is positive ($x \to +\infty$), the significant term in the numerator, $2x^3$, is positive. The significant term in the denominator, 3, is positive, so the limit will be positive. $\lim_{x\to\infty} \frac{2x^3+\frac{1}{x^2}-\frac{12}{x^4}}{3-\frac{1}{x}-\frac{5}{x^2}+\frac{1}{x^4}} = \infty$.

What would be different if Example 20.4 had $x \to -\infty$ instead of $x \to \infty$? The division would be the same, so the function would still be unbounded. But, since x is negative the numerator would be negative. Thus, this limit would be $-\infty$.

[37] Usually when we divide by x we need to stipulate that $x \neq 0$. It is not necessary here because in our limit we are dealing with x approaching infinity. So, x is nowhere close to zero.

Example 20.5. Evaluate $\lim\limits_{x\to-\infty}\dfrac{-5x^4+x^3-1}{-3x^2+x}$

Soluton: $\lim\limits_{x\to-\infty}\dfrac{-5x^4+x^3-1}{-3x^2+x}=\lim\limits_{x\to-\infty}\dfrac{-5x^2+x-\frac{1}{x^2}}{-3+\frac{1}{x}}$

Again, we have the numerator unbounded and the denominator a constant, so the function is unbounded. The numerator is negative because x^2 is positive, but when multiplied by -5, it becomes negative. The denominator is negative. So the limit is ∞.

Note that if Example 20.5 had $x\to+\infty$ the limit would still be positive infinity. Since the significant term in the numerator is $-5x^2$, this is a negative value regardless of the sign of x.

It is worth noting also in Example 20.5 that the numerator has two terms that do not go to zero as x approaches $-\infty$. Even though the x term does not get small, it is not significant. When x is very large, the term with the highest power will dominate all of the other terms.[38]

Example 20.6. Evaluate $\lim\limits_{x\to\infty}(-3x^5+2x^4+x^2-8)$ and $\lim\limits_{x\to-\infty}(-3x^5+2x^4+x^2-8)$

Solution: We factor the function by x^5 because 5 is the highest power:

$-3x^5+2x^4+x^2-8=x^5\left(-3+\frac{2}{x}+\frac{1}{x^3}-\frac{8}{x^5}\right)$.

Now we can see: $\lim\limits_{x\to\infty}x^5\left(-3+\dfrac{2}{x}+\dfrac{1}{x^3}-\dfrac{8}{x^5}\right)=\lim\limits_{x\to\infty}(-3x^5)=-\infty$ and

$\lim\limits_{x\to-\infty}x^5\left(-3+\dfrac{2}{x}+\dfrac{1}{x^3}-\dfrac{8}{x^5}\right)=\lim\limits_{x\to-\infty}(-3x^5)=\infty$.

In Example 20.7 we show why you cannot use the operations for real numbers on infinity. If you think that $\infty-\infty=0$, you will see that this just isn't so.

Example 20.7. Evaluate $\lim\limits_{x\to\infty}(\sqrt{x^4+6x^2}-x^2)$

Solution: $\lim\limits_{x\to\infty}(\sqrt{x^4+6x^2}-x^2)=\lim\limits_{x\to\infty}(\sqrt{x^4+6x^2}-x^2)\cdot\dfrac{\sqrt{x^4+6x^2}+x^2}{\sqrt{x^4+6x^2}+x^2}=\lim\limits_{x\to\infty}\dfrac{x^4+6x^2-x^4}{\sqrt{x^4+6x^2}+x^2}$

$=\lim\limits_{x\to\infty}\dfrac{6x^2}{\sqrt{x^4+6x^2}+x^2}=\lim\limits_{x\to\infty}\dfrac{6}{\sqrt{1+\frac{1}{x^2}}+1}=\dfrac{6}{\sqrt{1+0}+1}=\dfrac{6}{2}=3$

See? Example 20.7 shows that $\infty-\infty=3$....NOT! You really can't use real number operations on non-real numbers. What this limit does say is that the graph of this function has a horizontal asymptote $y=3$ on the right. What about on the left (as $x\to-\infty$)?

Not every limit uses Theorem 20.1. There are some limits that you should keep in mind. They are not hard to remember if you remember the graphs (page 36) of the exponential and logarithm functions:

$$\lim_{x\to\infty}e^x=\infty\qquad\lim_{x\to-\infty}e^x=0\qquad\lim_{x\to\infty}\ln x=\infty\qquad\lim_{x\to0^+}\ln x=-\infty$$

These limits are the same for other bases a where $a>1$.

[38]For example, when x is one thousand, x^2 is one million. When x is one million, x^2 is one trillion. When x approaches ∞, the lower powered term is indeed insignificant.

Finding Vertical and Horizontal Asymptotes

In general, a vertical asymptote occurs when the function value approaches ∞ or $-\infty$ as x approaches a constant, e.g. $\lim_{x\to a^+} f(x) = \infty$. A horizontal asymptote occurs when the function value approaches a constant as x approaches either ∞ or $-\infty$, e.g. $\lim_{x\to\infty} f(x) = c$. If you think about this for a moment you won't have to memorize anything. It should make sense. A vertical asymptote occurs at a finite x value; it is the y value that is unbounded. A horizontal asymptote occurs when the x value is unbounded and the y value nears a finite number.

Example 20.8. Find vertical and horizontal asymptotes for $f(x) = \dfrac{2x-3}{3x-1}$.

Solution: f is undefined at $x = \frac{1}{3}$, so we use limits to see if there is a vertical asymptote at $x = \frac{1}{3}$.

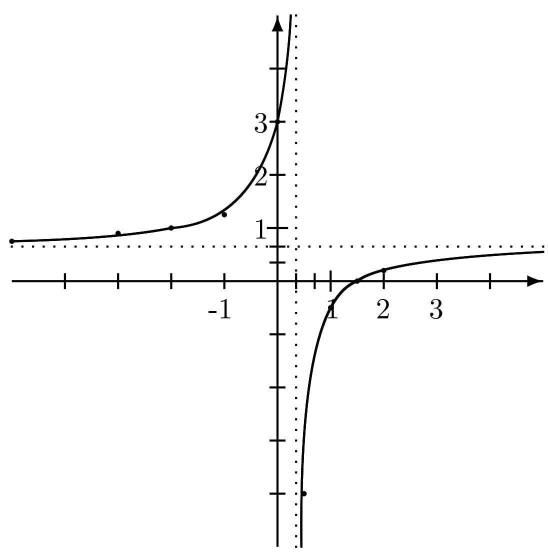
$\lim_{x\to \frac{1}{3}^-} \dfrac{2x-3}{3x-1} = \infty \qquad \lim_{x\to \frac{1}{3}^+} \dfrac{2x-3}{3x-1} = -\infty.$ So, there is a vertical asymptote[39], $x = \frac{1}{3}$.

To check for horizontal asymptotes we look at the two limits:

$\lim_{x\to -\infty} \dfrac{2x-3}{3x-1} = \dfrac{2}{3} \qquad \lim_{x\to \infty} \dfrac{2x-3}{3x-1} = \dfrac{2}{3}$

So there is the same horizontal asymptote, $y = \frac{2}{3}$ in both directions.

A graph of $f(x) = \dfrac{2x-3}{3x-1}$, the function from Example 20.8, is below. Compare the graph and asymptotes to the limits found in the example. The pictures can help you see the connection between the limits and the asymptotes.



$$f(x) = \frac{2x-3}{3x-1}$$

[39] Actually, finding either one of these limits to be infinite is sufficient to establish the existence of the vertical asymptote. But we might as well get used to doing both sides because in Section 21 we will need the complete limit information for graphing.

Example 20.9. Find vertical and horizontal asymptotes for $f(x) = \dfrac{x^2 - x - 6}{x^2 - 4}$

Solution: $f(x) = \dfrac{x^2 - x - 6}{x^2 - 4} = \dfrac{(x+2)(x-3)}{(x+2)(x-2)} = \dfrac{x-3}{x-2}$ when $x \neq -2$.

f is not defined at $x = -2$ and $x = 2$

$\lim_{x\to -2} \dfrac{x-3}{x-2} = \dfrac{-5}{-4} = \dfrac{5}{4}$ which is finite, so there is no vertical asymptote at $x = -2$.

$\lim_{x\to 2^+} \dfrac{x-3}{x-2} = -\infty$ and $\lim_{x\to 2^-} \dfrac{x-3}{x-2} = \infty$. So, there is a vertical asymptote at $x = 2$

$\lim_{x\to \infty} \dfrac{x-3}{x-2} = 1$ and $\lim_{x\to -\infty} \dfrac{x-3}{x-2} = 1$ So there is a horizontal asymptote $y = 1$ on both the left and the right ends of the graph.

Note: It is legitimate to use the reduced form of f for our limits. The reduced form is only invalid AT $x = 2$. None of our limits needed to use f at $x = 2$.

Example 20.10. Find vertical and horizontal asymptotes for $f(x) = \dfrac{\sqrt{5x^2+1}}{3x-5}$.

Solution: f is undefined at $x = \frac{5}{3}$ so we check there for vertical asymptotes:

$\lim_{x\to \frac{5}{3}^-} \dfrac{\sqrt{5x^2+1}}{3x-5} = -\infty \qquad \lim_{x\to \frac{5}{3}^+} \dfrac{\sqrt{5x^2+1}}{3x-5} = \infty$, so there is a vertical asymptote, $x = \frac{5}{3}$.

We now check for horizontal asymptotes. It helps first to note[40] that

$$\frac{\sqrt{5x^2+1}}{3x-5} = \frac{\sqrt{x^2(5 - \frac{1}{x^2})}}{x(3 - \frac{5}{x})} = \frac{|x|\sqrt{5 + \frac{1}{x^2}}}{x(3 - \frac{5}{x})} \quad \text{when } x \neq 0.$$

When $x > 0$, $|x| = x$ so $\frac{|x|}{x} = 1$, thus $\lim_{x\to\infty} f(x) = \dfrac{\sqrt{5}}{3}$.

When $x < 0$, $|x| = -x$ so $\frac{|x|}{x} = -1$, thus $\lim_{x\to-\infty} f(x) = -\dfrac{\sqrt{5}}{3}$.

So, we have two horizontal asymptotes: $y = \dfrac{\sqrt{5}}{3}$ on the right and $y = -\dfrac{\sqrt{5}}{3}$ on the left.

Example 20.11. Find vertical and horizontal asymptotes for $f(x) = \begin{cases} e^x + 2 & x < 0 \\ \frac{1}{x} & x > 0 \end{cases}$

Solution: The only point of discontinuity of f is at $x = 0$ so we look there for a vertical asymptote:

$\lim_{x\to 0^-} (e^x + 2) = 1 + 2 = 3 \qquad \lim_{x\to 0^+} \dfrac{1}{x} = \infty$, so there is a vertical asymptote, $x = 0$, but it is only used on the right side.

We now check for horizontal asymptotes:

$\lim_{x\to -\infty} (e^x + 2) = 0 + 2 = 2 \qquad \lim_{x\to\infty} \dfrac{1}{x} = 0.$

So, we have two horizontal asymptotes: $y = 2$ on the left, and $y = 0$ on the right.

Revisiting the number e

Way back in Section 5 we introduced the number e. The number was motivated by interest rates that were compounded continuously. At that time we looked at the expression $\left(1 + \frac{1}{n}\right)^n$ and asked what would happen to the values of that expression as n got very large. We did not have the

[40] Remember: $\sqrt{a^2} = |a|$. See page 13 if you need review.

formal notation of a limit. The statement we were really making was

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \tag{20.1}$$

We then made some algebraic manipulations (see Equation 5.3 on page 52) and concluded that if you accept Equation 20.1 , then you can accept a variation of it. In formal limit terminology we are saying that:

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$$

Simply changing to more formal notation doesn't go any further in proving Equation 20.1. In Section 34 we will present a geometric interpretation of e based on the function $f(x) = e^x$ and its relationship to the natural logarithm function. Some calculus text writers define the natural logarithm function and then use it to develop e. Other authors actually use Equation 20.1 as the <u>definition</u> of e and then use it to develop the natural logarithm. Without having presented a proof of Equation 20.1 we are leaning more toward the latter approach.

In Section 10 we made the claim that $\frac{d}{dx}e^x = e^x$. We now have the terminology to outline the proof.

$\frac{d}{dx}e^x = \lim_{h\to 0}\left(\frac{e^{x+h}-e^x}{h}\right)$ if this limit exists. We have to show the limit does exist and is e^x. We have:

$$\frac{e^{x+h}-e^x}{h} = e^x\frac{1}{h}(e^h-1)$$

$$\text{and}\quad e^h = \lim_{n\to\infty}\left(1+\frac{h}{n}\right)^n.$$

By the Binomial Theorem,

$$\left(1+\frac{h}{n}\right)^n = 1+n\left(\frac{h}{n}\right) + \text{ terms involving } h^2, h^3, \cdots, h^n.$$

So

$$\begin{aligned}\frac{1}{h}\left[\left(1+\frac{h}{n}\right)^n - 1\right] &= \frac{1}{h}(h + \text{ terms involving } h^2, h^3, \cdots, h^n)\\ &= 1 + \text{ terms involving } h, h^2, \cdots, h^{n-1}.\end{aligned}$$

As $h \to 0$ this $\to 1$, so $\lim_{h\to 0}\frac{1}{h}(e^h-1) = 1$, hence[41] $\lim_{h\to 0}\left(\frac{e^{x+h}-e^x}{h}\right) = e^x$.

[41] The hard part of this is $\lim_{h\to 0}\frac{e^h-1}{h} = 1$. What our "proof" really says is:

$$\lim_{h\to 0}\frac{e^h-1}{h} = \lim_{h\to 0}\lim_{n\to\infty}\frac{(1+\frac{h}{n})^n-1}{h} = \lim_{n\to\infty}\lim_{h\to 0}\frac{(1+\frac{h}{n})^n-1}{h} = \lim_{n\to\infty} 1 = 1.$$

It can be proved rigorously that in this case it is legitimate to interchange the order of taking limits ($h \to 0$, $n \to \infty$).

Section 20 - Exercises (answers follow)

1. Suppse that F is the rational function $F(x) = \dfrac{P(x)}{Q(x)}$. Suppose that p is the degree of polynomial P and that q is the degree of polynomial Q.

   (a) What can you say about $\lim_{x\to\infty} F(x)$ if:
   (1) $p < q$ (2) $p = q$ (3) $p > q$

   (b) Does the graph of the function F have a horizontal asymptote when:
   (1) $p < q$ (2) $p = q$ (3) $p > q$

   (c) How would the answers to part (a) change if the limit were $x \to -\infty$ instead of $x \to \infty$?

   (d) Use your results from parts (a), (b) and (c) to write some "shortcut rules" for finding $\lim_{x\to\pm\infty} R(x)$. This will save you the time of laboriously doing the appropriate justification division on each exercise.

2. Suppose P is a polynomial. Can you write "shortcut" rules for finding $\lim_{x\to\pm\infty} P(x)$ so that you do not have to factor the polynomial?

3. Suppose $0 < a < 1$. Use your knowledge of the graph of the exponential function $f(x) = a^x$ to evaluate $\lim_{x\to\infty} a^x$ and $\lim_{x\to-\infty} a^x$.

4. Find the limit if it exists. You may use your results from problems 1, 2 and 3 where appropriate.

   (a) $\lim_{x\to\infty} \dfrac{6x}{5x-1}$

   (b) $\lim_{x\to-\infty} \dfrac{x^3+x^2+1}{x^3+1}$

   (c) $\lim_{x\to+\infty} \dfrac{x^2+300x+8}{5x+2}$

   (d) $\lim_{x\to\infty} \dfrac{3x^3+2x-1}{2x^4-3x^3}$

   (e) $\lim_{x\to5} \dfrac{x+3}{5-x}$

   (f) $\lim_{x\to2} \dfrac{x^2-1}{x-3}$

   (g) $\lim_{x\to\infty} \dfrac{(3x+2)(2x-1)}{(x+3)(5x-4)}$

   (h) $\lim_{x\to\infty} e^{\frac{x^2+4x+2}{4x^2+5x+1}}$

   (i) $\lim_{x\to2^+} 10^{\frac{5}{x-2}}$

   (j) $\lim_{x\to2^-} 10^{\frac{5}{x-2}}$

   (k) $\lim_{x\to-\frac{1}{2}} 2^{6x+1}$

(l) $\lim_{x \to \infty} \dfrac{x}{\sqrt{4x^2 + x + 3}}$

(m) $\lim_{x \to -\infty} \dfrac{x}{\sqrt{4x^2 + x + 3}}$

(n) $\lim_{x \to \infty} (\sqrt{x^2 + 1} - \sqrt{x^2 - 1})$

(o) $\lim_{x \to \infty} (\sqrt{x^2 + x} - \sqrt{x^2 - 1})$

(p) $\lim_{x \to 4} \ln(3x - 4)$

(q) $\lim_{x \to \infty} [\ln(2 + x) - \ln(1 + x)]$

(r) $\lim_{x \to 5^+} \log_3(x^2 - 25)$

(s) $\lim_{x \to \infty} \dfrac{4x^3 - 3x + 5}{6x^4 - 4x - 2}$

(t) $\lim_{x \to -\infty} (5x^3 + 2x^2 - 2x - 7)$

(u) $\lim_{x \to -\infty} (x^2 + 3)$

(v) $\lim_{x \to \infty} \ln(\frac{1}{x})$

5. Find any horizontal and vertical asymptotes of the given function.

(a) $f(x) = \frac{1}{x}$

(b) $f(x) = -\frac{4}{x^3}$

(c) $f(x) = \frac{x-1}{x+1}$

(d) $f(x) = \frac{3x^2}{5x^2-6}$

(e) $f(x) = \frac{3x}{2x^2-x-1}$

(f) $f(x) = \frac{4x+3}{x-2}$

(g) $f(x) = \frac{1}{x} + \frac{1}{x-1}$

(h) $f(x) = \frac{3x+2}{x^2-6x+8}$

(i) $f(x) = \frac{x^2+2x+1}{5x^2+5x}$

(j) $f(x) = \begin{cases} \frac{1}{x} & x < 0 \\ 4 & x = 0 \\ \ln x & x > 0 \end{cases}$

6. (a) Explain the difference between $\lim_{x \to -3} f(x)$ and $\lim_{x \to 3^-} f(x)$.

(b) Explain the difference between $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to \infty^+} f(x)$.

7. Decide whether each of the following statements is True or False. In the statements, there is a distinction between "intersect" and "cross." Here "cross" means to intersect and continue on. For example, the letter V is the intersection of two line segments, whereas with the letter X the two line segments intersect and continue on (i.e., they "cross").

(a) A graph can intersect a horizontal asymptote.

(b) A graph can cross a horizontal asymptote.

(c) A graph can intersect a vertical asymptote.

(d) A graph can cross a vertical asymptote.

(e) A function with domain $\mathbb{R}$ can have a vertical asymptote.

(f) A continuous function with domain $\mathbb{R}$ can have a vertical asymptote.

(g) Graphing is fun.

Section 20 - Answers

1. (a) 1. The limit is zero. 2. The limit is the ratio of the coefficients of the lead terms (the terms with the highest power of x) of the numerator and denominator 3. The limit is ∞ or $-\infty$ depending on the ratio of the signs of the lead terms in the numerator and denominator.

   (b) Yes, Yes, No

   (c) The only change is when $p > q$. Here the calculation of the sign ($\pm\infty$) must incorporate the actual degrees of the lead terms and the fact that x is negative.

   (d) Check your "rules" in class.

2. Check your "rules" in class. The lead term is the only term in the polynomial that should figure into your shortcut.

3. $\lim_{x\to\infty} a^x = 0$ and $\lim_{x\to-\infty} a^x = \infty$.

4. (a) $\frac{6}{5}$

   (b) 1

   (c) ∞

   (d) 0

   (e) no limit (RHL is $-\infty$ and LHL is ∞)

   (f) -3

   (g) $\frac{6}{5}$

   (h) $e^{\frac{1}{4}}$

   (i) ∞

   (j) 0

   (k) $\frac{1}{4}$

   (l) $\frac{1}{2}$

   (m) $-\frac{1}{2}$

   (n) 0

(o) $\frac{1}{2}$

(p) $\ln 8$

(q) 0

(r) $-\infty$

(s) 0

(t) $-\infty$

(u) ∞

(v) $-\infty$

5. (a) $y = 0$; $x = 0$

(b) $y = 0$, $x = 0$

(c) $y = 1$; $x = -1$

(d) $y = \frac{3}{5}$, $x = \sqrt{\frac{6}{5}}$, $x = -\sqrt{\frac{6}{5}}$

(e) $y = 0$; $x = -\frac{1}{2}$; $x = 1$

(f) $y = 4$; $x = 2$

(g) $y = 0$; $x = 0$; $x = 1$

(h) $y = 0$; $x = 2$; $x = 4$

(i) $y = \frac{1}{5}$; $x = 0$

(j) $y = 0$ on left only; $x = 0$ (both sides)

6. (a) $\lim_{x\to -3} f(x)$ is a two-sided limit. The x values are approaching -3.
$\lim_{x\to 3^-} f(x)$ is a one-sided limit. The x values are approaching 3 from the left. The x values are all less than 3.

(b) $\lim_{x\to +\infty} f(x)$ describes the behavior of the function on the far far right of the graph. The x values are getting larger and larger. The expression "$+\infty$" means the same thing as "∞."
$\lim_{x\to \infty^+} f(x)$ makes no sense at all. It is suggesting that x is approaching ∞ from the right (i.e., that the x values are greater than ∞).

7. (a) , (b), (c) True (d) False (e) True (f) False (g) Your call.

21 How to Sketch a Graph

We give an enlarged version of the instructions given in Section 19.

To sketch the graph of $y = f(x)$:

1. Determine the domain of f and mark it on the x-axis. Pay attention to division by zero, taking the square root (or other even root) of a negative number, and taking the logarithm of a non-positive number. Values of x which would make you do these things are not in the domain. When you sketch your graph, make sure that you do not include any points whose x values are not in the domain. You cannot "cross" a vertical asymptote.

2. For each isolated point a not in the domain, check $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ to determine the behavior of the graph near a. This is where you might find vertical asymptotes. Your graph will certainly be discontinuous at these points.

3. Find where the graph crosses the y-axis (the y-intercept) by plugging in $x = 0$ to get $f(0)$. If convenient, find where the graph crosses the x-axis (the x-intercept(s) - there might be one, or more than one, or there might be none) by solving the equation $f(x) = 0$. This equation may be hard to solve; in that case don't bother with finding the x-intercepts.

4. If the domain is unbounded in either direction, check $\lim_{x\to\pm\infty} f(x)$, as appropriate. Here you will find the behavior of your function at the extreme ends of your graph. If a limit is finite, you have a horizontal asymptote in that direction. Remember that it is OK for a graph to cross a horizontal asymptote.

5. Find $f'(x)$. Find the critical points of f (those values of x in the domain of f where $f'(x) = 0$ or $f'(x)$ is undefined). Determine where f is increasing ($f'(x) > 0$) and where it is decreasing ($f'(x) < 0$). This will tell you the location of any local extrema.

6. Find $f''(x)$. Find the values of x in the domain for which $f''(x) = 0$ or $f''(x)$ is undefined. Determine where f is concave up ($f''(x) > 0$) and where it is concave down ($f''(x) < 0$). This will tell you the location of any points of inflection.

7. Sketch in any asymptotes. Plot the intercepts, the critical points, and the points of inflection.

8. Join the points with a smooth curve, making sure that the concavity is clear. Extend your curves toward any asymptotes, as called for by the limits found in steps 2 and 4. Again, make sure that this is done with the correct concavity.

9. Check your graph for inconsistencies against the increasing/decreasing information found in step 5. Recheck to be sure that your graph has the correct domain.

Example 21.1. Sketch the graph of $f(x) = \dfrac{x^2+x-2}{x^2-x}$

Solution: $f(x) = \dfrac{(x+2)(x-1)}{x(x-1)} = \dfrac{x+2}{x}$ when $x \neq 1$.

The domain of f is: $(-\infty,0)\cup(0,1)\cup(1,\infty)$.

$\lim\limits_{x\to 1} \dfrac{x+2}{x} = 3$, so there is a "hole" in the graph at the coordinates $(1,3)$.

$\lim\limits_{x\to 0^-} \dfrac{x+2}{x} = -\infty$ and $\lim\limits_{x\to 0^+} \dfrac{x+2}{x} = \infty$, so there is a vertical asymptote at $x=0$ (the y-axis).

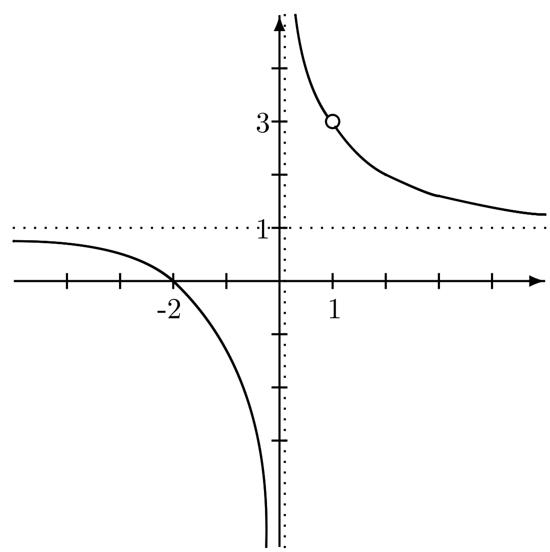
$f(0)$ does not exist, so there is no y-intercept.

$f(x)=0$ at $x=-2$, so there is one x intercept: $(-2,0)$.

$\lim\limits_{x\to\pm\infty} \dfrac{x+2}{x} = 1$, so there is a horizontal asymptote $y=1$ in both directions.

$f'(x) = \dfrac{-2}{x^2}$. There are no critical points. $f' < 0$ on each interval of the domain, so f is decreasing on each interval of the domain. There are no local extrema.

$f''(x) = \dfrac{4}{x^3}$. $f'' < 0$ on $(-\infty,0)$, so f is concave down on that interval. $f'' > 0$ on $(0,1)$ and on $(1,\infty)$ so it is concave up on those intervals. There are no POI.



$f(x) = \dfrac{x^2+x-2}{x^2-x}$

Example 21.2. Sketch the graph of $f(x) = \dfrac{2x^2 - 1}{x^2 - 1}$.

Solution: $f(x) = \dfrac{2x^2 - 1}{x^2 - 1} = \dfrac{2x^2 - 1}{(x+1)(x-1)}$

The domain of f is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

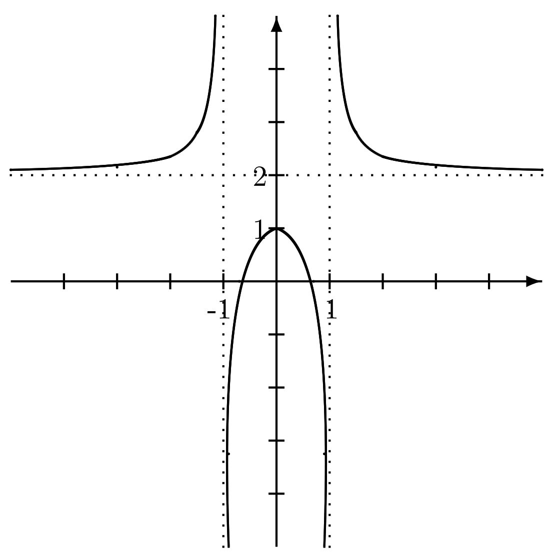
$\lim\limits_{x \to -1^-} \dfrac{2x^2 - 1}{x^2 - 1} = \infty$ and $\lim\limits_{x \to -1^+} \dfrac{2x^2 - 1}{x^2 - 1} - \infty$, so there is a vertical asymptote, $x = -1$.
$\lim\limits_{x \to 1^-} \dfrac{2x^2 - 1}{x^2 - 1} = -\infty$ and $\lim\limits_{x \to 1^+} \dfrac{2x^2 - 1}{x^2 - 1} = \infty$, so there is a vertical asymptote, $x = 1$.

$f(0) = 1$, so the y-intercept is 1. $f(x) = 0$ at $x = \pm\frac{\sqrt{2}}{2} \approx \pm.7071$, so there are two x-intercepts.

$\lim\limits_{x \to \pm\infty} \dfrac{2x^2 - 1}{x^2 - 1} = 2$, so there is a horizontal asymptote $y = 2$ in both directions.

$f'(x) = \dfrac{-2x}{(x^2+1)^2}$. The only critical point is $x = 0$. $f' > 0$ when x is in $(-\infty, -1)$ or $(-1, 0)$, so f is increasing on those intervals. $f' < 0$ when x is in $(0, 1)$ or $(1, \infty)$, so f is decreasing on those intervals. There is a local maximum at $(0, 1)$.

$f''(x) = \dfrac{2(3x^2 + 1)}{(x^2 - 1)^3}$. $f'' \neq 0$. $f'' < 0$ when x is in $(-1, 1)$, so it is concave down in this interval. $f'' > 0$ when x is in $(-\infty, -1)$ or $(1, \infty)$, so f is concave up on those intervals. There are no POI.



$$f(x) = \frac{2x^2 - 1}{x^2 - 1}$$

Section 21 - Exercises (answers follow)

Sketch the graph of the function using the instructions given in Section 21.

1. $f(x) = \dfrac{x-1}{x+1}$
2. $f(x) = \dfrac{x^2-3}{x+1}$
3. $f(x) = \dfrac{x^2}{1+x^2}$
4. $f(x) = x^2 + \dfrac{1}{x^3}$
5. $y = \ln x$
6. $y = x^{2/3} - x^{5/3}$
7. $f(x) = \dfrac{\sqrt{5x^2+1}}{3x-5}$ (See Example 20.10 in Section 20 for asymptote information. Skip concavity for this problem; f'' is too complex.)
8. $f(x) = \sqrt{x^2-2}$.
9. $h(x) = \dfrac{1+\sqrt{x}}{1-\sqrt{x}}$.
10. $f(x) = \dfrac{x-a}{x-b}$ where a and b are arbitrary constants.
11. $f(x) = \dfrac{x-a}{x+b}$ where a and b are arbitrary constants.

Section 21 - Answers

Answers for this section include derivatives and limits and other information so that you can easily find any errors that you might have made in trying to construct graphs from your derivatives and limits and subsequent conclusions. Graph sketches are printed at the end of the "data boxes" with the exception of $y = \ln x$, with which you should by now be very familiar.

1. $f(x) = \dfrac{x-1}{x+1}$

Domain: $x \neq -1$	y-intercept: -1	x-intercept: 1
$f'(x) = \dfrac{2}{(x+1)^2}$	$f' = 0$ nowhere	f' DNE at $x = -1$
Incr. $(-\infty, -1) \cup (-1, \infty)$	Decr. nowhere	
Loc. min: none	Loc. max: none	
$f''(x) = \dfrac{-4}{(x+1)^3}$	$f'' = 0$ nowhere	f'' DNE at $x = -1$
Conc. up $(-\infty, -1)$	Conc. down $(-1, \infty)$	P.O.I. none
$\lim_{x\to-\infty} f(x) = 1$	$\lim_{x\to\infty} f(x) = 1$	Horiz. Asymp: $y = 1$
$\lim_{x\to-1^-} f(x) = \infty$	$\lim_{x\to-1^+} f(x) = -\infty$	Vert. Asymp: $x = -1$

2. $f(x) = \dfrac{x^2 - 3}{x + 1}$

Domain: $x \neq -1$	y-intercept: -3	x-intercepts: $\pm\sqrt{3} \approx \pm 1.7$
$f'(x) = \dfrac{x^2 + 2x + 3}{(x+1)^2}$	$f' = 0$ nowhere	f' DNE at $x = -1$
Incr. $(-\infty, -1) \cup (-1, \infty)$	Decr. nowhere	
Loc. min: none	Loc. max: none	
$f''(x) = \dfrac{-4}{(x+1)^3}$	$f'' = 0$ nowhere	f'' DNE at $x = -1$
Conc. up $(-\infty, -1)$	Conc. down $(-1, \infty)$	P.O.I. none
$\lim\limits_{x \to -\infty} f(x) = -\infty$	$\lim\limits_{x \to \infty} f(x) = \infty$	Horiz. Asymp: none
$\lim\limits_{x \to -1^-} f(x) = \infty$	$\lim\limits_{x \to -1^+} f(x) = -\infty$	Vert. Asymp: $x = -1$

3. $f(x) = \dfrac{x^2}{1 + x^2}$

Domain: $(-\infty, \infty)$	y-intercept: 0	x-intercept: 0
$f'(x) = \dfrac{2x}{(1+x^2)^2}$	$f' = 0$ at $x = 0$	f' DNE: nowhere
Incr. $(0, \infty)$	Decr. $(-\infty, 0)$	
Loc. min. at $x = 0$	Loc. max: none	
$f''(x) = \dfrac{-2(3x^2 - 1)}{(1+x^2)^3}$	$f'' = 0$ at $x = \pm\sqrt{\frac{1}{3}} \approx \pm.58$	f'' DNE: nowhere
Conc. up $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$	Conc. down $\left(-\infty, -\sqrt{\frac{1}{3}}\right) \cup \left(\sqrt{\frac{1}{3}}, \infty\right)$	P.O.I. $\left(-\sqrt{\frac{1}{3}}, \frac{1}{4}\right)$ and $\left(\sqrt{\frac{1}{3}}, \frac{1}{4}\right)$
$\lim\limits_{x \to -\infty} f(x) = 1$	$\lim\limits_{x \to \infty} f(x) = 1$	Horiz. Asymp: $y = 1$
		Vert. Asymp: none

4. $f(x) = x^2 + \dfrac{1}{x^3} = \dfrac{x^5 + 1}{x^3}$

Domain: $x \neq 0$	y-intercept: none	x-intercept: -1
$f'(x) = 2x - 3x^{-4} = \dfrac{2x^5 - 3}{x^4}$	$f' = 0$ at $x = \sqrt[5]{\frac{3}{2}} \approx 1.1$	f' DNE at $x = 0$
Incr. $\left(\sqrt[5]{\frac{3}{2}}, \infty\right)$	Decr. $(-\infty, 0) \cup \left(0, \sqrt[5]{\frac{3}{2}}\right)$	
Loc. min. at $x = \sqrt[5]{\frac{3}{2}}$	Loc. max: none	
$f''(x) = 2 + 12x^{-5} = \dfrac{2(x^5 + 6)}{x^5}$	$f'' = 0$ at $x = -\sqrt[5]{6} \approx -1.4$	f'' DNE at $x = 0$
Conc. up $(-\infty, -\sqrt[5]{6}) \cup (0, \infty)$	Conc. down $(-\sqrt[5]{6}, 0)$	P.O.I. $\left(-\sqrt[5]{6}, \dfrac{5}{\sqrt[5]{6^3}}\right)$
$\lim\limits_{x \to -\infty} f(x) = \infty$	$\lim\limits_{x \to \infty} f(x) = \infty$	Horiz. Asymp: none
$\lim\limits_{x \to 0^-} f(x) = -\infty$	$\lim\limits_{x \to 0^+} f(x) = \infty$	Vert. Asymp: $x = 0$

5. $y = \ln x$

Domain: $(0, \infty)$	y-intercept: none	x-intercepts: 1
$y' = \frac{1}{x}$	$f' = 0$ nowhere	y' DNE nowhere in domain
Incr. $(0, \infty)$	Decr. nowhere	
Loc. min.: none	Loc. max.: none	
$y'' = -\frac{1}{x^2}$	$y'' = 0$ nowhere	f'' DNE nowhere in domain
Conc. up nowhere	Conc. down $(0, \infty)$	P.O.I.: none
	$\lim\limits_{x \to \infty} f(x) = \infty$	Horiz. Asymp: none
	$\lim\limits_{x \to 0^+} f(x) = -\infty$	Vert. Asymp: $x = 0$

6. $y = x^{2/3} - x^{5/3} = x^{2/3}(1 - x)$

Domain: $(-\infty, \infty)$	y-intercept: 0	x-intercepts: 0 and 1
$y' = \frac{2}{3}x^{-1/3} - \frac{5}{3}x^{2/3} = \frac{1}{3}x^{-1/3}(2 - 5x)$	$f' = 0$ at $x = \frac{2}{5}$	y' DNE at $x = 0$
Incr. $(0, \frac{2}{5})$	Decr. $(-\infty, 0) \cup (\frac{2}{5}, \infty)$	
Loc. min. at $x = 0$	Loc. max. at $x = \frac{2}{5}$	
$y'' = -\frac{2}{9}x^{-4/3} - \frac{10}{9}x^{-1/3} = -\frac{2}{9}x^{-4/3}(1 + 5x)$	$y'' = 0$ at $x = -\frac{1}{5}$	f'' DNE at $x = 0$
Conc. up $(-\infty, -\frac{1}{5})$	Conc. down $(-\frac{1}{5}, \infty)$	P.O.I. $\left(-\dfrac{1}{5}, \dfrac{6}{5\sqrt[3]{25}}\right)$
$\lim\limits_{x \to -\infty} f(x) = \infty$ (Think: $x^{\frac{2}{3}} > 0$ because of the square,	$\lim\limits_{x \to \infty} f(x) = -\infty$	Horiz. Asymp: none
and $(1 - x) > 0$ when x is negative. So, $+ \cdot + = +$).		Vert. Asymp: none

7. $f(x) = \dfrac{\sqrt{5x^2 + 1}}{3x - 5}$

Domain: $x \neq \frac{5}{3}$	y-intercept: $-\frac{1}{5}$	x-intercept: none
$f'(x) = \dfrac{-(25x + 3)}{(3x - 5)^2\sqrt{5x^2 + 1}}$	$f' = 0$ at $-\frac{3}{25}$	f' DNE at $x = \frac{5}{3}$
Incr. $\left(-\infty, -\frac{3}{25}\right)$	Decr. $\left(-\frac{3}{25}, \frac{5}{3}\right), \left(\frac{5}{3}, \infty\right)$	
Loc. min: None	Loc. max: at $x = -\frac{3}{25}$	
$f''(x)$ is too complex. Don't bother		
$\lim\limits_{x \to -\infty} f(x) = -\dfrac{\sqrt{5}}{3}$	$\lim\limits_{x \to \infty} f(x) = \dfrac{\sqrt{5}}{3}$	Horiz. Asymp: $y = -\frac{\sqrt{5}}{3} \approx -.75$ on LEFT
		Horiz. Asymp: $y = \frac{\sqrt{5}}{3} \approx .75$ on RIGHT
$\lim\limits_{x \to \frac{5}{3}^-} f(x) = -\infty$	$\lim\limits_{x \to \frac{5}{3}^+} f(x) = \infty$	Vert. Asymp: $x = \frac{5}{3}$

8. $f(x)=\sqrt{x^2-2}$.

Domain: $(-\infty,-\sqrt{2}]\cup[\sqrt{2},\infty)$	y-intercept: none	x-intercepts: $\pm\sqrt{2}\approx\pm1.4$
$f'(x)=\dfrac{x}{\sqrt{x^2-2}}$	$f'=0$ nowhere in domain	f' DNE at $x=\pm\sqrt{2}$
Incr. $(\sqrt{2},\infty)$	Decr. $(-\infty,-\sqrt{2})$	
Loc. min: none	Loc. max: none	
$f''(x)=\dfrac{-2}{(x^2-2)^{\frac{3}{2}}}$	$f''=0$ nowhere	f'' DNE at $x=\pm\sqrt{2}$
Conc. up nowhere	Conc. down $(-\infty,-\sqrt{2})\cup(\sqrt{2},\infty)$	P.O.I. none
$\lim\limits_{x\to-\infty}f(x)=\infty$	$\lim\limits_{x\to\infty}f(x)=\infty$	Horiz. Asymp: none
		Vert. Asymp: none

9. $h(x)=\dfrac{1+\sqrt{x}}{1-\sqrt{x}}$.

Domain: $[0,1)\cup(1,\infty)$	y-intercept: 1	x-intercept: none
$f'(x)=\dfrac{1}{\sqrt{x}(1-\sqrt{x})^2}=\dfrac{1}{x^{\frac{1}{2}}+2x+x^{\frac{3}{2}}}$	$f'=0$ nowhere	f' DNE at $x=0$ and $x=1$
Incr. $(0,1)\cup(1,\infty)$	Decr. nowhere	
Loc. min: none	Loc. max: none	
$f''(x)=\dfrac{-x^{-\frac{1}{2}}+3}{2x(1-x^{\frac{1}{2}})^3}=\dfrac{-1+3x^{\frac{1}{2}}}{2x^{\frac{3}{2}}(1-x^{\frac{1}{2}})^3}$	$f''=0$ at $x=\frac{1}{9}$	f'' DNE at $x=0$ and $x=1$
Conc. up $(\frac{1}{9},1)$	Conc. down $(0,\frac{1}{9})\cup(1,\infty)$	P.O.I. $(\frac{1}{9},2)$
	$\lim\limits_{x\to\infty}f(x)=-1$	Horiz. Asymp: $y=-1$ on RIGHT
$\lim\limits_{x\to1^-}f(x)=\infty$	$\lim\limits_{x\to1^+}f(x)=-\infty$	Vert. Asymp: $x=1$

10. $f(x)=\dfrac{x-a}{x-b}$ where a and b are arbitrary constants.

There are three cases to consider here: $a=b$, $a>b$, and $a<b$.

If $a=b$, then $f(x)=1$ with domain: $x\neq b$. The graph is the horizontal line $y=1$ with the point $(b,1)$ removed. The table below assumes that $a\neq b$.

Domain: $x\neq b$	y-intercept: $\frac{a}{b}$	x-intercept: a
$f'(x)=\dfrac{-b+a}{(x-b)^2}$	$f'=0$ nowhere, since $a\neq b$	f' DNE at $x=b$
Incr. on domain if $a>b$	Decr. nowhere if $a>b$	
Incr. nowhere if $a<b$	Decr. on domain if $a<b$	
Loc. min: none	Loc. max: none	
$f''(x)=\dfrac{-2(-b+a)}{(x-b)^3}$	$f''=0$ nowhere since $a\neq b$	f'' DNE at $x=b$
Conc. up $(-\infty,b)$ if $a>b$	Conc. down (b,∞) if $a>b$	P.O.I. none
Conc. up (b,∞) if $a<b$	Conc. down $(-\infty,b)$ if $a<b$	
$\lim\limits_{x\to-\infty}f(x)=1$	$\lim\limits_{x\to\infty}f(x)=1$	Horiz. Asymp: $y=1$
If $a>b$: $\lim\limits_{x\to b^-}f(x)=\infty$	$\lim\limits_{x\to b^+}f(x)=-\infty$	Vert. Asymp: $x=b$
If $a<b$: $\lim\limits_{x\to b^-}f(x)=-\infty$	$\lim\limits_{x\to b^+}f(x)=\infty$	Vert. Asymp: $x=b$

11. $f(x) = \dfrac{x-a}{x+b}$ where a and b are arbitrary constants.

There are three cases to consider here: $a = -b$, $a > -b$, and $a < -b$.

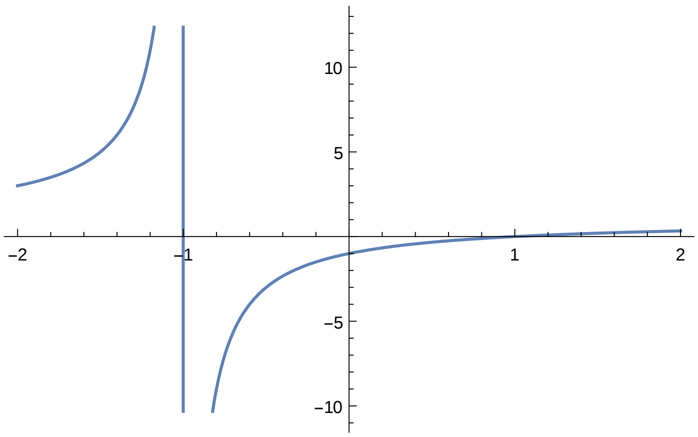
If $a = -b$, then $f(x) = 1$ with domain: $x \neq -b$. The graph is the horizontal line $y = 1$ with the point $(-b, 1)$ removed. The table below assumes that $a \neq -b$.

Domain: $x \neq -b$	y-intercept: $-\frac{a}{b}$	x-intercept: a
$f'(x) = \dfrac{b+a}{(x+b)^2}$	$f' = 0$ nowhere, since $a \neq -b$	f' DNE at $x = -b$
Incr. on domain if $a > -b$	Decr. nowhere if $a > -b$	
Incr. nowhere if $a < -b$	Decr. on domain if $a < -b$	
Loc. min: none	Loc. max: none	
$f''(x) = \dfrac{-2(b+a)}{(x+b)^3}$	$f'' = 0$ nowhere since $a \neq -b$	f'' DNE at $x = -b$
Conc. up $(-\infty, -b)$ if $a > -b$	Conc. down $(-b, \infty)$ if $a > -b$	P.O.I. none
Conc. up $(-b, \infty)$ if $a < -b$	Conc. down $(-\infty, -b)$ if $a < -b$	
$\lim\limits_{x\to-\infty} f(x) = 1$	$\lim\limits_{x\to\infty} f(x) = 1$	Horiz. Asymp: $y = 1$
If $a > -b$: $\lim\limits_{x\to-b^-} f(x) = \infty$	$\lim\limits_{x\to-b^+} f(x) = -\infty$	Vert. Asymp: $x = -b$
If $a < -b$: $\lim\limits_{x\to-b^-} f(x) = -\infty$	$\lim\limits_{x\to-b^+} f(x) = \infty$	Vert. Asymp: $x = -b$

Graphs for some of the exercises in this section can be found on the next two pages. They are done using Mathematica, a program you might want to explore.

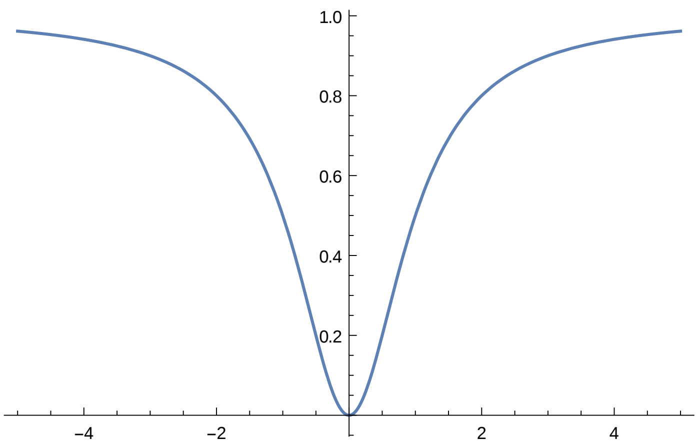
In[1]:= `Plot[{x - 1} / {x + 1}, {x, -2, 2}]`

Out[1]=



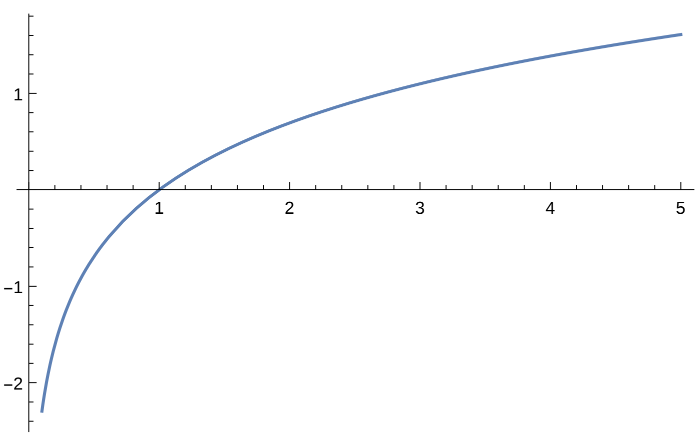
In[3]:= `Plot[{x^{2}} / {1 + {x^{2}}}, {x, -5, 5}]`

Out[3]=



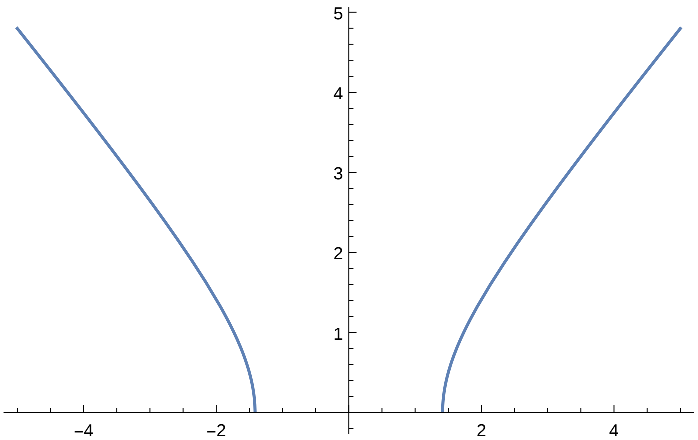
In[6]:= `Plot[Log[x], {x, 0.01, 5}]`

Out[6]=



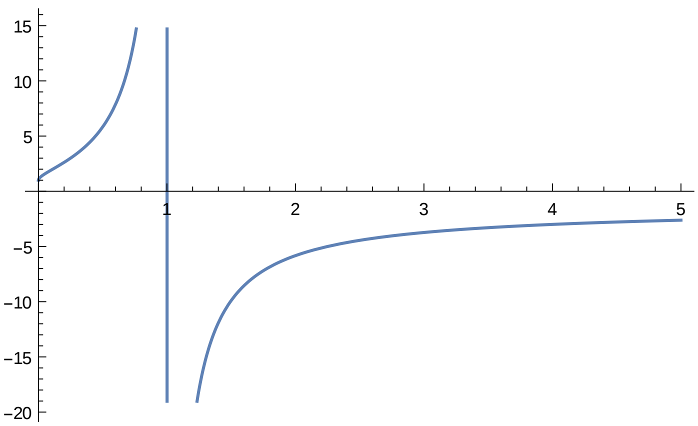
In[7]:= `Plot[{x^{2} - 2}^{1 / 2}, {x, -5, 5}]`

Out[7]=



In[9]:= `Plot[{1 + {x}^{1 / 2}} / {1 - {x}^{1 / 2}}, {x, 0, 5}]`

Out[9]=

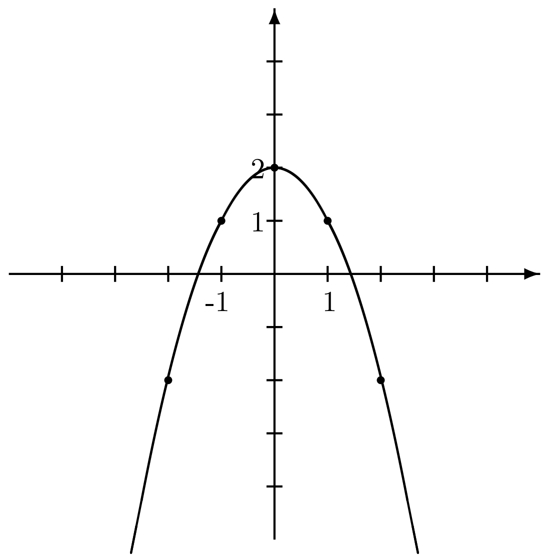


22 The Absolute Maximum and Minimum

The *absolute maximum* of f on a given interval I is M if (i) there is some a in I such that $f(a) = M$, and (ii) $f(x) \leq M$ for every x in I.
The *absolute minimum* of f on a given interval I is m if (i) there is some a in I such that $f(a) = m$, and (ii) $f(x) \geq m$ for every x in I.

We are saying that the absolute maximum is the highest y-value that function f attains on some specific interval subset of its domain. The absolute minimum is the lowest such y value.

Example 22.1. Find the absolute extrema (max and min) of the function $f(x) = -x^2 + 2$ on each of the following intervals: $(-\infty, \infty)$, $[1, 2]$, $(1, 2]$, $[-2, 2]$.
Solution: Sketch the graph of $f(x) = -x^2 + 2$. It is a concave down parabola with summit (vertex) at the point $(0, 2)$.



$f(x) = -x^2 + 2$

On the interval $(-\infty, \infty)$ the absolute maximum *value* of f is 2. It occurs *at* $x = 0$ (There is also a local maximum there). There is no absolute minimum value because the function is unbounded in the negative direction. Formally, $\lim_{x\to-\infty}(-x^2 + 2) = -\infty$ and $\lim_{x\to\infty}(-x^2 + 2) = -\infty$.

On the interval $[1, 2]$ the function is strictly decreasing, so there are no local extrema. $f(1) = 1$ and $f(2) = -2$, so the absolute maximum value of f is 1 and it occurs at $x = 1$. The absolute minimum value is -2 and it occurs at $x = 2$.

On the interval $(1, 2]$ there is no absolute maximum. There is no x value in $(1, 2]$ that achieves a highest y value. The absolute minimum value is -2 and it occurs at $x = 2$.

On the interval $[-2, 2]$ the absolute maximum value is 2 at $x = 0$. The absolute minimum value is -2 and it occurs at $x = -2$ AND at $x = 2$.

Notice in the last part of Example 22.1 that the absolute minimum value occurred at both $x = -2$ and $x = 2$.. Look back at the definition of absolute minimum. The requirement is that m

be less than *or equal to* all of the other y values. This allows for a tie. There will be at most one absolute maximum *value* and at most one absolute minimum *value*, but these extrema can occur in multiple places (at more than one x value). We want to find all of the places (x values) where absolute extrema occur.

Could an absolute maximum value ever equal an absolute minimum value?

Example 22.2. Find the absolute extrema of $f(x) = 4$ on the interval $(-\infty, \infty)$.
Solution. f is the constant function whose graph is the horizontal line $y = 4$.

The largest y value for f is 4, so 4 is the absolute maximum value of f. It occurs at all x in interval $(-\infty, \infty)$.

The smallest y value for f is 4, so 4 is the absolute minimum value of f. It occurs at all x in interval $(-\infty, \infty)$.

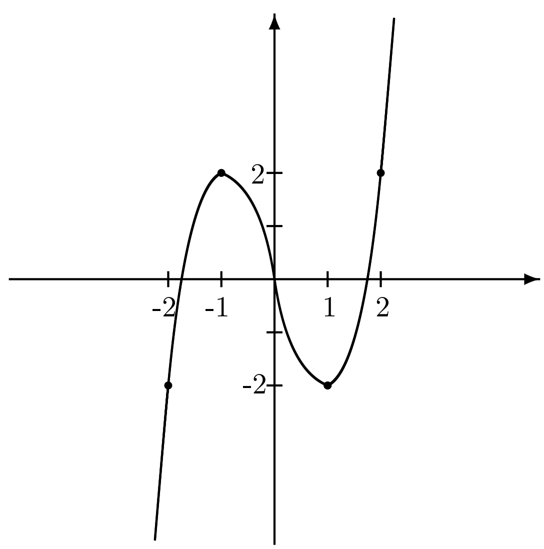
Example 22.3. Find the absolute extrema of $f(x) = x^3 - 3x$ on each of the following intervals: $(-\infty, \infty)$, $[-4, 4]$, $[-2, 2]$, $(-2, 3)$
Solution: Consider the graph of f below.

On $(-\infty, \infty)$ there are no absolute extrema. (There are local extrema, but no absolute extrema. There are y values higher than the $y = 2$ of the local max, and there are y values lower than the $y = -2$ of the local min).

On interval $[-4, 4]$ the absolute maximum value is 52 at $x = 4$. The absolute minimum value is -52 at $x = -4$.

On interval $[-2, 2]$ the absolute maximum value is 2 at $x = -1$ and at $x = 2$. The absolute minimum value is -2 at $x = -2$ and $x = 1$.

On $(-2, 3)$ there is no absolute maximum. The absolute minimum value is -2 at $x = 1$.



$f(x) = x^3 - 3x$

We will not always have a graph to help us find the absolute extrema. From the previous examples, you may have noticed that any absolute extrema occured at local extrema or at endpoints of the interval. In fact, these are the only places where we need to look. Does this make sense? If this isn't clear to you, sketch some arbitrary graphs on any interval and look for absolute extrema. Can you find any absolute extrema elsewhere on an interval?

The above paragraph tells us that to find an absolute maximum[42] on an interval I we need to:

1. Find all local maxima, as in Section 15.

2. If I has endpoints a and/or b, compare $f(a)$ and/or $f(b)$ and the y values at all local maxima to see what is the largest.

3. If I is not a closed interval, check what happens to $f(x)$ as x apporaches missing ends of the interval.

4. Then use common sense to decide where (if anywhere) f achieves its maximum value on I.

Now we know how to find absolute extrema of a function on some interval I. We are very often interested in finding the absolute extrema of a function over its entire domain. The process is just the logical extension of the process above. We need to find any local extrema and the endpoints of any intervals[43] in the domain. We also check the behavior of the function as x approaches any missing endpoints of intervals in the domain. This last includes looking at $x \to -\infty$ and $x \to \infty$ if applicable.

Example 22.4. Find the absolute extrema for $f(x) = -\frac{2}{3}x^3 + 4x^2 - 6x + 10$ on the interval $[0, 5]$.
Solution: $f'(x) = -2x^2 + 8x - 6 = -2(x^2 - 4x + 3) = -2(x-1)(x-3)$

We check the y values of the critical points and the endpoints:
$f(0) = 10 \qquad f(1) = 7\frac{1}{3} \qquad f(3) = 10 \qquad f(5) = -3\frac{1}{3}$.
So, f has abs. max. value of 10 at $x = 0$ and $x = 3$. f has abs. min. value of $-3\frac{1}{3}$ at $x = 5$.

In Example 22.4 we were guaranteed by the Extreme Value Theorem (see page 132) to find both an absolute maximum and an absolute minimum. This is not the case in Example 22.5.

Example 22.5. Find any absolute extrema for $f(x) = \dfrac{1-x^2}{x^3}$ on interval $[1, \infty)$.

Solution: $f'(x) = \dfrac{-2x \cdot x^3 - 3x^2(1-x^2)}{x^6} = \dfrac{-2x^4 - 3x^2 + 3x^4}{x^6} = \dfrac{x^4 - 3x^2}{x^6} = \dfrac{x^2-3}{x^4}$.

$f'(x) = 0$ at $x = \pm\sqrt{3}$. $f'(x)$ D.N.E. at $x = 0$. The only critical point is $x = \sqrt{3}$ because it is the only one in the domain.

We see from the f' that f decreases on $[1, \sqrt{3}]$ and f increases on $[\sqrt{3}, \infty)$. So, the absolute minimum value of f is $f(\sqrt{3}) = \dfrac{-2}{3\sqrt{3}}$.

$\lim\limits_{x\to\infty} f(x) = 0$ so on the right f is increasing *towards* the horizontal asymptote $y = 0$. Since $f(1) = 0$, the absolute maximum value is 0 at $x = 1$.

[42] An entirely similar procedure is used to find the absolute minimum.
[43] By "interval" here we mean a largest continuous segment of the domain.

In Example 22.5 we used intervals of increasing and decreasing to show that we had an absolute maximum at $x = 1$. We were essentially using the First Derivative Test on a global (rather than local) scale. The First Derivative Test and the Second Derivative Test tell us if we have a *local* extremum at a critical point. Those tests can be used under certain circumstances to claim an absolute extremum.

Theorem. *Suppose f is continuous on interval I and a is the only critical point of f in I. If there is a local maximum (or min) at a, then f has an absolute max (or min) at a.*

Proof. We will prove this theorem by contradiction. Suppose that f does not have an absolute maximum at a. Then there is some b in I such that $f(b) > f(a)$. Without loss of generality, assume the $a < b$.

Since there is a local max at $x = a$, there is some open interval containing a such that $f(a) \geq f(x)$ for all x in the interval. Clearly, b is not in this interval. If for every x in this interval it is true that $f(a) = f(x)$, then there are local maxima at each of these points. This means that these points are critical points, which is contrary to the given.

If it is not true that $f(a) = f(x)$for all x in the interval, then there must be some c in the interval such that $a < c < b$ and $f(c) < f(a)$. Since f is continuous on $[c, b]$ and $f(c) < f(a) < f(b)$ there must be some d in (c, b) such that $f(d) = f(a)$ (Intermediate Value Theorem). If f is differentiable on (a, d) then there must be some e in (a, d) such that $f'(e) = 0$ (Rolle's Theorem). If f is not differentiable on (a, d) then there is some k in (a, d) such that $f'(k)$ D.N.E. In either event (the existence of e or k) we have established that there must be a critical point in (a, d). This contradicts the given that a is the only critical point in I.

A similar proof handles the (minimum) case. □

Example 22.6. A manufacturer can make a profit of $P(q)$ (in hundreds of dollars) from the sale of q thousand items according to the formula $P(q) = -q^3 + 9q + 3$ How many items should be sold to maximize profit? What is that maximum profit?

Solution: The domain of the function $P(q)$ is $[0, \infty)$ because a negative value for q makes no sense in the context of this problem.

$P'(q) = -3q^2 + 9 = -3(q^2 - 3)$. $\quad P'(q) = 0$ at $q = \pm\sqrt{3}$. The only critical point is $q = \sqrt{3}$.

$P''(q) = -6q$. $P''(\sqrt{3}) = -6\sqrt{3} < 0$. So, P has a local maximum value at $q = \sqrt{3}$.

Since P is continuous (a polynomial) and $q = \sqrt{3}$ is the only critical point, this local max must be an absolute max.

Now $\sqrt{3}$ is approximately 1.732, so maximum profit occurs when 1,732 items are produced. $P(1.732) = 13.3923$. Thus, the profit is \$1,339.23. (It is always important to remember the units of measure).

Section 22 - Exercises (answers follow)

1. Prove that if the absolute maximum value, M, is equal to the absolute minimum value, m, for some function f, then f must be a constant function.

2. Why does the Extreme Value Theorem (page 132) not apply to Example 22.5?

3. Find the locations of all absolute maxima and minima for the functions defined as follows, with the specified domains.

   (a) $f(x) = x^3 - 6x^2 + 9x - 8$; $[0, 5]$

   (b) $f(x) = x^5 - 5x^4 + 2$; $[0, 4]$

   (c) $f(x) = \frac{x+1}{x-1}$ on $[2, 4]$

   (d) $f(x) = \frac{x}{x^2+3}$; $[0, 5]$

   (e) $f(x) = \frac{1}{x+1}$; $x \geq 0$

   (f) $g(x) = x^2 - 4\sqrt{x}$; $[0, 10]$

   (g) $h(x) = \sqrt{x^2 - 9}$

4. A manufacturer can produce widgets at a cost of \$5 apiece and estimates that if they are sold for x dollars apiece, consumers will buy $20 - x$ widgets per day. At what price per piece should she sell the widgets to maximize profit?

5. The total profit $P(x)$ (in thousands of dollars) from the sale of x hundred thousand items is $P(x) = -x^3 + 9x^2 + 120x - 400$ where $x \geq 5$. Find the number of items that must be sold to maximize profit. Find the maximum profit.

6. The estimated monthly profit (in dollars) realizable by a company for manufacturing and selling x units is: $P(x) = -0.04x^2 + 240x - 10,000$. How many units should they produce per month in order to maximize profits?

7. The profit $P(x)$ (in thousands of dollars) from the sale of x units of a certain commodity is given by $P(x) = \ln(-x^3 + 3x^2 + 72x + 1)$ for x in $[0, 10]$. Note: $-x^3 + 3x^2 + 72x + 1 \geq 0$ for all x in$[0, 10]$

   (a) Find the number of units that should be sold in order to maximize the total profit.

   (b) What is the maximum profit?

8. Given $f(x) = x^2e^{-x}$

   (a) Find any local extrema (max or min).

   (b) Does f have an absolute max? Justify your answer with a limit.

   (c) Does f have an absolute min? (We have not covered this limit in this course, but rewrite f as $f(x) = \frac{x^2}{e^x}$ and think how the numerator and denominator behave relative to each other as $x \to \infty$.

9. Use graphing techniques (first and second derivatives and limits) to graph each of the following functions. Then determine any absolute extrema.

   (a) $f(x) = \dfrac{(x+1)^2}{x^2+1}$

   (b) $g(x) = x^2 - |x|$ Hint: You could rewrite g into the equivalent piecewise defined form.

Section 22 - Answers

1. Hint: This proof follows directly from the definitions of absolute max and absolute min.

2. The domain $[1, \infty)$ is not a closed interval.

3. (a) absolute maximum at $x = 5$; absolute minimum at $x = 0$, $x = 3$.

   (b) absolute maximum at $x = 0$; absolute minimum at $x = 4$

   (c) absolute maximum at $x = 2$; absolute minimum at $x = 4$

   (d) absolute maximum at $x = \sqrt{3}$; absolute minimum at $x = 0$

   (e) absolute maximum at $x = 0$; no absolute minimum

   (f) absolute maximum at $x = 10$; absolute minimum at $x = 1$

   (g) absolute minimum at $x = 3$ and at $x = -3$; no absolute maximum ($x = 0$ is not in the domain)

4. \$12.50

5. Maximum occurs when $x = 10$. $P(10) = 700$. So, sell 1,000,000 items for a profit of \$700,000.

6. 3,000

7. (a) 6 (b) \$5784

8. (a) local min at $(0, 0)$ and local max at $(2, \frac{4}{e^4})$

   (b) No. $\lim\limits_{x\to-\infty} x^2e^{-x} = \infty$

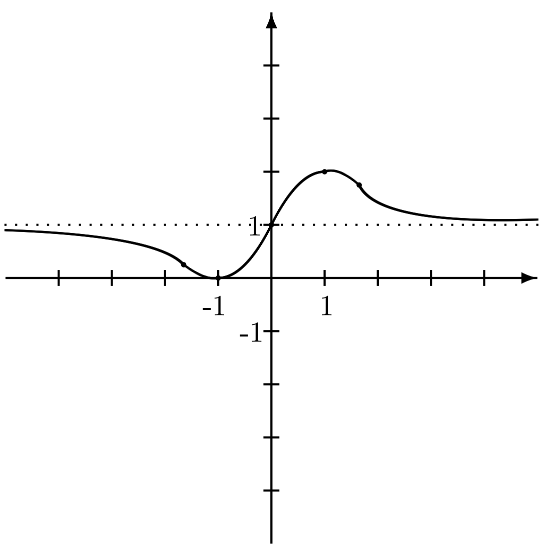
   (c) Yes. Absolute minimum value is 0 at $x = 0$.

9. (a) $f(x) = \dfrac{(x+1)^2}{x^2+1}$.

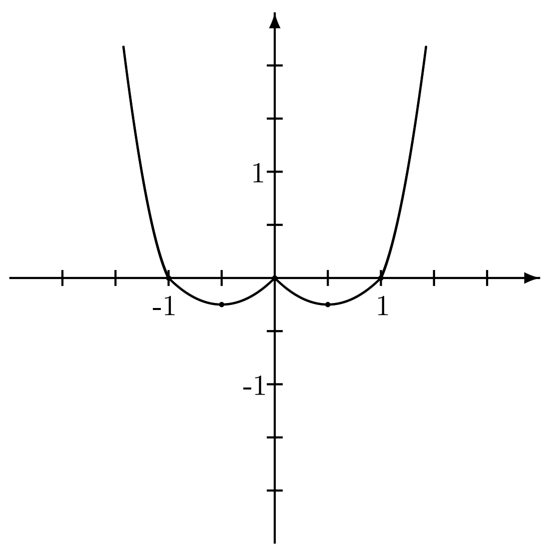
Domain: $(-\infty, \infty)$	y-intercept: 1	x-intercept: -1
$f'(x) = \dfrac{2-2x^2}{(x^2+1)^2} = \dfrac{2(1-x)(1+x)}{(x^2+1)^2}$	$f' = 0$ at $x = \pm 1$	f' DNE nowhere
Incr. $[-1, 1]$	Decr. $(-\infty, -1] \cup [1, \infty)$	
Loc. min: at $x = -1$	Loc. max: at $x = 1$	
$f''(x) = \dfrac{4x(x^2-3)}{(x^2+1)^3}$	$f'' = 0$ at $x = 0$ and $x = \pm\sqrt{3}$	f'' DNE nowhere
Conc. up $(-\sqrt{3}, 0) \cup (1, \infty)$	Conc. down $(-\infty, -\sqrt{3}) \cup (0, 1)$	P.O.I. at $0, \pm\sqrt{3}$
$\lim\limits_{x\to-\infty} f(x) = 1$	$\lim\limits_{x\to\infty} f(x) = 1$	Horiz. Asymp: $y = 1$
		Vert. Asymp: none
Absolute max value of 2 at $x = 1$	**Absolute min** value of 0 at $x = -1$	

(b) $g(x) = x^2 - |x| = \begin{cases} x^2 + x & x < 0 \\ x^2 - x & x \geq 0 \end{cases}.$

Domain: $(-\infty, \infty)$	y-intercept: 0	x-intercepts: 0, ± 1
$g'(x) = \begin{cases} 2x + 1 & x < 0 \\ 2x - 1 & x > 0 \end{cases}$	$f' = 0$ at $x = \pm\frac{1}{2}$	f' DNE at $x = 0$
Incr. $(-\frac{1}{2}, 0) \cup (\frac{1}{2}\infty)$	Decr. $(-\infty, -\frac{1}{2} \cup (0, \frac{1}{2})$	
Loc. min at $x = \pm\frac{1}{2}$	Loc. max at $x = 0$	
$g''(x) = \begin{cases} 2 & x < 0 \\ 2 & x > 0 \end{cases}$	$g'' = 0$ nowhere	g'' DNE at $x = 0$
Conc. up everywhere	Conc. down nowhere	P.O.I. none
$\lim\limits_{x \to -\infty} f(x) = \infty$	$\lim\limits_{x \to \infty} f(x) = \infty$	Horiz. Asymp: none
		Vert. Asymp: none
Absolute max: none	**Absolute min** value of $-\frac{1}{4}$ at $x = \pm\frac{1}{2}$	



$f(x) = \dfrac{(x+1)^2}{x^2+1}$



$g(x) = x^2 - |x|$

23 Optimization

In Section 22 we found out how to find the absolute maximum and/or absolute minimum values of functions. In this section we look at applications of this process. These are called optimization problems. We repeat, with some modifications, the General Strategy for Word Problems from Section 14.

1. Read the problem twice.

   (a) Identify in words what you are trying to compute or find. Often, but not always, you can find this in a sentence that ends with a question mark. Optimization problems will include some entity for which you want to find an extreme value or the location of an extreme value.

   (b) Identify in words the facts that are given.

   Get clear in your mind which parts are (a) and which parts are (b).

2. Translate (a) and (b) into mathematical statements using mathematical symbols.

3. Determine a mathematical relationship that connects the given (b) with the unknown (a). Often, drawing a picture can help immensely.

4. Restate the mathematical relationship into an equation. For optimization problems you will have some entity for which you want to find an extremum. This entity will become the dependent variable because its optimization will depend on one or more other entities (the independent variables). If you have multiple independent variables, look for sentences in the problem that connect the independent variables to each other mathematically. You will want to solve for one in terms of the other(s). Ultimately, you want to have an equation where the dependent variable is expressed in terms of only one independent variable.

5. Solve the equation. For optimization problems, this generally means taking the derivative to find critical points, and checking function behavior at the ends of domain intervals.

6. Verify that your answer yields the correct optimization (max? or min?).

7. Clearly identify your answer, including any units of measure.

8. Check your answer for reasonableness.

Example 23.1. If Toys-B-Us charges $p(q)$ cents for a toy, they are able to sell q thousand toys, where $p(q) = 200 - \frac{q}{30}$. How many toys must they sell in order to attain maximum revenue? What is the maximum revenue?
Solution: We want to maximize revenue, dependent on the number of toys sold. So, we want to find the absolute maximum of $R(q)$.

Revenue = price × quantity. $R = pq$ $R(q) = \left(200 - \frac{q}{30}\right) q = 200q - \frac{1}{30}q^2$.
$R'(q) = 200 - \frac{1}{15}q$. $R'(q) = 0$ when $q = 3,000$.

$R''(q) = -\frac{1}{15} < 0$, so R has a local max at $q = 3,000$.
Since R is continuous and there is only one critical point, we can say that R attains its absolute maximum at $q = 3,000$.
$R(3000) = 200(3000) - \frac{1}{30}(3000)^2 = 600,000 - 300,000 = 300,000$ cents
So, the maximum revenue is \$3,000, which is attained when 3,000 toys are sold.

Example 23.2. An open box (one without a top) is to be made from an 8 ft. $\times$ 8 ft. square sheet of metal. The box is to be made by cutting out identical squares from each of the four corners of the metal and then bending up the flaps. How large should the cut-out squares be if the box is to have maximal volume? What is the volume of the largest box?
Solution: We wish to maximize the volume of a box. The volume depends on the size of the squares cut from each corner of an 8×8 metal sheet.

It is helpful to draw a picture (see below). We will call x the length of the side of a cut-out square. We call y the length of the square remaining.

Volume = length $\times$ width $\times$ height. From the picture, imagine the flaps being folded up. So, $V = y \cdot y \cdot x = y^2 x$. Since the length of the original metal square was 8, we see from the picture that $y + 2x = 8, \Longrightarrow y = -2x + 8$.

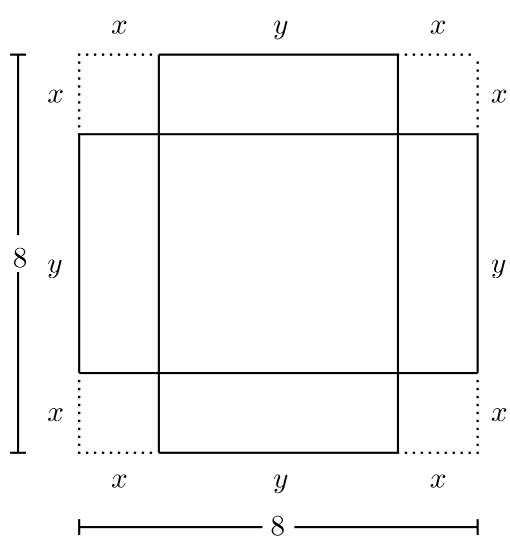
Substituting, we get $A(x) = (-2x+8)^2 x = (4x^2 - 32x + 64)x = 4x^3 - 32x^2 + 64x$.

$A'(x) = 12x^2 - 64x + 64 = 4(3x^2 - 16x + 16) = 4(3x-4)(x-4)$. $A'(x) = 0$ at $x = \frac{4}{3}$ and $x = 4$.

The domain of $A(x)$ is $0 \leq x \leq 4$.[44] $A(0) = 0$, $A(\frac{4}{3}) = (-2 \cdot \frac{4}{3} + 8)^2 \cdot \frac{4}{3} = \frac{1024}{27} \approx 38$, $A(4) = 0$.

So the area is maximal when the cut squares measure $\frac{4}{3}$ ft.$\times$ $\frac{4}{3}$ ft.

The volume of the box will be $\approx 38\text{ft}^3$.



Example 23.2

[44] One could argue that the endpoints $x = 0$ and $x = 4$ are not allowed. Then an application of the First Derivative Test or Second Derivative Test could be used to show a maximum volume at $x = \frac{4}{3}$.

Example 23.3. A community service organization has \$6,400 to spend on fencing for a rectangular playground. They want to put fancy fencing on the front and cheaper fencing on the back and sides. Fancy fencing costs \$6 per linear foot. Cheap fencing costs \$2 per linear foot. What are the dimensions of the largest area that can be fenced?
Solution: We want to find the rectangle with largest area that meets the cost restriction of \$6,400.

It can help to draw a picture. Let x be the length of the rectangle that represents the front and back of the playground. Let y be the length of the rectangle that represents the sides of the playground.

The cost of the playground is the sum of the cost of each side: $6x + 2y + 2x + 2y = 8x + 4y$.

$6,400 = 8x + 4y \Longrightarrow 1,600 = 2x + y \Longrightarrow y = 1,600 - 2x$.

Area = length $\times$ width. $A = xy = x(1600 - 2x) = 1600x - 2x^2$.

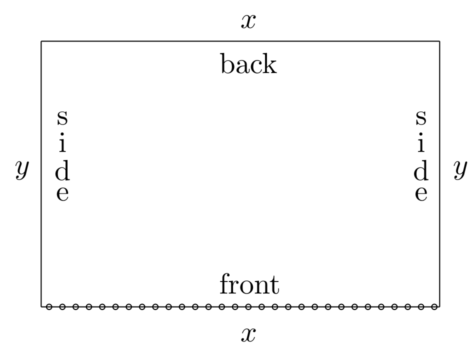
$A(x) = 1600x - 2x^2$. $\quad A'(x) = 1600 - 4x$. $\quad A'(x) = 0$ when $x = 400$.

$A''(x) = -4 < 0$, so there is a local maximum at $x = 400$.

Since A is continuous and $x = 400$ is the only critical point, there is an absolute maximum area at $x = 400$

When $x = 400, y = 1600 - 2(400) = 800$.

Build the playground with front and back each 400 ft.long and the sides 800 ft. long.



Example 23.3

Example 23.4. Of all rectangles with perimeter 26 cm., what are the dimensions of the one with the largest area?
Solution: We want to find the dimensions of a rectangle that has maximum area, so we want to maximize the area of a rectangle. We are told that the perimeter of the rectangle is 26 cm.

Area = length $\times$ width. $\quad$ Perimeter = 2(length) + 2(width).

$A = lw$ and $2l + 2w = 26$.

$2l + 2w = 26 \Longrightarrow l + w = 13 \Longrightarrow l = 13 - w$. So, $A = lw = (13 - w)w = 13w - w^2$.

$A(w) = 13w - w^2 \Longrightarrow A'(w) = 13 - 2w \quad A'(w) = 0$ when $w = 6.5$.

$A''(w) = -2 < 0$, so, by the Second Derivative Test, we have a local maximum when $w = 6.5$.

Since $A(w)$ is continuous and $w = 6.5$ is the only critical point, we have an absolute maximum there.

When $w = 6.5$, $l = 13 - w = 13 - 6.5 = 6.5$

The dimensions of the rectangle with maximum area are 6.5 cm $\times$ 6.5 cm.

Example 23.5. A manufacturer has been selling television sets. He sells 1,000 TVs per week if the price is \$450 each. A survey tells him that for each \$1 rebate he offers, the number of sets sold (q) will increase by 10 per week. (a) How much rebate should he offer in order to maximize revenue? (b) How much rebate should he offer in order to maximize profit if the cost function is $C(q) = 68,000 + 150q$?

Solution (a): We are trying to maximize revenue. Revenue depends on the amount of rebate, x (dollars). Revenue=(price)(quantity sold). With rebate x, price is $(450 - x)$ and quantity sold is $(1,000 + 10x)$.

$R(x) = (450 - x)(1,000 + 10x) = 450,000 + 3,500x - 10x^2$

$R'(x) = 3,500 - 20x \qquad R'(x) = 0$ at $x = 175$.

$R''(x) = -2 < 0$ so we have a local maximum when $x = 175$

Since R is continuous and $x = 175$ is the only critical point, we have absolute maximum revenue when the rebate is \$175.

Solution (b): We are trying to maximize profit. Profit depends on the amount of rebate.

Profit=Revenue-Cost. With rebate x, cost is $C(x) = 68,000 + 150(1,000 + 10x)$.

$P(x) = (450,000 + 3,500x - 10x^2) - (68,000 + 150,000 + 1,500x) = 232,000 + 2,000x - 10x^2$

$P'(x) = 2,000 - 20x. \qquad P'(x) = 0$ when $x = 100$.

$P''(x) = -20 < 0$, so we have a local maximum when the rebate is \$100.

Since P is continuous and $x = 100$ is the only critical point, we have an absolute maximum profit when the rebate is \$100.

Section 23 - Exercises (answers follow)

1. Find non-negative numbers x and y such that $x + y = 150$ and x^2y is maximized.

2. You are to enclose a rectangular garden having an area of 3,600 square meters and surround it by a fence. How can this be done using the least amount of fencing?

3. What are the dimensions of an open (no top) rectangular box that has a square base, a capacity of 32,000 cm^3, and is constructed using the least amount of material?

4. If a manufacturer charges $p(x)$ dollars per item, where $p(x) = 4 - \frac{x}{12}$, then x thousand items will be sold.

   (a) Find an expression for the total revenue from the sale of x thousand items.

   (b) Find the value of x that leads to maximum revenue.

   (c) Find the maximum revenue.

5. For the production of widgets the marginal revenue and marginal cost (in thousands of dollars per item) for producing x widgets are given by $R'(x) = 70 - x$ and $C'(x) = 0.1x^2 + 4x + 10$.

   (a) What is the number x at which these are equal?

   (b) Interpret the result: for what value of x is profit maximal?

6. If x units are produced, the cost of production is $C(x) = 400 + 2x + 0.05x^2$. In order to sell x units it is known that the price per unit should be $p(x) = 10 - \frac{x}{400}$ dollars. Find the production level that will maximize profit. .

7. A cylindrical can without a top is to be made to have volume 100 cubic centimeters. Find the radius of the base and the height of the can which will minimize the cost of the can.

8. Find the dimensions of the rectangle of largest area that can be inscribed in a semicircle of radius 6. What is the area of the rectangle? (Note: The equation of a semicircle, centered at the origin, of radius r is $y = \sqrt{r^2 - x^2}$

9. Draco spent \$1000 to purchase some stock. He tracked the performance of his investment for 20 days. A function that shows the value of his stock V (dollars) at time t days after his purchase is given by $V(t) = -\frac{1}{3}t^3 + 8t^2 - 60t + 1000$. At what time did the investment have the most value?

10. Barry runs a chalk manufacturing company. It costs the company \$1 to make each box of chalk. When the company sells the chalk for \$$p$ per box, they are able to sell $q(p) = \dfrac{p+1}{p^3}$ million boxes per year. Find the price that Barry should charge for a box of chalk in order to maximize his annual profit.

11. At what point on the curve $y = \frac{1}{3}x^3 - 3x^2 + 3x$ does the tangent to the curve have the smallest *slope*? What is the slope at this point?

12. Lamps are priced at \$30.00 and are selling at a rate 120 per month. A market survey has shown that for each \$2.00 reduction in price, ten more lamps will be sold each month. At what price will the revenue from the lamps be maximized?

13. Prove that the vertex of parabola $y = ax^2 + bx + c$ occurs at $x = \frac{-b}{2a}$

14. A man is in a rowboat 2 miles from the nearest point on a straight shoreline. He wishes to reach a point 6 miles further down shore. He can row at a rate of 3 mph and run at a rate of 5 mph. (a) How should he proceed (i.e., how far down shore should he land) in order to arrive at his destination in the least amount of time? (b) How would the strategy change if he had a motorboat that could go at a rate of 20 mph?

15. A company produces electronic parts. The price, (\$/unit) at which x units can be sold is given by the function $p(x) = 10 - 0.001x$ where $0 \le x \le 10,000$. Which production level will maximize the revenue of the company?

16. A farmer has an apple orchard that is planted with 30 trees per acre. The orchard yields an average of 12 bushels of apples per tree. The farmer estimates that for each additional tree planted (per acre), the average yield per tree is reduced by 0.1 bushels.

    (a) How many additional trees, per acre, should be planted in order to maximize the number of bushels of apples produced?

    (b) If the farmer's orchard consists of ten acres, what is the maximum number of bushels of apples he can expect to produce?

17. Alan and Joanna are designing their wedding photo. They want the rectangular page to have a total area of 60 in^2. The margin for framing at the bottom of the page is to be one inch and the margins for framing at the top and sides are to be $\frac{1}{2}$ inch each. What dimensions of paper will give them the largest area for printing the photo on the page within the margins?

Section 23 - Answers

1. $x = 100$, $y = 50$

2. 60m x 60m square garden

3. base: 40 cm $\times$ 40 cm   height: 20 cm

4. (a) $R(x) = 4x - \frac{x^2}{12}$ (b) $x = 24$ (c) $R(24) = \$48$

5. (a) $x = -60$ or $x = 10$, but only 10 makes sense. (b) 10.

6. $x = \frac{1600}{21} \approx 76$ units .

7. radius = height = $(\frac{100}{\pi})^{1/3}$.

8. Dimensions: $6\sqrt{2}$ x $3\sqrt{2}$ Area: 36

9. $t = 0$ (when he first bought it)

10. $p = \sqrt{3} \approx \$1.73$

11. Point : $(3, -9)$ Slope: -6

12. \$27.00

13. Hint: Remember that when you take your derivative, x is the only variable; a, b, c are all constants.

14. (a) Land 1.5 miles down shore (4.5 miles from destination) using rowboat.
    (b) Go directly to destination (no running on land) using motorboat.

15. 5,000 units

16. (a) 45 trees/acre (b) 5,625 bushels

17. Height: $2\sqrt{10} \approx 6.32$ inches, Width: $3\sqrt{10} \approx 9.49$ inches.

24 Elasticity

Relative Change

Suppose you buy one share of Stock A and one share of Stock B. You notice that over the next week stock A has increased by 50 cents and Stock B has increased by 10 cents. Which stock is the better performer? Of which stock would you be more inclined to purchase additional shares? Certainly Stock A has increased in value more than Stock B. But before deciding which stock is the better performer, you need to consider the price at which you bought each stock. You need to consider the *relative* change in the stock value.

Suppose that you purchased the share of Stock A for \$10 and you purchased the share of Stock B for \$2. Then Stock A has increased in value by $\frac{.50}{10} = 5\%$. On the other hand, Stock B has increased in value by $\frac{.20}{2} = 10\%$. So, relative to its purchase price, Stock B is performing better. If you had spent your original \$12 to purchase six shares of Stock B instead of one share each of stocks A and B, you would have earned more money. You would not see this if you looked only at the change in stock value and not at the relative change in value.

This idea of relative change is key to the concept of "Elasticity of Demand." Before discussing this further, we review some facts about price, demand and revenue.

Price, Demand and Revenue

By *price* we are referring to the amount of money charged by the seller to the buyer. It is the amount that the customer pays for an item. By *demand* we are referring to the quantity of items that the buyer purchases. We can reasonably assume then that *price* and *demand* are related [45]. It makes sense that the nature of this relationship is: As the price increases, the demand will decrease and as the price decreases, the demand will increase. The demand q depends on the price p. In mathematical terms, q is a function of p. We express the demand, then, as $q(p)$.

We have seen that *revenue*, the total amount of money received by the seller from the buyer, is calculated by the product: Revenue = Price $\times$ Demand. So revenue, R can be written as a function of p. $R(p) = p \cdot q(p)$.

How Will a Change in Price Affect Revenue?

Suppose you are a merchant and you raise the price of your product by \$5. Will this increase your revenue or decrease it? Remember, that if you increase your price, the quantity sold will be affected. The question is, will it be affected so much that your decrease in sales will lower the total revenue, or will the decrease in sales be insignificant enough that the revenue goes up. Think about this: If what you are selling is new cars, will an increase in the cost of the car by \$5 severely deter your customers? Probably not. So, your revenue should increase because you will be getting more money for each car that you sell. However, if what you are selling is pencils, and you increase the price of a pencil by \$5, what do you think will happen to your demand? You will likely have very few customers, so your revenue will go down.

[45] Do not confuse the relationship between price and demand with the relationship between supply and demand. Supply and demand is another issue entirely

Suppose now that raising the price of your product loses you 100 customers. Is this significant enough to adversely affect your revenue? It depends on your customer base, of course. If you historically have had 500 customers, then the change would have more impact than if you usually had 500,000 customers.

So, to answer the question of how a change in price will affect revenue, it makes sense to look at the *relative* change in price and its resulting *relative* change in the demand. Comparing the relative change in price to the relative change in demand is the essence of the concept of Elasticity of Demand.

Elasticity of Demand

To find the relative change in value, we take the actual change in value and divide it by the original value (as was done with the examples of stock at the beginning of this section). So, suppose you have a product, and you change the price (raise it or lower it) by a small amount h (h could be positive or negative). Then the new price is $(p+h)$. The demand would then change from $q(p)$ to $q(p+h)$.

The change in price is $(p+h)-p$, so the *relative change in price* is $\dfrac{(p+h)-p}{p} = \dfrac{h}{p}$.

The change in demand is $q(p+h)-q(p)$, so the *relative change in demand* is $\dfrac{q(p+h)-q(p)}{q(p)}$.

Economists measure the sensitivity of demand to changes in price as the ratio of these two numbers, with the demand figures in the numerator.

$$\frac{\frac{q(p+h)-q(p)}{q(p)}}{\frac{h}{p}} = \frac{q(p+h)-q(p)}{q(p)} \cdot \frac{p}{h} = \frac{q(p+h)-q(p)}{h} \cdot \frac{p}{q(p)}$$

As $h \to 0$ this approaches $q'(p) \cdot \dfrac{p}{q} = \dfrac{dq}{dp} \cdot \dfrac{p}{q}$. This is always negative since p and q represent positive values and $\dfrac{dp}{dq}$ is always negative (why?). Preferring to work with the convenience of positive numbers, we throw a negative sign on the front and define the *elasticity of demand* to be the positive function

$$E(p) = -\frac{p}{q}\frac{dq}{dp}.$$

When $E(p) < 1$, we say the demand is *inelastic*; when $E(p) > 1$ we say the demand is *elastic*; and when $E(p) = 1$ we say the demand has *unit elasticity*. Let's see why the number 1 is significant.

We are trying to assess the effect of a change in price on the revenue. The revenue function is $R(p) = p \cdot q(p)$. We can use the derivative, $\frac{dR}{dp}$, to determine when R is increasing and when R is decreasing. We use the product rule:

$$\frac{dR}{dp} = 1 \cdot q(p) + \frac{dq}{dp} \cdot p = q(p)\left(1 + \frac{p}{q}\frac{dq}{dp}\right) = q(p)\,(1 - E(p))$$

If $E(p) < 1$ (inelastic) then $\frac{dR}{dp}$ is positive so revenue increases as price is increased. If $E(p) > 1$ (elastic) then $\frac{dR}{dp}$ is negative so revenue decreases as price is increased. If $E(p) = 1$ (unit elasticity) then p is a critical point of R. At unit elasticity, if p is a local maximum of R then neither raising nor lowering the price per unit will increase revenue.

Example 24.1. For a certain product, it is known that the relationship between price and demand is given by $p = -.02q + 400$, where $0 \leq q \leq 20,000$. (a) Find the elasticity function, $E(p)$. (b) Compute $E(100)$ and interpret the result. (c) Compute $E(300)$ and interpret the result. (d) At what price do we have unit elasticity of demand?

Answer: It is important to remember that the elasticity function is a function of p. So we need to rewrite our demand function so that q is a function of p.

$$p = -.02q + 400 \Longrightarrow p - 400 = -.02q \Longrightarrow -50p + 20,000 = q.$$

So, we have $q(p) = -50p + 20,00$ and get from this that $\dfrac{dq}{dp} = -50$.

(a) The elasticity function is: $E(p) = -\dfrac{p}{q}\dfrac{dq}{dp} = -\dfrac{p}{-50p + 20,000} \cdot (-50) = \dfrac{p}{-p + 400}$

(b) $E(100) = \dfrac{100}{-100 + 400} = \dfrac{1}{3}$. Since $E(100) < 1$, we have an *inelastic* situation. A slight increase in price will increase revenue.

(c) $E(300) = \dfrac{300}{-300 + 400} = \dfrac{3}{1}$. Since $E(300) > 1$, we have an *elastic* situation. A slight increase in price will decrease revenue.

(d) $E(p) = 1$ when $p = -p + 400$. Soving for p we have *unit elasticity* at $p = 200$.

Example 24.2. The demand equation for a certain commodity is given by $q = 45 - \frac{1}{5}p^2$ where $0 \leq p \leq 15$. (a) If the price is lowered slightly from \$10, will the revenue increase or decrease? (b) Use elasticity to find the maximum revenue.

Answer: $E(p) = -\dfrac{p}{q} \cdot \dfrac{dq}{dp} = -\dfrac{p}{45 - \frac{1}{5}p^2} \cdot -\frac{2}{5}p = \dfrac{\frac{2}{5}p^2}{45 - \frac{1}{5}p^2} = \dfrac{2p^2}{225 - p^2}$

(a) $E(10) = \dfrac{2(10^2)}{225 - 10^2} = \dfrac{200}{125} > 1$. This is an *elastic* situation. An increase in price would mean a decrease in revenue. So, a decrease in price would yield an increase in revenue.

(b) $E(p) = 1$ when $2p^2 = 225 - p^2$. Solving for p we have *unit elasticity* when $p = \sqrt{75} \approx 8.66$. The maximum revenue is approximately $R(8.66) = (8.66)(45 - \frac{1}{5}(8.66)^2)$.

Elasticity of Demand from Another Perspective

Recall that the elasticity function $E(p)$ was derived from a ratio of the relative change in demand and the relative change in price. The relative change in demand was in the numerator. When a positive fraction is less than 1 we know that the denominator is greater than the numerator. In the case of $E(p)$, we can interpret that the relative change in price is more significant than the relative change in demand. So the change in the revenue will follow the direction of the change in the price. If the price increases, the revenue will increase. If the price decreases, the revenue will decrease. Again, this is the *inelastic* condition.

By the same reasoning, if the value of positive fraction $E(p)$ is greater than 1, then we can interpret that the numerator (the relative change in demand) is more significant than the denominator (the relative change in price). So the direction of the change in revenue will follow the direction of the change in demand. This is opposite the direction of the change in price. If the price is increased, the revenue will decrease and if the price is decreased, the revenue will increase. This is the *elastic* condition.

One final hint/warning

The elasticity function $E(p) = -\frac{p}{q}\frac{dq}{dp}$ is always a positive function. It is easy to forget the minus sign at the beginning of the equation. If you do forget the sign, you will get a negative result. If you get a negative result, stop right there! All negative numbers are less than 1. Your result and conclusions will make no sense. If you get a negative value for $E(p)$, you have made an algebra error or you have forgotten the leading negative sign. Find your error and fix it.

Section 24 - Exercises (answers follow)

1. Suppose that the demand equation for a certain commodity is $q = 60 - p$ (for $0 \leq p \leq 60$).

   (a) Express the elasticity of demand as a function of p.

   (b) Calculate the elasticity of demand when the price is $p = 20$. Interpret your answer.

   (c) At what price is the elasticity of demand equal to 1?

2. A company finds that the demand equation for a product, is given by $p = \dfrac{30}{q^{2.1}}$ where p is the price (in dollars) per item and q is the number of items that can be sold per hour at this price. Express q as a function of p, and find the elasticity of demand when the price is set at \$4 per item. Interpret the result.

3. For the demand function $q(p) = 100 - \dfrac{p}{4}$, find

   (a) the elasticity of demand function $E(p)$

   (b) the values of q (if any) at which total revenue is maximized

4. A commodity is to be sold at unit price p (in dollars). The quantity q of items sold per month is $q = \sqrt{25 - p^2}$ where $0 \leq p \leq 5$. Currently, the unit price is \$2/item.

   (a) Is the demand elastic or inelastic at this price?

   (b) If the price is increased, will the revenue increase or decrease?

5. For each of the following demand functions, Find the elasticity of demand function $E(p)$. Then evaluate $E(20)$ and $E(40)$ and interpret your results.

   (a) $q = 400 - .2p^2$

   (b) $q = 1000p^{-\frac{1}{2}}$.

   (c) $q = \frac{500}{p^2}$.

   (d) $q = 625e^{-.025p}$

6. The quantity of items sold depends on the price, as expressed by the function $q(p) = \sqrt{24 - 3p^2}$.

   (a) What is the domain of $q(p)$?

   (b) Find the elasticity function $E(p)$.

   (c) For what values of p is the demand elastic? inelastic?

   (d) If the currect price is \$1 per item, what would a slight increase in price do to revenue?

7. A certain company (which will remain nameless) manufactures a certain commodity (which is also top secret). The company has determined that when it sets the price at p thousand pesos per liter, it can sell a quantity of $q = (700 - 5p)^2$ liters of its product. The current price of the commodity is $40,000$ pesos per liter. If the company managers want to increase revenue, should they increase the price or decrease the price?

8. The owner of the Showplace video store has estimated that the rental price p (in dollars) of new-release DVDs is related to the quantity q (in thousands) rented each day by the demand equation $p = \frac{3}{2}\sqrt{16 - q^2}$.

   (a) What is the elasticity function $E(p)$ for this demand function?

   (b) If the store owner increased the current \$4 price slightly, can she expect her revenue to increase or decrease?

   (c) Use your elasticity function to determine the price that will yield the maximum revenue. What is the maximum revenue?

9. Without looking, write the formula for $E(p)$.

Section 24 Answers

1. (a) $E(p) = \frac{p}{60-p}$ (b) $\frac{1}{2}$ (c) 30

2. $E(p) = \frac{10}{21}$ $E(4) = \frac{10}{21} < 1$, so inelastic: revenue increases as price increases

3. (a) $E(p) = \frac{p}{400-p}$ (b) $q = 50$

4. (a) $E(p) = \frac{p^2}{25-p^2}$ $E(2) = \frac{4}{21}$, inelastic (b) increase

5. (a) $E(p) = \frac{2p^2}{2000-p^2}$ $E(20) = \frac{1}{2}$, inelastic: revenue increases as price increases.
   $E(40) = 8$, elastic: revenue decreases as price increases.

   (b) $E(p) = \frac{1}{2}$ $E(20) = \frac{1}{2}$ $E(40) = \frac{1}{2}$, always inelastic: revenue increases as price increases

   (c) $E(p) = 2$ $E(20) = 2$ $E(40) = 2$, always elastic: revenue decreases as price increases

   (d) $E(p) = .025p$ $E(20) = .05$, inelastic: revenue increases as price increases.
   $E(40) = 1$; unit elasticity. This is price for maximum revenue.

6. (a) $0 \le p \le \sqrt{8}$ (b) $E(p) = \dfrac{p^2}{8 - p^2}$ (c) elastic: $2 < p \le \sqrt{8}$ inelastic: $0 \le p < 2$
   (d) revenue would increase

7. $E(p) = \dfrac{2p}{140 - p}$. $E(40) < 1$, so they should increase the price.

8. (a) $E(p) = \dfrac{p^2}{36 - p^2}$ (b) increase (c) $p = 3\sqrt{2} \approx \$4.24$ yields revenue \$1,333.33

9. $E(p) = -\dfrac{p}{q}\dfrac{dq}{dp}$

25 3-space

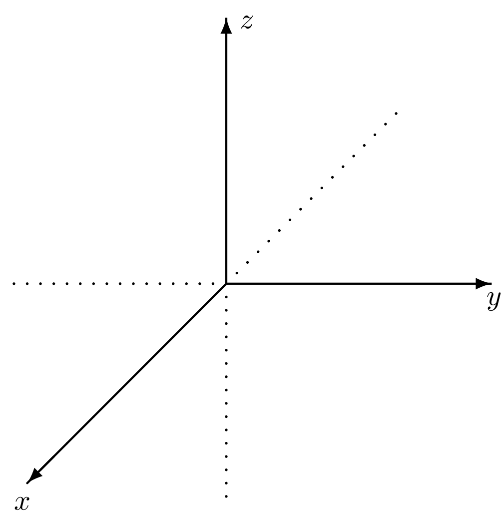
Up to this point, we have been working with functions that have had only one independent variable. Even when we did implicit differentiation or related rates, we had only one independent variable at a time. We are now going to work in three dimensions " 3-*space*" for short. We will have functions of the form $z = f(x, y)$ where there are two independent variables.

The graph of a function with one independent variable is a set of ordered pairs (x, y) where for each x there is a unique y. In 3-space, the graph of a function is a set of ordered triples (x, y, z) where for each ordered pair (x, y) there is a unique z. So, instead of $y = f(x)$, we have $z = f(x, y)$.

The domain of a function with one independent variable is some set of x values. It is a subset of the x-axis. If there are no restrictions, the domain is $\mathbb{R}$. In 3-space the domain is a set of ordered pairs (x, y). It is a subset of the xy plane. If there are no restrictions, the domain is $\mathbb{R} \times \mathbb{R}$ (any real number for x and any real number for y). We will discuss domain more in Section 26.

Even if you have never done any study in 3-space, you will see that most concepts are analogous to those in 2-space (those with one independent variable) with which you are very familiar.

When we graph functions in 3-space, we need three mutually perpendicular axes: west-east, south-north and down-up. It is customary when graphing to have the dependent variable on the vertical axis. The traditional way of drawing 3-D axes is below. In this illustration, the positive ends of the axes are drawn with solid lines and the negative ends are drawn with dotted lines.



Axes in 3-Space

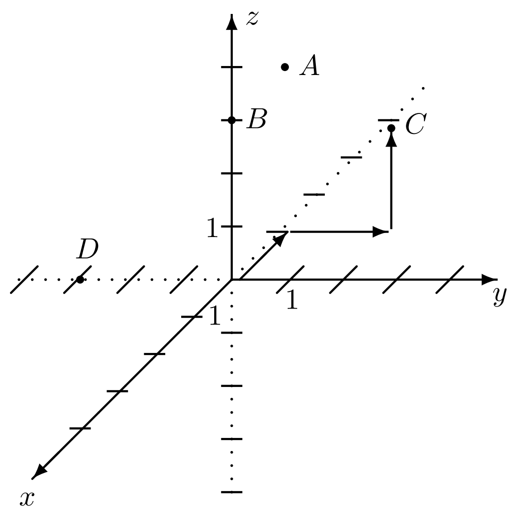
The three axes are calibrated lines (see Section 1). Note that in 3-space it is traditional to have the y and z axes in the plane of your paper with the positive x-axis coming out of the paper towards you and the negative x-axis going away from you behind the paper. Thus, the domain of the function is a flat horizontal surface (an xy plane) and the range is the vertical measures above (+) or below (-) this plane. The equation of the domain plane is simply $f(x, y) = 0$, or $z = 0$.

Example 25.1. Given $z = f(x, y) = x^2 + 2xy + y + 3$.

Find $f(0, 1)$, $f(0, 0)$, $f(-1, 2)$ and plot these points on 3-D axes.

Solution: $f(0, 1) = 0^2 + 2 \cdot 0 \cdot 1 + 1 + 3 = 4 \quad f(0, 0) = 3 \quad f(-1, 2) = 2$

The points $A = (0, 1, 4), B = (0, 0, 3)$ and $C = (-1, 2, 2)$ are graphed below:



Example 25.1

As you can see, it is difficult to simply look at a point in 3-space when the drawing is on a flat piece of paper. In the above graph, if there were no direction arrows indicating the plot path of point C (back one, right 2, up 2), it looks like C could just as easily be the point $(-3.8, 0, 0)$. We will not be sketching entire graphs in 3-space!

Recall that when working with graphs of functions in 2-space, we can find the y-intercept by letting $x = 0$. A function has at most one y-intercept. The graph crosses the x axis when the y value is zero, so we find the x-intercept(s) by setting $y = 0$ and solving the equation. There can be more than one x intercept.

We have an analogous situation in 3-space. The graph will intersect the z axis when the x and y coordinates are both zero. Since z is a *function* of x and y, $z = f(0, 0)$ must have a unique solution. So, there will be at most one z-intercept. However, there can be multiple x-intercepts and y intercepts. A graph will cross the x axis when the y and z values are both zero. Can you see from the sketch of the axes that this must be true? Similarly, to find the y-intercept(s) we set x and z equal to zero.

Example 25.2. What are the coordinates of the axes intercepts for $z = f(x, y) = x^2 + 2xy + y + 3$ (the function in Example 25.1)? Plot these intercepts on a graph.

Answer: The z-intercept occurs when x and y are both zero. $z = 0 + 0 + 0 + 3$. So, $z = 3$. The coordinates are $(0, 0, 3)$. This is point B in the graph of Example 25.1.

The y-intercept occurs when x and z are both zero. $0 = 0 + 0 + y + 3$. So, $y = -3$ The coordinates are $(0, -3, 0)$. This is the point D in the graph for Example 25.1.

The x-intercept occurs when z and y are both zero. $0 = x^2 + 0 + 0 + 3$. This equation has no solution. There is no x intercept.

Planes in 3-Space

A plane in 3-space is the analogue of a line in 2-space. A plane is like a stiff, flat piece of cardboard with no thickness that goes on forever in all directions (as opposed to a line which has no thickness and goes on forever in only two directions). You can determine a specific line and its equation from only two distinct points, but you must have three non-colinear points to determine a plane and its equation.

Recall from Section 3 that all lines in 2-space have equations of the form $px + qy + r = 0$ where p, q and r are numbers. (Examples: $2x - 3y + 4 = 0$, $x + 6 = 0$) Recall too that if $p = 0$ then the line $qy + r = 0$ is parallel to the x-axis, and if $q = 0$ then the line $px + r = 0$ is parallel to the y-axis.

In the same way, all planes in 3-space have equations of the form $px + qy + rz + s = 0$ where p, q, r and s are all numbers. If $p = 0$ and $q = 0$ the plane $rz + s = 0$ is parallel to the xy plane; if $p = 0$ and $r = 0$ it is parallel to the xz plane; and if $q = 0$ and $r = 0$, it is parallel to the yz plane.

Suppose you need to find the equation of the line in 2-space that goes through the points $(2, 3)$ and $(-1, -4)$. It is not likely that you would do it this way, but you *could* start from the general equation of a line and do it as follows: the general equation is $px + qy + r = 0$; substitute the x and y values from the given points to get:

$$\begin{aligned} 2p + 3q + r &= 0 \\ -p - 4q + r &= 0 \end{aligned}$$

We solve these equations simultaneously. Hence $3p + 7q = 0$ (subtracting the second equation from the first), and so $q = -\dfrac{3p}{7}$. Now substiuting:

$$2p + 3 \cdot \frac{-3p}{7} + r = 0 \implies \frac{14p}{7} - \frac{9p}{7} + r = 0 \implies \frac{5p}{7} + r = 0 \implies r = -\frac{5p}{7}.$$

Thus the equation of the required line, $px + qy + r = 0$, becomes

$$\begin{aligned} px - \frac{3p}{7}y - \frac{5p}{7} &= 0 \\ \text{or} \quad p(7x - 3y - 5) &= 0 \\ \text{or} \quad 7x - 3y - 5 &= 0. \qquad \text{if } p \neq 0 \end{aligned}$$

We are sure that $p \neq 0$ because the line through the given points is not parallel to the x axis. You can check your work by substituting the original points in the final equation to verify that both statements are true: $7(2) - 3(3) - 5 = 0$ ($\surd$) and $7(-1) - 3(-4) - 5 = 0$ ($\surd$).

This strange way of finding the equation of a line is given here as a model for finding the equation of a plane. Using this process, you can get the equation of the plane in 3-space when given three points, provided the points don't all lie on the same line. The algebra can become more complicated, and there are several correct ways to approach a problem. In the above example, we solved q and r for p and then substituted. We could have solved p and r for q or we could have solved p and q for r and it would have worked also. We give one example of finding the equation of a plane. The calculations are intimidating, but the process is simple.

Example 25.3. Find the equation of the plane that contains points $(1,0,2)$, $(2,1,4)$ and $(5,-2,0)$. Solution: We substitute the x, y and z values of each point into $px+qy+rz+s=0$ and get:

$$\begin{aligned}
&(1) \qquad & p+2r+s&=0\\
&(2) \qquad & 2p+q+4r+s&=0\\
&(3) \qquad & 5p-2q+s&=0
\end{aligned}$$

We decide to write q, r, and s in terms of p.
From (1), we get: $s=-2r-p$. Substituting this into (2) we get:
$2p+q+4r+(-2r-p)=0 \implies p+q+2r=0 \implies q=-p-2r$. Substituting into (3):
$5p-2(-p-2r)+(-2r-p)=0 \implies 5p+2p+4r-2r-p=0 \implies 6p+2r=0 \implies r=-3p$.
Now, $r=-3p$ gives us $q=-p-2r=-p-2(-3p)=-p+6p=5p$, and
$r=-3p$ gives us $s=-2r-p=-2(-3p)-p=5p$.
We now have all of the unknowns in terms of p and substitute them into the linear equation:

$$\begin{aligned}
px+qy+rz+s&=0\\
px+(5p)y+(-3p)z+(5p)&=0\\
p(x+5y-3z+5)&=0\\
x+5y-3z+5&=0
\end{aligned}$$

Check: $1+0-3(2)+5=0\ \surd \quad 2+5(1)-3(4)+5=0\ \surd \quad 5+5(-2)-3(0)+5=0\ \surd$

Example 25.4. How would you set up the solution for finding the equation of the plane that contains the points: $(2,3,1)$, $(1,2,3)$ and $(3,1,2)$?
Answer: Substituting the x, y and z coordinates into the general $px+qy+rz+s=0$ you get

$$\begin{aligned}
2p+3q+r+s&=0\\
p+2q+3r+s&=0\\
3p+q+2r+s&=0
\end{aligned}$$

If you care to actually find the equation, you should get $x+y+z-6=0$. (Check it).

A *linear function (of two variables)* has the form $f(x,y)=mx+ny+b$ where m, n and b are given numbers:

Examples.

$$\begin{aligned}
f(x,y)&=3x-2y+e\\
f(x,y)&=\pi x\\
f(x,y)&=x+\sqrt{2}\,y+\sqrt{7}
\end{aligned}$$

Since $z = f(x, y)$, we have $z = mx + ny + b$, or $mx + ny - z + b = 0$. This is in the form $px + qy + rz + s = 0$. So, the graph of a linear equation in 3-space is a plane.[46]

Just as with functions of one variable, most functions of two variables are non-linear. We discuss these in section 26.

[46] $px + qy + rz + s = 0$ is a linear equation even though its graph is not a line. In general, a linear equation is one where the highest power of any variable term is one. There are no variables squared or multiplied together or put in denominators or used as exponents or manipulated in any way except being multiplied by a constant. A linear equation in 4-space is $pw + qx + ry + sz + t = 0$, where the variables are w, x, y, z and constants are p, q, r, s, t. What would you guess it looks like?

Section 25 - Exercises (answers follow)

1. Which of the following 3-space equations are linear (have a graph that is a plane)?

   (a) $6x + 2y + z - 9 = 0$

   (b) $x + y + z = 0$

   (c) $xyz + 7 = 0$

   (d) $z = 3x + 2y - 3$

   (e) $f(x, y) = -5x + y - 1$

   (f) $x^2 + y^2 + z - 4 = 0$

   (g) $\dfrac{2x - 8y + z - 4}{3x + 7y + 2z - 9} = 0$

   (h) $z = \sqrt{x + y + 5}$

   (i) $-x + y - z = 17$

   (j) $y = 4$

   (k) $x + 1 = y + z$

   (l) $e^{x+y-z} = 2$

2. The points $(2, 0, t)$ and $(u, 8, u)$ belong to the plane $x + 2y + 3z + 4 = 0$. Find the value of t and the value of u.

3. Find the equation of the plane with the given points.

   (a) $(3, 0, 0), (0, 6, 0), (0, 0, 6)$

   (b) $(1, 2, 1), (0, 0, 3), (2, 0, 2)$

   (c) $(1, 1, 13), (-1, -1, 1), (0, -2, -1)$

4. Each of the following equations represents a plane. For each equation (1) Find the axes intercepts. (2) Plot the intercepts on 3-space axes (3) Connect the three points into a triangle to get a view of the orientation of the plane in 3-space. (4) What is the point on the plane where $x = 1$ and $y = 1$? Plot this point. Can you see how it would "fit" on the extended plane?

   (a) $x + y + z = 3$

   (b) $2x + 2y + z = 8$

   (c) $x - 2y + 3z = 7$

Section 25 - Answers

1. a, b, d, e, i, j, k

2. $t = -2$, $u = -5$

3. (a) $2x + y + z = 6$

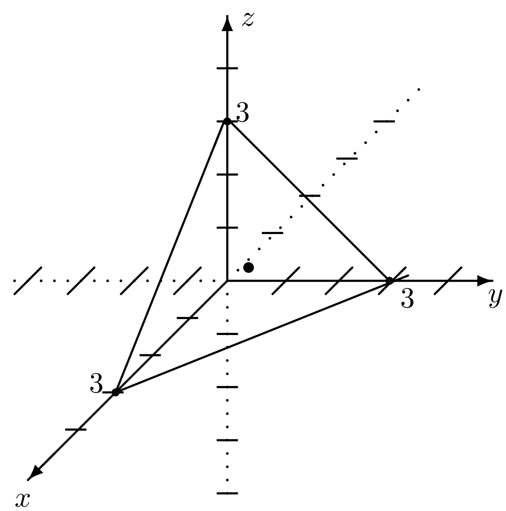
(b) $2x + 3y + 4z - 12 = 0$

(c) $2x + 4y - z + 7 = 0$

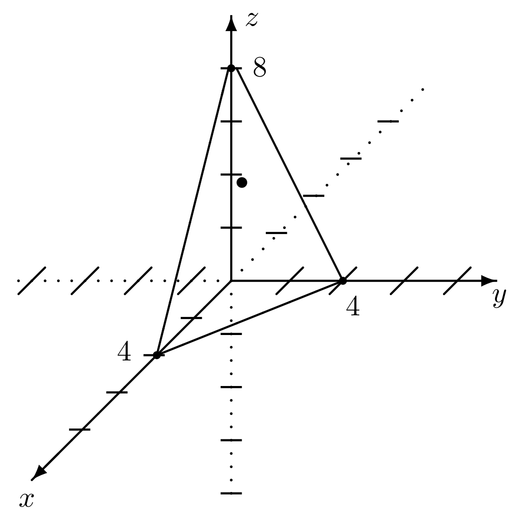
4. (a) Intercepts at $(3,0,0)$, $(0,3,0)$ and $(0,0,3)$ Point: $(1,1,1)$

   (b) Intercepts at $(4,0,0)$, $(0,4,0)$ and $(0,0,8)$ Point: $(1,1,4)$

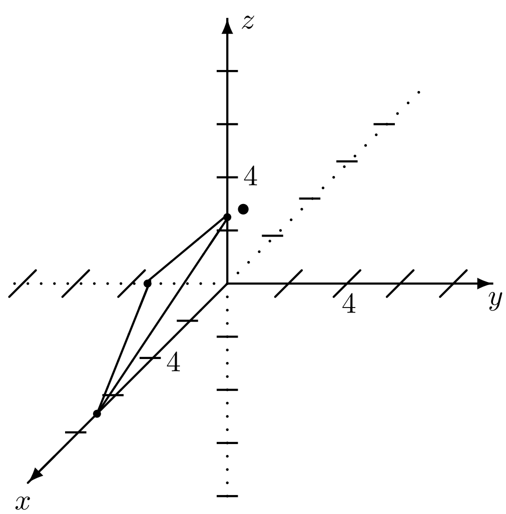
   (c) Intercepts at $(7,0,0)$, $\left(0,-\frac{7}{2},0\right)$ and $\left(0,0,\frac{7}{3}\right)$ Point $\left(1,1,\frac{8}{3}\right)$



Exercise 4a



Exercise 4b



Exercise 4c

26 Functions of Two Variables

In Section 25 we introduced functions of two (independent) variables, but focused mostly on linear functions, whose graphs are planes in 3-space. Now we will broaden our scope.

The function $f(x, y) = x^2 + 2xy + y^3$ is a non-linear function in two (independent) variables. The domain of the function is the set of all ordered pairs (x, y) that make sense algebraically in the function. In this case there are no even roots or denominators or logarithms to concern us, so the domain is $\mathbb{R} \times \mathbb{R}$. That is, x can be any value in $\mathbb{R}$ and y can be any value in $\mathbb{R}$. So, any point in the xy plane is in the domain of f. The dependent variable, usually called z, is the result one gets from evaluating $f(x, y)$. When dealing with only one independent variable we stressed that $y = f(x)$. Here we emphasize that you need to understand that $z = f(x, y)$.

Some points (x, y, z) that belong to this function are:

$(2, 3, 43)$ because $f(2, 3) = 2^2 + 2 \cdot 2 \cdot 3 + 3^3 = 43$.
$(0, -\pi, -\pi^3)$ because $f(0, -\pi) = -\pi^3$
$\left(\frac{1}{2}, 0, \frac{1}{4}\right)$ because $f\left(\frac{1}{2}, 0\right) = \frac{1}{4}$.

We won't be attempting to sketch a graph of f, but we can still think about what it looks like in general terms. We have said that the domain of f is all of the points in the xy plane. For each point (x, y) in the xy plane there is a unique z value associated with it, as defined by $f(x, y)$. So, for each ordered pair (x, y) in the plane, there is some point hovering above it (if z is positive) or hanging below it (if z is negative) or is sitting right on the ordered pair (if $z = 0$).

When a function is continuous,[47] as our function f happens to be, all of these "hovering" and "hanging" points are connected, forming a wavy sheet. One could imagine the xy plane as being a piece of plastic wrap held flat and taut. The graph of f might look like this sheet if fingers poked at it from above and below so that it formed hills and valleys. Indeed a more familiar way to think of a 3-space continuous graph is to think of a relief map: sea level is the xy plane, and the relief map would be the graph of the function $g(x, y) =$ the height above sea level of the map grid point (x, y).

Not all functions are continuous on their domains. Consider the piecewise defined function:

$$g(x) = \begin{cases} 1 & x \text{ and } y \text{ are both integers} \\ 0 & x \text{ or } y \text{ is not an integer} \end{cases}$$

The domain of g is $\mathbb{R} \times \mathbb{R}$. The range (the set of z values) is the set $\{0, 1\}$. What does the graph look like? Think about this for a moment and try to visualize the graph before reading on. Then see if your idea matches this description: Think about the flat xy plane (the function domain) as a piece of graph paper, centered at $(0, 0)$, with the grid lines having unit measure. The intersections of each of the grid lines would have coordinates where both x and y are integers. At all other places on the graph paper (the blank squares and the line segments between grid intersection points) at least one of the coordinates is a non-integer. So, the graph of g would have z values of 1 "hovering"

[47] There is a specific definition of *continuity* that involves limits in a way analogous to and consistent with our definition in Section 9. We will not go into this detail.

over each of the grid intersection points. All of the other points on the graph paper would have z values of 0, so the graph of g sits right on the paper there. Essentially, the graph will look like the entire xy plane, but with orderly rows of individual points plucked from it and suspended in air directly above the teeny, tiny holes created when the points were removed.

Functions like g above are fun to think about, but we will be mostly dealing with functions that are continuous *on their domains*.[48] However, we will be dealing with functions that don't necessarily have the entire xy plane ($\mathbb{R} \times \mathbb{R}$) for a domain.

If the domain of a function $f(x, y)$ is not explicitly given, we must determine the natural domain. The natural domain of a function is simply all of the ordered pairs (x, y) that are valid algebraically and that make sense in the context of the problem. For a pair (x, y) to be in the domain both x and y must have acceptable values. We determine that a value is algebraically acceptable in the same way that we would for single-variable functions: we throw out any values that would give us zeros in denominators or even roots of negative numbers or non-positive arguments for logarithms. What is left is the natural domain of the function.

Example 26.1. Find the domain of each of the following functions.

(a)	$f(x,y) = x^2\sqrt{y}$	$D_f = \{(x,y) : y \geq 0\}$
(b)	$f(x,y) = \dfrac{x+3}{x-y}$	$D_f = \{(x,y) : x \neq y\}$
(c)	$f(x,y) = \sqrt{25-(x^2+y^2)}$	$D_f = \{(x,y) : x^2+y^2 \leq 25\}$
(d)	$f(x,y) = \dfrac{\sqrt{1-x}}{\sqrt{2+y}}$	$D_f = \{(x,y) : x \leq 1 \text{ and } y > -2\}$
(e)	$f(x,y) = \ln(xy-1)$	$D_f = \{(x,y) : xy > 1\}$

In each of these examples, it is relatively easy to look at the function and see which values would not be allowed for either x or for y. In examples (a) and (d) we have individual restrictions on x and y. In the other three examples the restrictions involve x and y operating together. In example (b) we can have any values for x and y as long as they aren't the same. Example (d) is interesting in that neither x nor y can be bigger than 5 or less than -5 (why?), but even that is not sufficient for a restriction (another "why?"). In example (e) we can have any value for x (except zero) as long as when we choose y the product of x and y is greater than 1.

While we will not be sketching the 3-space graphs of $f(x, y)$, it is useful to sketch the 2-space domain of f. This way we have a visual idea of what subset of the xy plane will have z values. To graph a subset of the xy plane we graph the boundaries of the domain regions and then shade in the portion that is included in the domain. If the boundary points are included in the domain, the boundary is drawn with a solid line. If the boundary points are not included in the domain, the boundary is drawn with a dotted line. Pay attention to where boundaries intersect. Use a closed circle to indicate an intersection point that is included in the domain; use an open circle to indicate an intersection point that is not in the domain.

Example 26.1 (a) The boundary is the x-axis (solid line) and we shade the entire region above the x-axis. This gives us a picture of all of the points whose y values are non-negative.

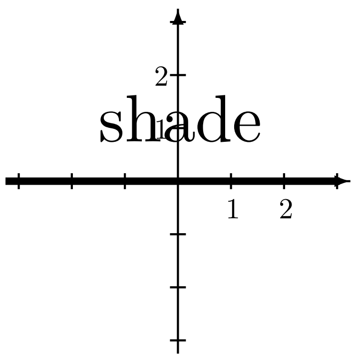
[48] Foreshadowing: A function isn't differentiable where it isn't continuous!

Example 26.1 (b) We wish to eliminate only the values where $y = x$. We graph the line $y = x$, make it a dotted line, and shade the entire rest of the plane.

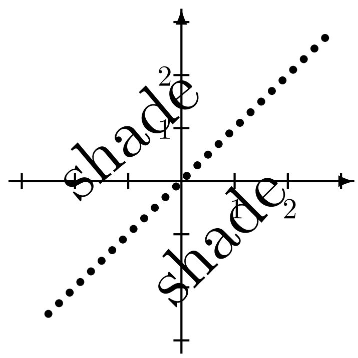
Example 26.1 (c) The boundary is the graph of $x^2+y^2 = 25$. This is the equation of a circle that has center at $(0, 0)$ and radius 5. The boundary is included so we draw the circle with a solid line. We shade the interior of the circle because that is where the sum of the squares of the coordinates is less than 25.

Example 26.1 (d) We have two boundary lines. There is a solid vertical line $x = 1$ and a dotted horizontal line $y = -2$. This divides the xy plane into four pieces. We shade the upper left section because that is where the points all have x values less than 1 and y values greater than -2.

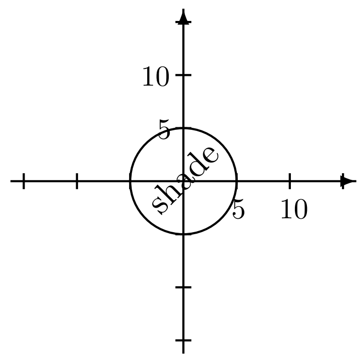
Example 26.1 (e) Since logarithm functions can only operate on positive values, we had to be sure that $(xy - 1) > 1$. That is why $xy > 1$. How can we graph this? What points (x, y) on the plane have the property that $xy > 1$? Certainly we will have to have x and y both positive or both negative. We look at each case separately. Case 1: If x and y are both positive, we are only dealing with points in the first quadrant of the xy plane. Also, since $y > 0$ we know that $xy > 1$ is the same as $y > \frac{1}{x}$. The curve $y = \frac{1}{x}$ then is the boundary (dotted, due to the strict inequality) for the domain in the first quadrant. Since we want the points to have y values greater than the line $y = \frac{1}{x}$ we shade the region above the line. Case 2: If x and y are both negative, we are only dealing with the third quadrant of the xy plane. This time $y < 0$, so $xy > 1$ is equivalent to $y < \frac{1}{x}$ (We must change the sign when multiplying by a negative value). The boundary is the dotted line $y = \frac{1}{x}$ in the third quadrant, and we shade the region *below* the boundary because y is less than $\frac{1}{x}$.



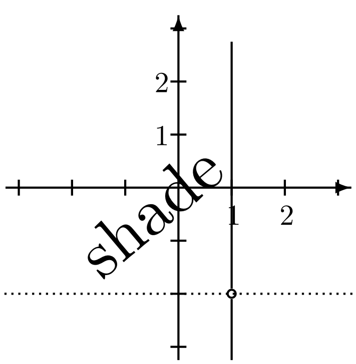
(a) $y \geq x$



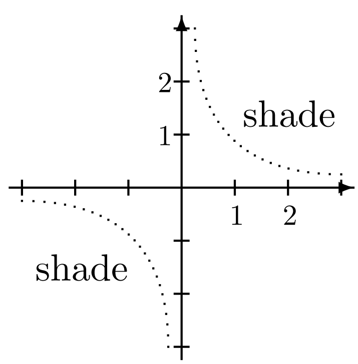
(b) $y \neq x$



(c) $x^2 + y^2 \leq 25$



(d) $x \leq 1$ and $y > -2$



(e) $xy > 1$

Multivariate Applications

Why do we need to have functions with more than one independent variable? The answer is simple. There are times in life where more than one entity influences the outcome of a situation. Indeed limiting ourselves to only two independent variables is rather restrictive in the grand scheme of things. However, working with two variables is a very good introduction into working with several variables. When you can relate the strangeness of working in two variables with the familiar working in one variable, further expansion into more than two variables isn't that difficult (well, except maybe for drawing graphs).

Example 26.2. Suppose you have a company that manufactures toy xylophones and yo-yos. You have a daily fixed expense of \$500. It costs you \$7 to make each xylophone and \$2 to make each yo-yo. Write a function $C(x, y)$ to express your daily cost, where x is the number of xylophones made and y is the number of yo-yos made. Then find the cost to make 30 xylophones and 80 yo-yos.

Answer: $C(x, y) = 500 + 7x + 2y$

$C(30, 80) = 500 + 7(30) + 2(80) = 500 + 210 + 160 = 870$ dollars.

Section 26 - Exercises (answers follow)

1. Let $f(x, y) = x^2 + y^2 - x + 2$. Find the following:
   (a) $f(0, 0)$ (b) $f(1, 0)$ (c) $f(0, -1)$ (d) $f(a, 2)$

2. Let $h(x, y) = \sqrt{x^2 + 2y^2}$. Find the following.
   (a) $h(2, 1)$ (b) $h(3, 4)$ (c) $h(-1, 6)$ (d) $h(-6, 1)$.

3. Find the domain of the function and sketch the domain on an xy plane.
   (a) $f(x, y) = 2x^2 + 3y^3$
   (b) $f(x, y) = \dfrac{6x + 5y}{5x + 6y}$
   (c) $f(u, v) = \dfrac{uv}{u - v}$
   (d) $f(x, y) = \dfrac{x^2 - y^2}{x + 2y}$
   (e) $f(x, y) = \sqrt{xy} - 5x + 2y - 3$
   (f) $f(x, y) = \dfrac{5x + \ln y}{x + y}$
   (g) $f(x, y) = 2y \ln x + e^y$

4. The IQ (intelligence quotient) of a person whose mental age is m years and whose physical age is p years is defined as $q(m, p) = \dfrac{100m}{p}$. What is the IQ of a 9-year-old child who has a mental age of 13.5 years?

5. Using x skilled workers and y unskilled workers, a manufacturer can produce $f(x, y) = 50x + 5y^2$ units per day. Currently there are 20 skilled workers and 40 unskilled workers on the job.
   (a) How many units are currently being produced each day?
   (b) By how much will the daily production level change if one more unskilled worker is added to the current work force?
   (c) By how much will the daily production level change if one more skilled worker is added to the current work force?
   (d) By how much will the daily production level change if one more skilled worker *and* one more unskilled worker are added to the current work force?

6. Production of an aircraft part is given by $P(x, y) = 100\left(\frac{3}{5}x^{-\frac{2}{3}} + \frac{2}{5}y^{-\frac{2}{3}}\right)^{-3}$, where x is the amount of labor in work-hours and y is the amount of capital. Find the following. Don't spend time trying to simplify these answers.
   (a) What is the production when 64 work-hours and 4 units of capital are provided?
   (b) If 236 units of capital and 16 work-hours are used, what is the production output?

7. A company that makes all-wood furniture knows that the cost (\$) to make a piece of their furniture depends on the amount of wood used (measured in linear feet) and the amount of labor required for assembly and finishing (measured in hours). Wood costs \$5 per linear foot and labor costs \$22 per hour. There is also a fixed warehousing cost of \$15 for each piece of furniture.

   (a) Write a cost function $C(x, y)$ to describe the cost of a piece of furniture that requires x linear feet of wood and y hours of labor to produce and store.

   (b) What is the cost to make a bookcase that needs 20 feet of wood and 2.5 hours of labor to complete?

8. Suppose the company described in Exercise 7 decides to put decorative hardware on their furniture. Each piece of harware costs \$1. Let z represent the number of pieces of hardware placed on a piece of furniture.

   (a) Write a cost function $C(x, y, z)$ to describe the cost of producing and storing a piece of furniture.

   (b) Evaluate $C(10, 5, 2)$.

   (c) What is meant by $C(10, 5, 2)$?

Section 26 - Answers

1. (a) 2 (b) 2 (c) 3 (d) $a^2 - a + 6$

2. (a) $\sqrt{6}$ (b) $\sqrt{41}$ (c) $\sqrt{73}$ (d) $\sqrt{38}$

3. (a) All real values of x and y ($\mathbb{R} \times \mathbb{R}$)

   (b) $\left\{(x, y) : x \neq -\frac{6}{5}y\right\}$

   (c) $\{(u, v) : u \neq v\}$

   (d) $\left\{(x, y) : y \neq -\frac{1}{2}x\right\}$

   (e) $\{(x, y) : xy \geq 0\}$

   (f) $\{(x, y) : y > 0 \text{ and } y \neq -x\}$

   (g) $\{(x, y) : x > 0\}$

4. 150

5. (a) 9,000 (b) increase by 405 (c) increase by 50 (d) increase by 455

6. (a) $P(64, 4) = 100\left(\dfrac{3}{5}64^{-\frac{2}{3}} + \dfrac{2}{5}4^{-\frac{2}{3}}\right)^{-3}$

   (b) $P(16, 236) = 100\left(\dfrac{3}{5}16^{-\frac{2}{3}} + \dfrac{2}{5}236^{-\frac{2}{3}}\right)^{-3}$

7. (a) $C(x, y) = 5x + 22y + 15$ (b) $C(20, 2.5) = \$170$

8. (a) $C(x, y, z) = 5x + 22y + z + 15$ (b) $C(10, 5, 2) = \$177$
(c) $C(10, 5, 2)$ is the cost to produce and store a piece of furniture that requires 10 linear feet of wood, 2 pieces of decorative hardware and 5 hours of labor to make.

27 Partial Derivatives

Recall from Section 8 that for a function of one variable, $y = f(x)$, its derivative at $x = a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The derivative gave us the instantaneous rate of change between the independent variable x and the dependent variable y at the specific point $(a, f(a))$.

In the case of a function of two variables, $z = f(x, y)$, we could also ask the question of how z is changing, but at any given point the graph of f takes off in many directions. We will concern ourselves with how the function changes in the direction parallel to the x-axis and how it changes in the direction parallel to the y-axis.

Consider the function $z = f(x, y) = 3x^2 + 7y + x^3y^4 + 10$. Its graph is some continuous wavey surface in 3-space. Suppose we want to know the rate of change of z compared to x at the specific point $(a, b, f(a, b))$. We would be looking at the slope of the line that is tangent to f at the point $(a, b, f(a, b))$ and that is parallel to the x-axis. One way to visualize this is to think of the plane $y = b$ that cuts through 3-space. This plane is parallel to the xz plane, b units from it. This plane cuts through the graph of f. The intersection of f and this plane is precisely all of the points in f where the y value is b. This intersection looks like a curve in 2-space. It is essentially a function of two variables, x and z. In fact, the function would be $z = 3x^2 + 7b + x^3b^4 + 10$ since y is equal to some selected constant b. So, if we want to know the rate of change of z compared to x at the point $(a, b, f(a, b))$ we would find the derivative of $z = 3x^2 + 7b + x^3b^4 + 10$ with respect to x (so, b is a constant) and substitute a for x. Thus, the slope of the tangent line to $(a, b, f(a, b))$ in the direction parallel to the x-axis is $6x + 0 + (3b^4)x^2 + 0$ at $x = a$: $6a + (3b^4)a^2$.

What does this derivative mean? Suppose the above function represents a cost function for a manufacturer of xylophones (x) and yo-yos (y). The expression $6a + (9b^4)a^2$ tells us the instantaneous rate of change between the *cost* and the *quantity of xylophones* when we are making a xylophones and b yo-yos. This gives us an idea of how making a change in the number of xylophones produced would affect the cost. It says nothing about changes in the quantity of yo-yos made. In this case the number of yo-yos is fixed at b.

Suppose we wanted to know the rate of change between the cost and the quantity of yo-yos when we are making a xylophones and b yo-yos. This time we are saying that x is fixed; x is the constant a. Our function $z = f(x, y)$ really becomes the function $z = f(a, y) = 3a^2 + 7y + a^3y^4 + 10$. The instantaneous rate of change between z and y is $0 + 7 + 4a^3y^3 + 0 = 7 + 4a^3b^3$ when we are making a zylophones and b yo-yos.

For functions of two independent variables, $z = f(x, y)$ we look at the rate of change of z in one direction at a time. These derivatives are called *partial derivatives*. We use the notation f_x to denote a partial derivative with respect to x and the notation f_y to denote a partial derivative with respect to y. The equivalent Leibnitz notations are $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. The symbol ∂ is called "del" and one says "del z by del x," or simply "del z, del x." When z is a function of more than one variable, you should never write $\dfrac{dz}{dx}$ or $\dfrac{dz}{dy}$; always use the del. Like our previous Leibnitz notation, the dependent variable goes on the top and the independent variable goes on the bottom.

The formal definitions of partial derivatives are very much like the definition that we had for single variable functions. They might look imposing, but take some time to make sense of them.

The derivative of $f(x, y)$ at (a, b), in the x-direction is

$$f_x(a,b) = \left.\frac{\partial z}{\partial x}\right|_{(a,b)} = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h} \tag{27.1}$$

and the derivative in the y-direction is

$$f_y(a,b) = \left.\frac{\partial z}{\partial y}\right|_{(a,b)} = \lim_{k\to 0}\frac{f(a,b+k)-f(a,b)}{k}. \tag{27.2}$$

Example 27.1. Given $f(x, y) = 5x + 7y - 2x^2y^4 + 1$, find f_x and f_y.
Answer: $f_x(x, y) = 5 - 4xy^4$ and $f_y(x, y) = 7 - 8x^2y^3$.

You can see from this example and from the general definition of the functions f_x and f_y that you calculate the partial derivatives of $f(x, y)$ by holding one of the variables constant and differentiating with respect to the other. So in the example we got $f_x(x, y)$ by holding y constant and applying our rules of differentiation to the variable x, and, similarly, we got $f_y(x, y)$ by holding x constant.

Warning: Do not confuse partial deriviatives with implicit differentiation. Partial derivatives are derivatives of multivariate functions. Implicit differentiation is used to find derivatives of expressions that may or may not be functions. With implicit differentiation we have only one independent variable, x, and we think of y as some function of x. With implicit differentiation an expression like $2x^2y^4$ would require a product rule and a $\frac{dy}{dx}$ because y is a function of x. When finding the partial derivative f_x we don't need a product rule because y is a constant.

It is important that you practice and become adept at finding partial derivatives.

Example 27.2. Find the partial derivatives for each of the following functions:

$f(x, y) = y^2 - y^4 + x^2$ Answer: $f_x = 2x$ $f_y = 2y - 4y^3$

$g(x, y) = 3x^2y + 2\ln y + e^{y^2x}$ Answer: $g_x = 6xy + 0 + e^{y^2x}y^2$ $g_y = 3x^2 + 2\cdot\frac{1}{y} + e^{y^2x}2yx$

$h(x, y) = \dfrac{y^2}{y-3x}$ Answer: $h_x = \dfrac{3y^2}{(y-3x)^2}$ $h_y = \dfrac{2y(y-3x) - 1\cdot y^2}{(y-3x)^2}$

$j(x) = \dfrac{x+y}{x-y}$ Answer: $\dfrac{\partial z}{\partial x} = \dfrac{1(x-y)-1(x+y)}{(x-y)^2}$ $\dfrac{\partial z}{\partial y} = \dfrac{1(x-y)+(x+y)}{(x-y)^2}$

$k(x) = 3x^2y\cdot\ln(2xy^5)$ Answer: $k_x = 6xy\cdot\ln(2xy^5) + \frac{1}{2xy^5}\cdot 2y^5(3x^2y)$
$k_y = 3x^2\ln(2xy^5) + \frac{1}{2xy^5}\cdot 10xy^4(3x^2y)$

When working in 2-space we found some functions that were not differentiable over their entire domains. This can happen in 3-space also.

Example 27.3. Given $z = f(x, y) = \sqrt{xy}$, find the domain of f, $\frac{\partial z}{\partial x}$, and the domain of $\frac{\partial z}{\partial x}$.
Answer: The domain of f is $\{(x, y) : xy \geq 0\}$. This is all of the points in the first and third quadrants of the xy plane, and includes all of the points on the x and y axes.

$\frac{\partial z}{\partial x} = \frac{1}{2}(xy)^{-\frac{1}{2}} y = \frac{y}{2\sqrt{xy}}$

The domain of $\frac{\partial z}{\partial x}$ is $\{(x, y) : xy > 0\}$ This includes all of the points in the first and third quadrants of the xy plane, but does not include the points on the x or y axes.

This example shows that the natural domain of a partial derivative is sometimes smaller than the natural domain of the original function .

Second Partial Derivatives

The function $f(x, y)$ yields the two partial derivatives $f_x(x, y)$ and $f_y(x, y)$. Each of these two derivatives is a function (of two variables) in its own right and so could have partial derivatives of its own. These are the *second partial derivatives* of f. The function f has two partial derivatives, four second partial derivatives, eight third partial derivatives, etc.

The notation for the second derivatives is as would be expected. To find the derivative of f_x with respect to x we would want $(f_x)_x$ and we write: f_{xx}. Using Leibnitz notation for this we would want $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)$ and we write: $\dfrac{\partial^2 z}{\partial x^2}$.

To find the derivative of f_x with respect to y, we would want $(f_x)_y$ and we write: f_{xy}. Using Leibnitz notation we would want $\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)$ and we write $\dfrac{\partial^2 z}{\partial y \partial x}$. Notice that the x and y switch first and second positions in the two notations. If you think about how they come about, though, the positions make sense.

The derivative of f_y with respect to y is f_{yy} or $\dfrac{\partial^2 z}{\partial y^2}$. The derivative of f_y with respect to x is f_{yx} or $\dfrac{\partial^2 z}{\partial x \partial y}$.

The derivatives f_{xx} and f_{yy} are sometimes called the "pure" partial derivatives. The derivatives f_{xy} and f_{yx} are called the "mixed" partial derivatives.

Example 27.4. In Example 27.1 we were given $f(x, y) = 5x + 7y - 2x^2y^4 + 1$ and we found: $f_x(x, y) = 5 - 4xy^4$ and $f_y(x, y) = 7 - 8x^2y^3$. Find the second partial derivatives of f.
Answers: $f_{xx} = -4y^4 \quad f_{xy} = -16xy^3 \quad f_{yy} = -24x^2y^2 \quad f_{yx} = -16xy^3$

You may have noticed in the example above that $f_{xy} = f_{yx}$. This is most often the case. There are weird functions for which the mixed partial derivatives are different, but you won't meet them in this course. So YOU MAY ASSUME THE TWO MIXED PARTIAL DERIVATIVES ARE EQUAL[49]. However, rather than think that you now have only one mixed partial derivative to calculate, it is advised that you still calculate both and use this as a checking mechanism for your work. If you get two different answers, you have made a mistake somewhere in the differentiation process.

It is important that you practice finding second partial derivatives also.

[49]In a more advanced course you'd be told what properties the function f must have for this to be true. Any function f you'll meet here will have those properties.

Example 27.5. Find the second partial derivatives for $f(x, y) = 2x^3y + x - e^y$.
Answer: $f_x = 6x^2y + 1$ and $f_y = 2x^3 - e^y$, so:

$f_{xx} = 12xy$ and $f_{xy} = 6x^2$. $f_{yy} = -e^y$ and $f_{yx} = 6x^2$.

We check and are encouraged because the mixed partial derivatives match. $\surd$

When higher order derivatives are being sought it is sometimes helpful to simplify derivatives before going on to further differentiation.

Example 27.6. Find the second partial derivatives for $z = (2x + y^2 - 1)^4$.
Answer: $\frac{\partial z}{\partial x} = 4(2x+y^2-1)^3(2) = 8(2x+y^2-1)^3$ and $\frac{\partial z}{\partial y} = 4(2x+y^2-1)^3(2y) = 8y(2x+y^2-1)^3$.
So,

$\frac{\partial^2 z}{\partial x^2} = 24(2x + y^2 - 1)^2(2)$ and $\frac{\partial^2 z}{\partial y \partial x} = 24(2x + y^2 - 1)^2(2y)$.

$\frac{\partial^2 z}{\partial y^2} = 8(2x + y^2 - 1)^3 + 3(2x + y^2 - 1)^2(2y)(8y)$ and $\frac{\partial^2 z}{\partial x \partial y} = (8y)3(2x + y^2 - 1)^2(2)$

The mixed partial derivatives are equal. $\surd$

Example 27.7. Find the second partial derivatives for $g(x, y) = 20x^3 + 30y^2 + \ln(xy) + 8$
Answer: $g_x = 60x^2 + \frac{1}{xy} \cdot y = 60x^2 + \frac{1}{x}$ and $g_y = 60y + \frac{1}{xy} \cdot x = 60y + \frac{1}{y}$. So,

$g_{xx} = 120x - \frac{1}{x^2}$ and $g_{xy} = 0$. $g_{yy} = 60 - \frac{1}{y^2}$ and $g_{yx} = 0$.

The mixed partial derivatives match. $\surd$

Directional derivatives:

In this course we will be dealing only with partial derivatives (rates of change in a direction parallel to axes). But it is of sufficient interest and importance to mention that partial derivatives are a basis for a much broader view of 3-D rates of change. A brief description is given here so that students who are so intrigued or who find later that this could be useful can follow up independently.

Partial derivatives involve a subject called linear algebra. In more advanced work, you may need to know the rate of change of $f(x, y)$ at (a, b) in a particular direction that is neither the direction of the x-axis nor the direction of the y-axis. It's best to think of a *direction* as a point lying on the circle of radius 1 whose center is $(0, 0)$. Then the direction of the x-axis is $(1, 0)$, that of the y-axis is $(0, 1)$, and a general direction is $(u, \pm\sqrt{1 - u^2})$ where $-1 \leq u \leq 1$. The directions involving the plus sign point into the upper half plane, while those involving the minus sign point into the lower half plane.

The *directional derivative of* f *in the direction* $(u, \sqrt{1 - u^2})$ at (a, b) is

$$\lim_{h \to 0} \frac{f(a + hu, b + h\sqrt{1 - u^2}) - f(a, b)}{h}$$

while the corresponding formula for the direction $(u, -\sqrt{1 - u^2})$ is

$$\lim_{h \to 0} \frac{f(a + hu, b - h\sqrt{1 - u^2}) - f(a, b,)}{h}.$$

It is a deep fact of calculus that these turn out to be $uf_x(a, b) + \sqrt{1 - u^2}f_y(a, b)$ in the first case, and $uf_x(a, b) - \sqrt{1 - u^2}f_y(a, b)$ in the second. This says that if you know the two partial derivatives of $f(x, y)$ you automatically know all the directional derivatives of $f(x, y)$. They are "linear combinations" of the two partial derivatives, where the multiplying constants are given by the specified direction.

Section 27 - Exercises (answers follow)

1. Find $f_x(x, y)$ and $f_y(x, y)$
   (a) $f(x, y) = 4x^4 - y^3 + 2x - 4$
   (b) $f(x, y) = (x + xy + y)^5$
   (c) $f(x, y) = e^{xy+1}$
   (d) $f(x, y) = \dfrac{x^4 + y^4}{x^2 - y^2}$
   (e) $f(x, y) = \ln(3y^8 - 2x)$
   (f) $f(x, y) = y^3 e^x + x^3 e^y$
   (g) $f(x, y) = (3x^2 + xy)^{\frac{2}{3}}$
   (h) $f(x, y) = e^{3x^2} + 2y^3$
   (i) $f(x, y) = y$

2. Find: $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$, $\dfrac{\partial^2 f}{\partial x \partial y}$, $\dfrac{\partial^2 f}{\partial y \partial x}$ (If the last two are not equal, you have made an error.)
   (a) $f(x, y) = 1000 + 5x - 4y^2 - 3xy$
   (b) $f(x, y) = 2x^3 + 3xy^2 + 4y^5$
   (c) $z = 4x^2 e^y$
   (d) $f(x, y) = \sqrt{2x + 3y}$
   (e) $f(x, y) = \dfrac{2x}{5 - 3y}$
   (f) $f(x, y) = e^{x+y} \ln x$

3. Your weekly cost (\$) to manufacture x bicycles and y tricycles is $C(x, y) = 16,000 + 6x + 20y$. Calculate and interpret $\frac{\partial C}{\partial x}$ and $\frac{\partial C}{\partial y}$.

4. Suppose that $M(x, y) = 40x^2 + 30y^2 - 10xy + 30$, approximates the manufacturing cost of a computer, where x is the cost of components and y is the cost of labor. Find the following partial derivatives and evaluate them at the given points:
   (a) $M_y(4, 2)$ (b) $M_x(3, 6)$ (c) $\frac{\partial M}{\partial x}(2, 5)$ (d) $\frac{\partial M}{\partial y}(6, 7)$

5. A company produces two types of calculators: scientific and graphing. The marginal cost to produce scientific calculators is \$10. The marginal cost to produce graphing calculators is \$15. The fixed weekly cost of calculator production is \$1,200. (a) Write the company's weekly cost function $C(x, y)$ where x is the number of scientific calculators made and y is the number of graphing calculators made. (b) Compute $C(40, 20)$ and tell what it means.

6. Suppose f is a function of three variables: $f(x, y, z) = 2xy + 5x^2y^2z + 4z - 3$.
   Find: f_x, f_y, f_z, f_{xy}, f_{yy}, f_{zy}, f_{yz}, f_{xyz}

7. Sal Monella runs a butcher shop. The profit she makes from selling a pounds of antelope meat, b pounds of beef and c pounds of chicken is given by the unlikely function $P(a,b,c) = 3a + 2bc + c^3$. Find the derivative that would express the instantaneous change in profit compared to change in the number of pounds of chickens she sells.

8. Suppose f is a function of three variables: $w = f(x,y,z)$. Use the model for Definitions 27.1 and 27.2 to write the limit definitions for f_x, f_y and f_z.

Section 27 - Answers

1. (a) $f_x = 16x^3 + 2 \qquad f_y = -3y^2$
   (b) $f_x = 5(x + xy + y)^4(1 + y) \qquad f_y = 5(x + xy + y)^4(x + 1)$
   (c) $f_x = ye^{xy+1} \qquad f_y = xe^{xy+1}$
   (d) $f_x = \dfrac{4x^3(x^2 - y^2) - 2x(x^4 + y^4)}{(x^2 - y^2)^2} \qquad f_y = \dfrac{4y^3(x^2 - y^2) + 2y(x^4 + y^4)}{(x^2 - y^2)^2}$
   (e) $f_x = \dfrac{-2}{3y^8 - 2x} \qquad f_y = \dfrac{24y^7}{3y^8 - 2x}$
   (f) $f_x = y^3e^x + 3x^2e^y \qquad f_y = 3y^2e^x + x^3e^y$
   (g) $f_x = \frac{2}{3}(3x^2 + xy)^{-\frac{1}{3}}(6x + y) \qquad f_y = \frac{2}{3}(3x^2 + xy)^{-\frac{1}{3}} \cdot x$
   (h) $f_x = e^{3x^2} \cdot 6x \qquad f_y = 6y^2$
   (i) $f_x = 0 \qquad f_y = 1$

2. (a) $\frac{\partial^2 f}{\partial x^2} = 0 \qquad \frac{\partial^2 f}{\partial y^2} = -8 \qquad \frac{\partial^2 f}{\partial x \partial y} = -3 \qquad \frac{\partial^2 f}{\partial y \partial x} = -3 \quad \surd$
   (b) $\frac{\partial^2 f}{\partial x^2} = 12x \qquad \frac{\partial^2 f}{\partial y^2} = 6x + 80y^3 \qquad \frac{\partial^2 f}{\partial x \partial y} = 6y \qquad \frac{\partial^2 f}{\partial y \partial x} = 6y \quad \surd$
   (c) $\frac{\partial^2 z}{\partial x^2} = 8e^y \qquad \frac{\partial^2 z}{\partial y^2} = 4x^2e^y \qquad \frac{\partial^2 z}{\partial x \partial y} = 8xe^y \qquad \frac{\partial^2 z}{\partial y \partial x} = 8xe^y \quad \surd$
   (d) $\frac{\partial^2 f}{\partial x^2} = -(2x + 3y)^{-\frac{3}{2}} \qquad \frac{\partial^2 f}{\partial y^2} = -\frac{9}{4}(2x + 3y)^{-\frac{3}{2}} \qquad \frac{\partial^2 f}{\partial x \partial y} = -\frac{3}{2}(2x + 3y)^{-\frac{3}{2}} \qquad \frac{\partial^2 f}{\partial y \partial x} = -\frac{3}{2}(2x + 3y)^{-\frac{3}{2}} \quad \surd$
   (e) $\frac{\partial^2 f}{\partial x^2} = 0 \qquad \frac{\partial^2 f}{\partial y^2} = \dfrac{36x}{(5 - 3y)^3} \qquad \frac{\partial^2 f}{\partial x \partial y} = \dfrac{6}{(5 - 3y)^2} \qquad \frac{\partial^2 f}{\partial y \partial x} = \dfrac{6}{(5 - 3y)^2} \quad \surd$
   (f) $\frac{\partial^2 f}{\partial x^2} = e^{x+y}\ln x + \frac{1}{x}e^{x+y} - \frac{1}{x^2}e^{x+y} + e^{x+y}\frac{1}{x} \qquad \frac{\partial^2 f}{\partial y^2} = e^{x+y}\ln x \qquad \frac{\partial^2 f}{\partial x \partial y} = e^{x+y}\ln x + \frac{1}{x}e^{x+y}$ $\frac{\partial^2 f}{\partial y \partial x} = e^{x+y}\ln x + \frac{1}{x}e^{x+y} \quad \surd$

3. $\frac{\partial C}{\partial x} = 6$ If you change (increase or decrease) the number of bicycles manufactured by 1, the cost will correspondingly change by \$6.
   $\frac{\partial C}{\partial y} = 20$ If you change (increase or decrease) the number of tricycles manufactured by 1, the cost will correspondingly change by \$20.

4. (a) 80 (b) 180 (c) 110 (d) 360

5. (a) $C(x,y) = 10x + 15y + 1,200$ (b) $C(40,20) = 1,900$ means that the cost to produce 40 scientific calculators and 20 graphing calculators in a week is \$1,900.

6. $f_x = 2y + 10xy^2z \qquad f_y = 2x + 10x^2yz \qquad f_z = 5x^2y^2 + 4 \qquad f_{xy} = 2 + 20xyz$
   $f_{yy} = 10x^2z \qquad f_{zy} = 10x^2y \qquad f_{yz} = 10x^2y \qquad f_{xyz} = 20xy$

7. $\dfrac{\partial P}{\partial c} = 2b + 3c^2$

8. $$f_x(x,y,z) = \left.\frac{\partial w}{\partial x}\right|_{(x,y,z)} = \lim_{h\to 0}\frac{f(x+h,y,z)-f(x,y,z)}{h}$$
   $$f_y(x,y,z) = \left.\frac{\partial w}{\partial y}\right|_{(x,y,z)} = \lim_{k\to 0}\frac{f(x,y+k,z)-f(x,y,z)}{k}$$
   $$f_z(x,y,z) = \left.\frac{\partial w}{\partial z}\right|_{(x,y,z)} = \lim_{j\to 0}\frac{f(x,y,z+j)-f(x,y,z)}{j}$$

28 Local Maxima and Minima (Two Variables)

This section should be compared with Section 15 because it deals with the same basic question. Here we consider $z = f(x, y)$, assuming that f is "nice" in the sense that first and second partial derivatives of f make sense (i.e. the limits exist). This tells us that the graph of f is continuous and flows smoothly with no sharp edges.

We say $f(x, y)$ has a *local maximum* (or *relative maximum*) at domain point (x_0, y_0) if there is a number $\epsilon > 0$ such that $f(x, y) \leq f(x_0, y_0)$ for all points (x, y) whose distance from (x_0, y_0) is less than ϵ. You can guess the definition of *local minimum* (see Section 15 for comparison): just replace $\leq$ by $\geq$.

This definition essentially says that we have a local maximum at the domain point (x_0, y_0) if for all of the domain points really close to (x_0, y_0) their z values are less than or equal to the z value for (x_0, y_0). This time "close to" can be from any direction. Think of a small circle in the domain with the point (x_0, y_0) as the center. We have a local maximum at (x_0, y_0) if the z values for all of the points in the circle are less than or equal to the z value for (x_0, y_0). A similar idea applies to local minimum.

Understand that we are discussing *local* extrema. We are not finding absolute extrema here. As with functions of one variable, absolute extrema can be found by looking for them at the locations of local extrema and at the "boundaries" of the domain. However, these boundaries can be difficult to deal with, especially those involving limits. So, we content ourselves with finding local extrema.

By analogy with Theorem 15.1 (page 126) we have:

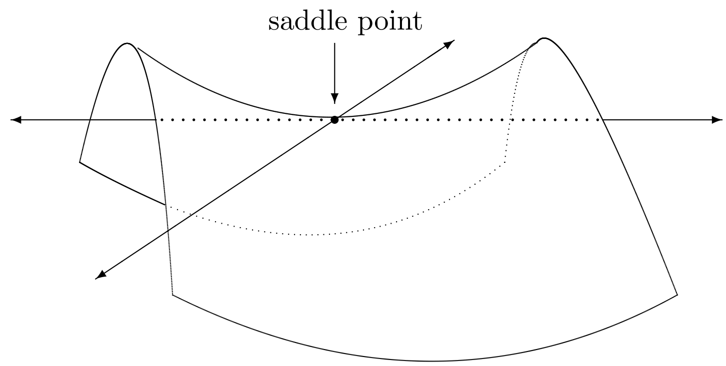
Theorem 28.1. *If ("nice" function) f has a local maximum or a local minimum at domain point (x_0, y_0), then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$.*

If $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ we say that (x_0, y_0) is a *critical point* of f.

Another difference between this section and Section 15 is that here we are restricting ourselves to "nice" functions where derivatives exist. In our study of functions of one variable we included as critical points those values $x = a$ in the domain where $f'(a)$ did not exist.

Does Theorem 28.1 make sense? If the derivatives f_x and f_y are both zero at (x_0, y_0), then we have horizontal tangent lines in those two directions. From the discussion of directional derivatives in Section 27 we can conclude that all of the tangent lines to f at (x_0, y_0) are horizontal. Thus, we have a horizontal tangent *plane* touching the graph of f at the critical point. If the plane is above the graph we have a local maximum. If the plane is below the graph, we have a local minimum.

Be careful! While the above paragraph is true, it does not tell the entire story. Just as with Theorem 15.1, we need to make sure that we don't reverse the "if" and "then" statements. Neither theorem says that "If there is a critical point, then we have a local extremum." This statement is backwards. The theorems say, "If there is a local extremum, then we have a critical point." It is possible for $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ and still not have a local extremum at (x_0, y_0). This occurs when the horizontal tangent line from one direction hits (x_0, y_0) from below the graph and the horizontal tangent line from the other direction hits the point from above the graph. In this case, we do not have a local extremum. We have a *saddle point*.



Saddle Point Illustration

Example 28.1. Find the critical points for $f(x, y) = xy + x - y$
Solution: $f_x(x, y) = y + 1$ and $f_y = x - 1$.
$f_x = y + 1 = 0$ when $y = -1$.
$f_y = x - 1 = 0$ when $x = 1$.
So the only critical point is $(1, -1)$

Example 28.2. Find the critical points for $f(x, y) = \frac{1}{3}x^3 - y^3 + \frac{5}{2}x^2 + 3y^2 + 1$
Solution: $f_x = x^2 + 5x$ and $f_y = -3y^2 + 6y$
$f_x = x^2 + 5x = x(x + 5) = 0$ when $x = 0$ or $x = -5$
$f_y = -3y^2 + 6y = y(-3y + 6) = 0$ when $y = 0$ or $y = 2$.
So, critical points are: $(0, 0)$, $(-5, 0)$, $(0, 2)$ and $(-5, 2)$.

Example 28.3. Find the critical points for $f(x, y) = 3xy - x^2y - xy^2$.
Solution: $f_x = 3y - 2xy - y^2$ and $f_y = 3x - x^2 - 2xy$.
$f_x = 3y - 2xy - y^2 = y(3 - 2x - y) = 0$ when $y = 0$ or $y = 3 - 2x$.
$f_y = 3x - x^2 - 2xy = x(3 - x - 2y) = 0$ when $x = 0$ or $x = 3 - 2y$.
Since we must have *both* $f_x = 0$ and $f_y = 0$ we look at cases and solve simultaneously:
Case 1: $y = 0$ and $x = 0$, and thus the critical point is $(0, 0)$
Case 2: $y = 0$ and $x = 3 - 2y$. So, $x = 3 - 2(0) = 3$, and thus the crit. pt. is $(3, 0)$
Case 3: $y = 3 - 2x$ and $x = 0$. So $y = 3 - 2(0) = 3$, and thus, the crit. pt. is $(0, 3)$
Case 4: $y = 3-2x$ and $x = 3-2y$. So, $y = 3-2(3-2y) = 3-6+4y \implies y = -3+4y \implies y = 1$.
Since $y = 1$, $x = 3 - 2y = 3 - 2(1) = 1$. Thus the crit pt is (1,1)
There are four critical points: $(0, 0), (3, 0), (0, 3)$ and $(1, 1)$.

Example 28.4. Find the critical points for $f(x) = xe^{y^2-4} - x$

Solution: $f_x = e^{y^2-4} - 1$ and $f_y = xe^{y^2-4}2y$

$f_x = e^{y^2-4} - 1 = 0$ when $y^2 - 4 = 0$. So, $y = 2$ or $y = -2$.

$f_y = xe^{y^2-4}2y = 0$ when $x = 0$ or $y = 0$.

So, the critical points are: $(0, 2)$ and $(0, -2)$. *Note that $(0, 0)$ is not a critical point because $f_x(0, 0) \neq 0$.*

Now that we can find critical points, how do we analyze them? How do we determine if the critical point is the location of a local extremum or a saddle point? There is an analog to the Second Derivative Test which we will use:

Theorem 28.2. *Suppose (x_0, y_0) is a critical point of f. Let $D(x_0, y_0)$ (or D for short) be the number $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$.*

(i) *If $D > 0$ and $f_{xx}(x_0, y_0) < 0$ then f has a local maximum at (x_0, y_0);*

(ii) *If $D > 0$ and $f_{xx}(x_0, y_0) > 0$ then f has a local minimum at (x_0, y_0);*

(iii) *If $D < 0$ then f has a saddle point at (x_0, y_0);*

(iv) *If $D = 0$ this test gives no information.*

We will not prove this theorem, but let's look at the number D at least a little bit. We have some idea from our study of single variable calculus that the second derivative is related to concavity. In this case we can think of concavity as being related to the pure partial second derivatives, f_{xx} and f_{yy}. If we have critical point at (x_0, y_0) we have a horizontal tangent there. If f_{xx} is negative, we are suggesting that there is downward concavity in the x-direction. If f_{yy} is also negative there, then there is downward concavity in the y-direction also. If at the critical point there is downward concavity everywhere, then we reasonably visualize that there is a local maximum at that point. Similarly, if both pure partial second derivatives are positive, we can reasonably visualize a local minimum at the critical point.

This says that in order to have a local extremum the signs of the two pure partial second derivatives should be the same. If you look at the number D in Theorem 28.2 the ONLY way that D can be positive is if both f_{xx} and f_{yy} have the same sign. Do you see that? So, $D > 0$ (cases i and ii) means that the concavity of the graph is the same in both directions. Whether the concavity is up or down depends on whether both derivatives are negative (case i) or both are positive (case ii).

Look again at the drawing of the saddle point. Can you see that in that case the concavity in one direction is up and the concavity in the other direction is down? In this case the signs of the pure second partial derivatives are different. This would definitely make D negative, hence case (iii) of the theorem.

This is only an intuitive discussion of how the theorem works. It is not a very rigorous discussion at that. Indeed, having both f_{xx} and f_{yy} have the same sign does not guarantee that there is an extremum at the critical point. It could be that $f_{xx} \cdot f_{yy}$ is less than $(f_{xy})^2$. In that case, there is a saddle point. (case iii). The mixed partial second derivatives are not irrelevant and can sometimes have sufficient impact that the simple, vague, concavity argument above doesn't hold up.

We will finish this section by taking Examples 28.1 through 28.4 and analyzing their critical points to check for local extrema.

Example 28.5. Find the local extrema for $f(x, y) = xy + x - y$
Solution: We know: $f_x(x, y) = y + 1$ $\quad f_y = x - 1$ $\quad$ crit pts: $(1, -1)$
$f_{xx} = 0$ and $f_{yy} = 0$ $\quad f_{xy} = 1$ and $f_{yx} = 1$ $\surd$
$D(1, -1) = 0 \cdot 0 - (1)^2 = -1 < 0$, so there is a saddle point at $(1, -1)$

Example 28.6. Find the local extrema for $f(x, y) = \frac{1}{3}x^3 - y^3 + \frac{5}{2}x^2 + 3y^2 + 1$
Solution: We know: $f_x = x^2 + 5x$ $\quad f_y = -3y^2 + 6y$ $\quad$ crit pts: $(0, 0)$, $(-5, 0)$, $(0, 2)$, $(-5, 2)$.
$f_{xx} = 2x + 5$ and $f_{yy} = -6y + 6$ $\quad f_{xy} = 0$ and $f_{yx} = 0$ $\surd$
$D(x, y) = (2x + 5)(-6y + 6) - (0)^2$
$D(0, 0) = (0 + 5)(0 + 6) - 0^2 = 30 > 0$ and $f_{xx}(0, 0) = 5 > 0$, so, local min. at $(0, 0)$.
$D(-5, 0) = (-5)(6) - (0)^2 < 0$, so saddle point at $(-5, 0)$
$D(0, 2) = (5)(-6) - (0)^2 < 0$, so, saddle point at $(2, 0)$
$D(-5, 2) = (-5)(-6) - (0)^2 > 0$ and $f_{xx}(-5, 2) = -5 < 0$, so local max at $(-5, 2)$.

Example 28.7. Find the local extrema for $f(x, y) = 3xy - x^2y - xy^2$.
Solution: We know: $f_x = 3y - 2xy - y^2$ $\quad f_y = 3x - x^2 - 2xy$ $\quad$ crit pts: $(0, 0), (3, 0), (0, 3), (1, 1)$.
$f_{xx} = -2y$ and $f_{yy} = -2x$ $\quad f_{xy} = 3 - 2x - 2y$ and $f_{yx} = 3 - 2x - 2y$ $\surd$
$D(x, y) = (-2y)(-2x) - (3 - 2x - 2y)^2$
$D(0, 0) = (0)(0) - (3 - 0 - 0)^2 < 0$, so saddle point at $(0, 0)$.
$D(3, 0) = (0)(-6) - (3 + 6 - 0)^2 < 0$, so saddle point at $(3, 0)$
$D(0, 3) = (-6)(0) - (3 - 0 - 6)^2 < 0$, so saddle point at $(0, 3)$
$D(1, 1) = (-2)(-2) - (3 - 2 - 2)^2 = 4 - 1 > 0$ and $f_{xx}(1, 1) = -2 < 0$, so local max at $(1, 1)$

Example 28.8. Find the local extrema for $f(x) = xe^{y^2-4} - x$.
Solution: We know: $f_x = e^{y^2-4} - 1$ $\quad f_y = xe^{y^2-4}2y$ $\quad$ crit pts:$(0, 2), (0, -2)$
$f_{xx} = 0$ (so, f_{yy} doesn't matter) $\quad f_{xy} = e^{y^2-4}2y$ and $f_{yx} = e^{y^2-4}2y$. $\surd$
$D(x, y) = 0 - (e^{y^2-4}2y)^2$
$D(0, 2) < 0$ and $D(0, -2) < 0$ so saddle points at both places.

Section 28 - Exercises (answers follow)

1. Find all points where the functions below have any relative maxima or minima. Identify any saddle points.
   (a) $f(x,y) = 6 - 2x^2 - 3y^2$
   (b) $f(x,y) = x^3 - 3y^2$
   (c) $f(x,y) = x^2 - 2xy + 2y^2 + x$
   (d) $f(x,y) = x^2 + y^2 - y^2x^2 - 4$
   (e) $f(x,y) = xy + \frac{4}{x} + \frac{2}{y} + 8$
   (f) $f(x,y) = (x-1)^2 + y^3 - y^2 - y + 1$
   (g) $f(x,y) = x^2 + 4y^3 - 6xy - 1$
   (h) $f(x,y) = x^4 + y^3 + \frac{3}{xy}$
   (i) $f(x,y) = e^{x^2+y^2}$
   (j) $f(x,y) = (x-8)\ln(x^2y)$

2. Suppose the labor cost for manufacturing an item is $L(x,y) = \frac{3}{2}x^2 + y^2 - 2x - 2y - 2xy + 68$ dollars, where x is the number of hours required by a skilled worker and y is the number of hours required by a semiskilled worker. Find values of x and y that minimize the labor cost. Find the minimum labor cost.

3. The total daily revenue (in dollars) that a company realizes from selling granola is given by $R(x,y) = -0.008x^2 - 0.004y^2 - 0.003xy + 25x + 18y$, where x denotes the number of small boxes and y denotes the number of large boxes sold daily. The total daily cost (in dollars) of production is given by $C(x,y) = 2x + 5y + 100$. How many small boxes and how many large boxes should be produced per day to maximize profits. What is the maximum profit realizable?

4. A factory makes widgets and bidgets. Each week the total cost (in dollars) to make x widgets and y bidgets is $C(x,y) = 10,000 + 50x + 70y - xy$. The manager at the factory wants to make between 100 and 150 widgets and between 80 and 120 bidgets. What combination will cost the least? What combination will cost the most? (Hint: Draw a picture of the domain of the problem.)

Section 28 - Answers

1. (a) Local maximum at $(0,0)$
   (b) Only critical pt. is $(0,0)$. Since $D_{(0,0)} = 0$, no conclusions can be drawn.
   (c) Local minimum at $(-1, \frac{-1}{2})$
   (d) Local minimum at $(0,0)$ and Saddle points at: $(1,1), (1,-1), (-1,1), (-1,-1)$
   (e) Local minimum at $(2,1)$

(f) Local minimum at $(1, 1)$ and Saddle point at $(1, -\frac{1}{3})$

(g) Local minimum at $(\frac{9}{2}, \frac{3}{2})$ and Saddle point at $(0, 0)$

(h) Local minimum at $\left(\sqrt[19]{\left(\frac{3}{4}\right)^4}, \sqrt[19]{\frac{4}{3}} \right)$ (A little algebra fun, huh?)

(i) Local Minimum at (0,0)

(j) Saddle point at $(8, \frac{1}{64})$

2. Minimum cost of $59 when $x = 4$ and $y = 5$

3. The only critical point is approximately (1218.49, 1168.07). Since x and y must both be integers, check the closest integer valued points for maximum value of profit. Substitute the points (1218, 1168), (1219, 1168), (1218, 1169), and (1219, 1169) into the profit function. The maximum profit is $21,605.04, which occurs at both (1218, 1168) and (1219, 1168). As long as you understand this concept of using the closest integer valued points, it probably isn't worth your time to do the actual calculations.

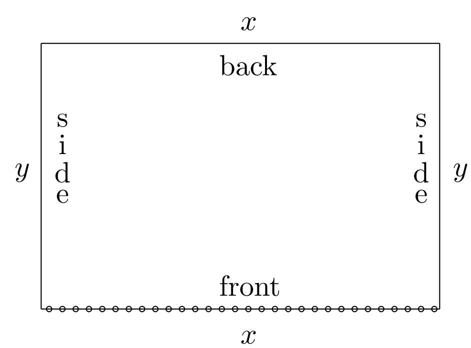
4. minimum at $x = 150$ and $y = 120$ maximum at $x = 100$ and $y = 80$

29 Lagrange Multipliers (Constrained Optimization)

This is an important meeting point of calculus and economics.

In this section we have a function of two variables, $f(x, y)$ and we wish to find the local maxima and minima. But, unlike the problems in section 28 these functions have an additional restriction, or constraint. This means that instead of considering all points (x, y) in the natural domain of f we are only considering those in a subset of that domain. We are only considering those points that meet the conditions of the constraint. Let's look at an example. We will take Example 23.3 from Section 23 and approach it in a new way.

Example 29.1. (Example 23.3 revisited): A community service organization has \$6,400 to spend on fencing for a rectangular playground. They want to put fancy fencing on the front and cheaper fencing on the back and sides. Fancy fencing costs \$6 per linear foot. Cheap fencing costs \$2 per linear foot. What are the dimensions of the largest area that can be fenced?



Example 23.3

We found the solution to this (see page 190) to be: $x = 400$ and $y = 800$. Build the playground with front and back each 400 ft. long and the sides each 800 ft. long.

This problem has two unknowns, the length and width of the desired playground. We were able to manipulate this problem into a question of only one unknown by using the cost information and doing some algebraic manipulation. The problem can be solved, however, by thinking of it as the optimization of a multivariate function $f(x, y)$ (length and width are the two variables) subject to a constraint (the cost restriction). The process that we will use is called the method of Lagrange Multipliers. The proof that this method works is too hard for this course but you can (and must) learn to use the method.

Method of Lagrange Multipliers

1. *Identify what entity is to be optimized and what that entity depends on.*
   We wish to maximize the area of the playground. Area depends on length and width.

2. *Write an equation that mathematically connects the items in Step 1. The dependent variable should be the item being optimized.*
   $A(x, y) = xy$, where A is the area, x is the width, and y is the length of the playground.

3. *Identify the constraint (restriction) involved in the situation.*
   We are restricted by the cost. We only have \$6,400 to spend.

4. *Write the constraint as a function,* $g(x, y)$ *and **write it so that*** $\underline{g(x, y) = 0}$.
   We know that $C(x, y) = 6,400 = 6x + 2y + 2x + 2y$, so $g(x, y) = 0 = 8x + 4y - 6400$.

5. *Create a new function, F of three variables, $F(x, y, \lambda)$. The new function is the sum of the optimization function and λ times the constraint function.*[50]
   $F(x, y, \lambda) = A(x, y) + \lambda g(x, y) = xy + \lambda(8x + 4y - 6400)$.

6. *Find the partial derivatives F_x, F_y, and F_λ.*
   $F_x = y + 8\lambda \qquad F_y = x + 4\lambda \qquad F_\lambda = 8x + 4y - 6400$

7. *Set each of the partial derivatives equal to zero and solve the system simultaneously.*

$$\begin{aligned} (1) \qquad & y + 8\lambda = 0 \\ (2) \qquad & x + 4\lambda = 0 \\ (3) \qquad & 8x + 4y - 6400 = 0 \end{aligned}$$

   From (1) we know that $y = -8\lambda$. From (2) we know that $x = -4\lambda$.
   We can simplify (3) to become $2x + y - 1600 = 0$.
   Substituting the first two results into the simplified third equation, we get:
   $2(-4\lambda) + (-8\lambda) = 1600.$ So $-16\lambda = 1600$, or $\lambda = -100$.
   Substituting back, we get: $x = -4\lambda = -4(-100) = 400$ and $y = -8\lambda = -8(-100) = 800$.
   These are the same answers that we got using the method in Section 23.

When we solved this problem in the earlier section, an important step was to verify that we actually had found a maximum value. This is not part of the Lagrange multiplier method. However, it would be prudent in real life situations to choose a point near the solution and compare its value to the value of the solution point. In the above example, the area of the 400×800 playground is 320,000 ft^2. If we choose a point near there, say having the front be 401 feet and the side be 798 feet,[51] the area would be 319,998 ft^2 — clearly an inferior playground. In our answer of 400×800 we have found a maximum area rather than a minimum area.

Example 29.2. A toymaker makes exactly eleven dolls per day. He has found that the cost when he makes r red-haired dolls and b brown-haired dolls is given by the function $C(b, r) = b^2 - br - 3b - 2r + 250$. How many of each kind of doll should he make in order to minimize his cost?

[50] The variable λ is called a "dummy variable." Its value has no significance to the problem. It is merely a tool required for the Lagrange Multiplier method. Hint: "Lambda" looks fine in print. However, a hastily handwritten λ looks a lot like an X, and this can cause confusion to both the writer and a reader. Write legibly.

[51] The number 401 was chosen randomly as a number close to 400, and the 798 was obtained by the constraint $2x+y-1600 = 0$.

Solution: We want to minimize the cost. The cost depends on the number of red-haired dolls and number of brown-haired dolls that are made. The cost function $C(b, r)$ is given.

We are constrained by the quantity of dolls made daily, 11. So $b + r = 11$, or $g(b, r) = 0 = b + r - 11$.

$F(b, r, \lambda) = b^2 - br - 3b - 2r + 250 + \lambda(b + r - 11)$.

$F_b = 2b - r - 3 + \lambda \qquad F_r = -b - 2 + \lambda \qquad F_\lambda = b + r - 11$

$$
\begin{aligned}
&(1) \qquad 2b - r - 3 + \lambda = 0 \\
&(2) \qquad -b - 2 + \lambda = 0 \\
&(3) \qquad b + r - 11 = 0
\end{aligned}
$$

From (2) we know that $\lambda = b + 2$. From (3) we know that $r = 11 - b$

Substituting this information in (1) we have $2b - (11 - b) - 3 + (b + 2) = 0$.

So, $2b - 11 + b - 3 + b + 2 = 0$, or $b = 3$

We substitute $b = 3$ into equation (3) and conclude that $r = 8$. So, to minimize the cost, the toymaker should make 3 brown-haird dolls and 8 red-hairded dolls each day.

Like the example of the playground, this example is as easily done using the methods of Section 23. We could have solved the equation $b + r = 11$ for either variable, substituted the result in the optimization function and proceded with the single variable optimization process. Using the single variable method, we are able to verify that we have indeed gotten the correct optimization (maximum vs. minimum). There are situations, however, where it is difficult to solve the constraint equation for one variable. In those cases, the Lagrange Multiplier method allows us to work in two variables.

In the following example it is not impossible to solve for one variable, but it is awkward due to the square root that would be involved and the $\pm$ that must go with it. So, we use the method of Lagrange Multipliers.

Example 29.3. The temperature at the point (x, y) of a flat surface is xy degrees. A bug is walking on the surface in the exact elliptical pattern described by $4x^2 + 8y^2 = 16$. What are the maximum and minimum temperatures the bug encounters?

Solution: We want to find both the maximum and minimum temperatures. The temperature depends on the location [the (x, y) coordinates] of the bug. We know that the temperature at the point (x, y) is given by $T(x, y) = xy$.

We know that the bug only travels on points that are on the prescribed ellipse. So, our constraint fuction is $g(x, y) = 0 = 4x^2 + 8y^2 - 16 = 0$.

$F(x, y, \lambda) = xy + \lambda(4x^2 + 8y^2 - 16)$.

$F_x = y + 8\lambda x \qquad F_y = x + 16\lambda y \qquad F_\lambda = 4x^2 + 8y^2 - 16$

$$
\begin{aligned}
&(1) \qquad y + 8\lambda x = 0 \\
&(2) \qquad x + 16\lambda y = 0 \\
&(3) \qquad 4x^2 + 8y^2 - 16 = 0
\end{aligned}
$$

From equations (1) and (2) we get[52] $\lambda = -\frac{y}{8x}$ and $\lambda = -\frac{x}{16y}$.

Hence, $-\frac{y}{8x} = -\frac{x}{16y} \implies -16y^2 = -8x^2 \implies 2y^2 = x^2$.

Substituting this result into (3) we get $4(2y^2) + 8y^2 = 16 \implies 16y^2 = 16 \implies y = \pm 1$.

Since $2y^2 = x^2$, if $y = 1, x = \pm\sqrt{2}$ and if $y = -1, x = \pm\sqrt{2}$.

Our possible coordinates for maximum and minimum are: $(x, y) = (\sqrt{2}, 1), (-\sqrt{2}, -1), (\sqrt{2}, -1)$ and $(-\sqrt{2}, 1)$.

Evaluating these in $T(x, y)$, we get the maximum temperature of $\sqrt{2}$ degrees at $(\sqrt{2}, 1)$ and at $(-\sqrt{2}, -1)$ and the minimum temperature of $-\sqrt{2}$ degrees at $(\sqrt{2}, -1)$ and at $(-\sqrt{2}, 1)$.

The method of Lagrange multipliers works with any number of variables:

Example 29.4. Find the points on the sphere in 3-space $x^2 + y^2 + z^2 = 1$ which are closest to and farthest from the point $(1, 2, 3)$ outside the sphere.

Solution: The distance between (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

It is easier to deal with the square of the distance. We can work with D^2 because the number D is at maximum (or minimum) precisely when D^2 is at maximum (or minimum). So we use $f(x, y, z) = (1 - x)^2 + (2 - y)^2 + (3 - z)^2$ for our optimization function.

Since we are only considering points on the given sphere, we have $g(x, y, z) = x^2 + y^2 + z^2 - 1$ as our constraint function.

So, $F(x, y, z, \lambda) = f(x, y, z) + \lambda g(x, y, z) = (1 - x)^2 + (2 - y)^2 + (3 - z)^2 + \lambda(x^2 + y^2 + z^2 - 1)$.

$F_x = -2(1 - x) + 2\lambda x \quad F_y = -2(2 - y) + 2\lambda y \quad F_z = -2(3 - z) + 2\lambda z \quad F_\lambda = x^2 + y^2 + z^2 - 1$

$$\begin{aligned}
&(1) \qquad -2(1 - x) + 2\lambda x = 0 \\
&(2) \qquad -2(2 - y) + 2\lambda y = 0 \\
&(3) \qquad -2(3 - z) + 2\lambda z = 0 \\
&(4) \qquad x^2 + y^2 + z^2 - 1 = 0
\end{aligned}$$

From (1), (2), (3) we get $x = \frac{1}{1+\lambda}$, $y = \frac{2}{1+\lambda}$ and $z = \frac{3}{1+\lambda}$.

This tells us that $y = 2x$ and $z = 3x$.

Substituting into (4), we get $x^2 + (2x)^2 + (3x)^2 - 1 = 0 \implies 14x^2 = 1 \implies x = \pm\sqrt{14}$.

So our two points are $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$ and $\left(-\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right)$.

Checking these values in $f(x, y, z)$ we see that the minimum occurs at $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$ and the maximum occurs at $\left(-\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right)$

Final Remarks:

As we have said, the beauty of this method is that it works in any number of variables. In economics, the problem is likely to involve many variables.

[52] Provided neither x nor y is zero, but it's easy to see that if x or y is zero then both would have to be zero and the third equation would not be satisfied.

If there are 25 variables and one constraint function $g(x_1, x_2, \ldots, x_{25})$ then there will be 26 equations to be solved (λ being the 26^{th} variable). Computer packages exist to help with this.

We can also have more than one constraint. If there are (for example) four variables, x, y, z, w and two constraint functions $g(x, y, z, w)$ and $h(x, y, z, w)$ then you set

$$F(x, y, z, w, \lambda, \mu) = f(x, y, z, w) + \lambda g(x, y, z, w) + \mu h(x, y, z, w)$$

and you solve the six equations $F_x = 0$, $F_y = 0$, $F_z = 0$, $F_w = 0$, $F_\lambda = 0$ and $F_\mu = 0$.

Again, we appreciate the help of computers for this.

Section 29 - Exercises (answers follow)

1. In each of the examples given in this section, the λ partial derivative was equal to the constraint function. Will this always be the case?

   <u>Note</u>: *Several of the remaining problems can be solved without Lagrange multipliers but you are to use the Lagrange Multiplier process for all of them.*

2. Find the local maxima or minima:

   (a) Maximum of $f(x,y) = 2xy$, subject to $x + y = 12$

   (b) Minimize the function $f(x,y) = x^2 + 3y^2$ subject to the constraint $x - y + 1 = 0$.

   (c) Find the minimum value of the function $f(x,y) = x^2 - y^2$ subject to the constraint $x^2 + y^2 = 4$.

   (d) Maximize the function $f(x,y) = x + y - x^2 - y^2$ subject to the constraint $x + 2y = 6$.

   (e) Maximize $f(x,y) = x^2 - 10y^2$, subject to $x + y = 9$

   (f) Find the maximum and minimum values of the function $f(x,y) = 2x^2 + y^2 - 4y$ subject to the constraint $x^2 + y^2 = 1$.

3. Suppose x and y are positive numbers whose sum is 35. Find the values for x and y that make the product x^2y a maximum.

4. A company has two plants that produce diamond necklaces. At plant A, it costs $x^2 + 1200$ dollars to make x necklaces. At plant B, it costs $3y^2 + 800$ dollars to make y necklaces. An order has come to the company for 1,200 necklaces. (a) How many necklaces should be made in plant A and how many in plant B if the company wishes to minimize the cost? (b) If the company charges the customer $1,000 for each necklace, how much *profit* will they have for this sale?

5. A closed rectangular box is made with two kinds of materials. The top and bottom are made with heavy-duty cardboard costing 20¢ per square foot, and the sides are made with lightweight cardboard costing 10¢ per square foot. Given that the box is to have a capacity of 2 cubic feet, what should its dimensions be if the cost is to be minimized?

6. The total cost to produce x widgets and y bidgets is given by $C(x,y) = 3x^2 + 4y^2 + 2xy + 3$. If a total of ten items must be made, how should production be allocated so that total cost is minimized?

7. You wish to fence off a rectangular area along the bank of a river. The area is to be 3,200 square meters, and no fencing is needed along the river bank. Find the dimensions of the rectangle that will require the least amount of fencing.

8. Find the dimensions that will minimize the surface area (and hence the cost) of a rectangular tank, open on top, with a volume of 32 cubic feet.

9. A large corporation has decided to audit the finance records for its branches in Tokyo and Gabarone. the corporation has determined that the cost, in thousands of dollars, for this will be $C(x, y) = 2x^2 + xy + y^2 + 100$ where x is the number of audits performed in Tokyo and y is the number of audits performed in Gabarone. (a) What will be the cost if the company performs 5 audits in Tokyo and 10 audits in Gabarone? (b) If the company has enough people to perform a total of 16 audits, how many should be done in each city (how many in Tokyo and how many in Gabarone?) in order to minimize the cost?

Section 29 - Answers

1. Yes
2. (a) 72 at $(x, y) = (6, 6)$
   (b) $\frac{3}{4}$ at $(x, y) = \left(-\frac{3}{4}, \frac{1}{4}\right)$
   (c) -4 at $(x, y) = (0, 2)$ and also at $(x, y) = (0, -2)$
   (d) -3.55 at $(x, y) = (1.4, 2.3)$
   (e) 90 at $(x, y) = (10, -1)$
   (f) maximum value 5 at $(x, y) = (0, -1)$ minimum value -3 at $(x, y) = (0, 1)$
3. $x = 23\frac{1}{3}$ and $y = 11\frac{2}{3}$
4. (a) Make 900 at plant A and 300 at plant B. (b) \$118,000
5. 1 ft $\times$ 1 ft $\times$ 2 ft. (length $\times$ width $\times$ height)
6. $x = 6$ widgets and $y = 4$ bidgets
7. 40×80 (40 meters perpendicular to the river; 80 meters parallel to the river)
8. 4 ft. $\times$ 4 ft. $\times$ 2 ft. (length $\times$ width $\times$ height)
9. (a) \$300,000 (b) Do 4 audits in Tokyo and 12 audits in Gabarone.

Part III
INTEGRAL CALCULUS

Integral calculus is the mathematics of summing up very many very small numbers, or, more accurately, the limiting case of this: summing infinitely many infinitesimally small numbers. After you have read the last part of this book, Sections 30-36, it should be clear what these mysterious words mean. In Sections 35 and 36 we illustrate how integral calculus can be related to finance by discussing some examples based on a simple but non-trivial financial model for estimating the future value of a current investment.

We should point out that integral calculus and differential calculus are really two aspects of the same set of ideas, not separate subjects.

We will limit our study of integral calculus to functions with only one independent variable.

30 Antiderivatives or Indefinite Integrals

Now we return to a function $f(x)$ of one variable. Consider

$$f(x) = x^3 + 7x + 1$$

By an *antiderivative* of a function $f(x)$ we mean a function $F(x)$ whose derivative is $f(x)$, i.e., $F'(x) = f(x)$. In the case of our example

$$F(x) = \frac{x^4}{4} + \frac{7x^2}{2} + x$$

is an antiderivative because the derivative of the function $\frac{x^4}{4} + \frac{7x^2}{2} + x$ is the function $x^3 + 7x + 1$. Of course the derivative of $\frac{x^4}{4} + \frac{7x^2}{2} + x + 2$ is also $x^3 + 7x + 1$ and the derivative of $\frac{x^4}{4} + \frac{7x^2}{2} + x - 10$ is also $x^3 + 7x + 1$ and the derivative of $\frac{x^4}{4} + \frac{7x^2}{2} + x + \pi$ is also $x^3 + 7x + 1$. Indeed, for any constant value C, the derivative of $\frac{x^4}{4} + \frac{7x^2}{2} + x + C$ is $x^3 + 7x + 1$ because $\frac{d}{dx}C = 0$. That is why we speak of *an* antiderivative rather than *the* antiderivative.

A reasonable question would be, "Aside from the difference of a constant, are there other antiderivatives for the function $f(x) = x^3 + 7x + 1$?" The answer is "No." An important fact is that, when the domain is an interval, adding a constant to an antiderivative is the *only* way to get another antiderivative. This is stated in the following theorem and proved with the help of the Mean Value Theorem (see Section 16).

Theorem 30.1. *Assume the domain of the function $f(x)$ is an interval (possibly $\mathbb{R}$). If $F(x)$ and $G(x)$ are antiderivatives of f then $G(x) = F(x) + C$ for some constant C.*

Proof. Suppose this were false. Then there would be two points x_1 and x_2 at which the function $F(x) - G(x)$ takes different values; say $F(x_1) - G(x_1) = u$ and $F(x_2) - G(x_2) = v$ where $u \neq v$. By the Mean Value Theorem there is some number c between x_1 and x_2 such that

$$F'(c) - G'(c) = \frac{u - v}{x_1 - x_2} \neq 0.$$

But $F'(c) - G'(c) = f(c) - f(c) = 0$. This is a contradiction so our "Suppose" must have been wrong. $\square$

Notation and Vocabulary

There is a more commonly used term for "antiderivative." It is *indefinite integral.* Finding an antiderivative is called *integrating.* The process of finding an antiderivative is called *integration.* The function being integrated (i.e., the function for which an antiderivative is being sought) is called the *integrand.*

The notation for integration uses an integral sign, $\int$, paired with a *differential.* The differential is written "dx" (or "dy" or "dt," etc.) and indicates that the variable is x (or y, or t, etc.). The differential is an important part of the notation and should not be omitted even when the intended variable seems obvious.

We write $\int f(x)dx$ to mean "the antiderivative of function f with respect to x."[53]

Using our example above we would write $\int (x^3 + 7x + 1)dx = \frac{x^4}{4} + \frac{7x^2}{2} + x + C$.

Notice the "$+C$" at the end of the antiderivative above. It is customary to use an upper case C to represent an *arbitrary constant.* We include "$+C$" in the answer to an indefinite integral question to indicate that there are infinitely many antiderivatives, although they differ only by a constant. Leaving the "$+C$" off would be an incomplete answer.

Example 30.1. Find: $\int 2x\,dx$.

Answer: $x^2 + C$

One very nice feature of integration problems is that they can be easily checked. By now you should be competent at finding derivatives. To check the answer to Example 30.1 we need only differentiate: $\frac{d}{dx}(x^2+C) = 2x+0 = 2x$ and verify that the derivative is the integrand of the original problem.

Some Rules of Integration

(1) $\int 1dx = x + C$.

(2) $\int (k \cdot f(x))dx = k\int f(x)dx$ for any constant k.
e.g. $\int 4(x^3 + 7x + 1)dx = 4\int (x^3 + 7x + 1)dx$.

(3) $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$
e.g. $\int (x^3 + 7x + 1)dx = \int x^3dx + 7\int xdx + \int 1.dx$.

(4) $\int x^n dx = \frac{1}{n+1}x^{n+1} + C = \frac{x^{n+1}}{n+1} + C$ when $n \neq -1$.
e.g. $\int x^3dx = \frac{1}{4}x^4 + C$ and $\int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}} + C$ and $\int \frac{1}{x^5}dx = \int x^{-5}dx = -\frac{1}{4}x^{-4} + C$.

(5) $\int x^{-1}dx \left(= \int \frac{1}{x}dx \right) = \ln|x| + C$. Note that without the absolute value sign we would not be getting the entire solution. The integrand is defined for all $x \neq 0$ but $\ln x$ is not defined for negative numbers. By using the absolute value we account for the entire domain of $\frac{1}{x}$.

[53] We use the same terminology "with respect to x" that we use when are indicating which variable to use for differentiating. Just as $\frac{d}{dx}f$ tells us to differentiate f with respect to x, $\int f\,dx$ tells us to integrate f with respect to x.

(6) $\int e^x dx = e^x + C$.

These rules are not magic. They are rules of differentiation in reverse. The first three are just Rules 2, 3,and 4 of Section 10 in reverse. The fourth is the Power Rule (Section 12) in reverse. The fifth is Exercise 14 from Section 11 in reverse. The sixth is Theorem 19.3 in reverse.

Since the Product Rule for differentiation is NOT simply: $\frac{d}{dx}(f \cdot g) = \frac{d}{dx}f \cdot \frac{d}{dx}g$, it is not easy to integrate a product of functions. $\int f(x)g(x)dx \neq \int f(x)dx \cdot \int g(x)ds$. Similarly $\int \frac{f(x)}{g(x)}dx \neq \frac{\int f(x)dx}{\int g(x)dx}$. It is much harder to integrate products and quotients. There is no easy formula for $\int f(x)g(x)dx$ or $\int \frac{f(x)}{g(x)}dx$ in terms of $\int f(x)dx$ and $\int g(x)dx$.

Notation:

(1) Instead of $\int 1 \cdot dx$ we write $\int dx$.

(2) Instead of $\int \frac{1}{f(x)}dx$ we often write $\int \frac{dx}{f(x)}$
e.g. $\int \frac{dx}{x}$ rather than $\int \left(\frac{1}{x}\right) dx$ and $\int \frac{xdx}{x^2+1}$ rather than $\int \left(\frac{x}{x^2+1}\right) dx$.

(3) We write the differential dx as a multiplier. Instead of $\int 2x + 3dx$ we write $\int (2x+3)dx$.

Example 30.2.

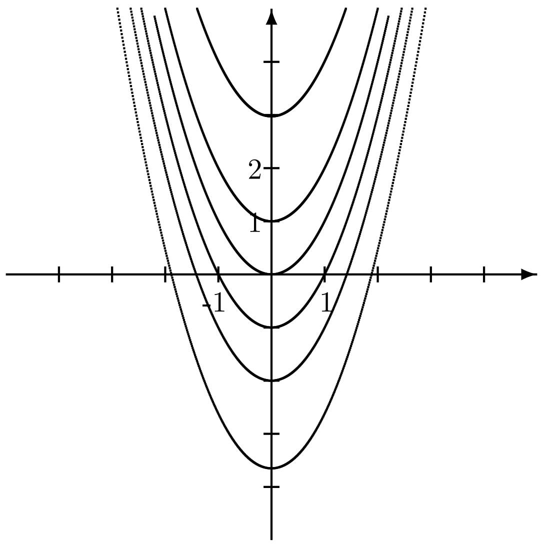
1. $\int (x^5 + 1)dx = \frac{1}{6}x^6 + x + C$
2. $\int (3x^2 + 2\sqrt[3]{x} - 8e^x + 6)dx = 3 \cdot \frac{1}{3}x^3 + 2 \cdot \frac{3}{4}x^{\frac{4}{3}} - 8 \cdot e^x + 6 \cdot x + C = x^3 + \frac{3}{2}x^{\frac{4}{3}} - 8e^x + 6x + C$
3. $\int \left(\frac{5}{x} - \frac{x}{5} + 5\right) dx = \int \left(5 \cdot \frac{1}{x} - \frac{1}{5}x + 5\right) = 5\ln|x| - \frac{1}{10}x^2 + 5x + C$

Example 30.3.

1. $\int (x+1)(x-1)dx = \int (x^2 - 1)dx = \frac{1}{3}x^3 - x + C$
2. $\int \frac{3x^2 + 2x + 1}{x^2}dx = \int \left(3 + \frac{2}{x} + x^{-2}\right) dx = 3x + 2\ln|x| - \frac{1}{x} + C$

Boundary or Initial Conditions:

Recall that the indefinite integral $\int 2x\ dx$ has inifinitely many solutions. We write the general solution as $F(x) = x^2 + C$ where C represents any constant. So, some solutions are $x^2 + 1$, $x^2 - 2$, $x^2 + \pi$ and even simply x^2. Below are graphs of some of the solutions.



$F(x) = x^2 + C$ for various values of constant C

Look at the graphs. None of the graphs shown intersect. Will this be true if we were to graph ALL of the solutions? Think about it. Because the derivative of $F(x) = x^2 + C$ is $2x$, we know that at any given point $(a, a^2 + C)$, the slope at that point is $2a$. For example, at $x = 1$, the slope for each of the graphs is $F'(1)$, or $2(1) = 2$. At $x = 0$, all of the graphs have slope 0. At $x = -2$ all of the graphs have slope -4. If for any value of x two graphs consistently have the same slope, those graphs are parallel. They will not intersect. We can conclude from this that any point on the x, y plane will belong to at most one solution curve.

Look again at the graphs. For each graph, the C value is the y-intercept. Since C can be any real number, if we were to graph ALL of the solutions to $\int 2x\,dx$ every point on the y-axis would belong to one solution (and *only* one solution). While this is easy to see for the y-axis, that is for the line $x = 0$, it is true for any vertical line. Can you see, for example, that every point on the vertical line $x = 1$ belongs to exactly one solution? We can conclude from this that every point on x, y plane belongs to exactly one solution. If we were able to draw in all of the parabolas that are solutions to $\int 2x\,dx$, every point in the plane would be covered, and covered only once.

Example 30.4. Find the specific solution to $\int 2x\,dx$ that goes through the point (3,5).

Solution: $\int 2x\,dx = F(x) = x^2 + C$

$F(3) = 5 = 3^2 + C$. So, $5 = 9 + C$, which means $C = -4$.

The solution is $F(x) = x^2 - 4$.

Example 30.5. Find a function f whose derivative is $3x^5 + \frac{1}{x}$ and that satisfies $f(-1) = -2$.

Solution: $\displaystyle\int \left(3x^5 + \frac{1}{x}\right) dx = \frac{1}{2}x^6 + \ln|x| + C.$

$f(-1) = -2 = \frac{1}{2}(-1)^6 + \ln|-1| + C = \frac{1}{2} + 0 + C$. So, $C = -\frac{5}{2}$.

The solution is $f(x) = \frac{1}{2}x^6 + \ln|x| - \frac{5}{2}$.

The extra condition $f(-1) = -2$ is called an *initial condition* or *side condition* or *boundary condition.*

Example 30.6. Find a function f such that $f(1) = 4$ and $f'(-1) = 2$ and $f''(x) = x + 3$.

Solution: $f'(x) = \displaystyle\int (x+3)dx = \frac{1}{2}x^2 + 3x + C.$

$f'(-1) = 2 = \frac{1}{2}(-1)^2 + 3(-1) + C = \frac{1}{2} - 3 + C = -\frac{5}{2} + C$. So, $C = \frac{9}{2}$.

This tells us that $f'(x) = \frac{1}{2}x^2 + 3x + \frac{9}{2}$.

$f(x) = \displaystyle\int \left(\frac{1}{2}x^2 + 3x + \frac{9}{2}\right) dx = \frac{1}{6}x^3 + \frac{3}{2}x^2 + \frac{9}{2}x + D.$

$f(1) = 4 = \frac{1}{6} + \frac{3}{2} + \frac{9}{2} + D = \frac{37}{6} + D$. So, $D = -\frac{13}{6}$.

The solution is $f(x) = \frac{1}{6}x^3 + \frac{3}{2}x^2 + \frac{9}{2}x - \frac{13}{6}$.

Reasonable questions at this point would be, "Why do we care about antiderivatives? Why would we *want* to work backwards?" When we have a function, we have an expression that gives the relationship between two variables. When we take the derivative of that function, we get an expression that tells the rate of change of those two variables. Sometimes we could be given a rate of change, a derivative, (such as a velocity) and want to know how the variables themselves (distance and time) relate to each other. For this we use integration.

As a simple illustration of this, consider the question: Suppose water is pumped into an empty swimming pool at the rate of 20 gallons per minute. How much water would be in the pool after one hour? This is an easy question that an elementary school student should be able to answer: $20 \times 60 = 120$ gallons. We will look at this using calculus.

We are given a rate of change between the two variables, $w=$ gallons of water and t=time (in minutes). In Leibnitz notation, this is $\frac{dw}{dt} = 20$. We are also given an initial condition. Since the pool is empty at the beginning, we have $w = 0$ when $t = 0$. We want to answer the question "How much water is there in the pool when $t = 60$?" so we want to find a function $w(t)$ that relates the two variables w and t, and then evaluate $w(60)$.

$$\begin{aligned}\frac{dw}{dt} &= 20\\ dw &= 20\,dt\\ \int dw &= \int 20\,dt\\ w &= 20t + C\end{aligned}$$

Using the initial condition (when $t = 0$, $w = 0$) we get $C = 0$.

So our function that relates w and t is simply: $w(t) = 20t$.

To answer the question, we evaluate $w(60) = 20 \cdot 60 = 120$.

Of course it is silly to use calculus for this problem. The problem is easy because our rate of change is constant. The power of calculus is that we can do this when the rates of change are not constant.

At the risk of being redundant, we look again at how calculus takes an expression $\frac{dy}{dx}$ and creates an expression $y = f(x)$ by integrating:

$$\int \frac{dy}{dx}\, dx = \int dy = y + C$$

Example 30.7. The velocity of a particle moving along a straight path is $v(t) = 4e^t - 3t^2$ inches/second. The position of the particle at time t is $s(t)$. The position of the particle at time $t = 0$ is 3 inches to the right of the path's center. Where will the particle be at $t = 1$ (one second later)?

Solution: We know that the derivative of the position function is the velocity function (i.e., $\frac{d}{dt}s(t) = v(t)$). Therefore, $\displaystyle\int v(t)\, dt = s(t)$. We also have the initial condition: $s(0) = 3$. We want to find $s(1)$.

$s(t) = \displaystyle\int (4e^t - 3t^2)dt = 4e^t - t^3 + C$.

$s(0) = 3 = 4e^0 - 0 + C = 4 + C$. So, $C = -1$.

$s(t) = 4e^t - t^3 - 1$, so $s(1) = 4e - 1 - 1 = 4e - 2 \approx 8.87$.

At $t = 1$ the particle is approximately 8.87 inches to the right of the center of the path. It is approximately 5.87 to the right of where it started.

Example 30.8. The marginal cost (dollar cost per item) of producing x items is given by the expression $1.92 - .002x$. If the cost of producing one item is \$562, find the cost of producing 100 items.

Solution: The marginal cost is the derivative of the cost function, so the cost function is the antiderivative of the marginal function.

$C(x) = \displaystyle\int (1.92 - .002x)dx = 1.92x - .001x^2 + C$.

$C(1) = 562 = 1.92 - .001(1) + C = 1.919 + C$. So, $C = 560.081$.

$C(x) = 1.92x - .001x^2 + 560.081$.

$C(100) = 192 - 10 + 560.081 = 742.081$. So, the cost to produce 100 items is \$742.08.

Section 30 - Exercises (answers follow)

1. Find:
   (a) $\int 2dx$
   (b) $\int (x + x^3)dx$
   (c) $\int (12 - 3x)dx$
   (d) $\int \frac{3}{\sqrt{t}}dt$
   (e) $\int (x^{2/3} + x^{-1/3})dx$
   (f) $\int (-9t^{-2} - 2t^{-1})dt$
   (g) $\int \left(\sqrt[3]{x^2} - \frac{1}{x^2}\right) dx$
   (h) $\int \left(\sqrt{x^3} - \frac{1}{\sqrt{x}} + \sqrt{6}\right) dx$
   (i) $\int \left(\sqrt{x} + \frac{3}{x} - e^x\right) dx$
   (j) $\int \frac{1 - 2\sqrt[3]{u}}{\sqrt[3]{u}}du$
   (k) $\int \sqrt{x}(x^2 - 1)dx$
   (l) $\int \left(\frac{5}{x} + 2x^3(x^2 + 1)\right) dx$

2. Suppose $f(x) = x^2 + 6$ and $g(x) = x - 5$
   (a) Find $f(x) \cdot g(x)$. (multiply and simplify)
   (b) Use your result from part (a) to find $\int f(x)g(x)dx$.
   (c) Find $\int f(x)dx$ and $\int g(x)dx$.
   (d) Use your results from parts (b) and (c) to show that $\int f(x)g(x)dx \neq \int f(x)dx \cdot \int g(x)dx$.

3. Find $f(x)$ given that:
   (a) $f'(x) = 1 + e^x + \frac{1}{x}$ and $f(1) = 3 + e$.
   (b) $f'(x) = 2x^2 - e^x + 5$ and $f(0) = 7$
   (c) $f'(x) = 3x^2 + x - 1 - e^x$ and the graph of f goes through the point $(0, 3)$

4. Find the cost function, $C(x)$ when $C'(x) = 6x^2 + 5x$ and the fixed cost is \$12.

5. The marginal cost of producing the xth item is $5 + 2x + 1/x$. The total cost to produce one item is \$500. (a) Find the cost function $C(x)$. (b) How much does it cost to produce 20 items?

6. Suppose the marginal profit from the sale of x hundred items is $P'(x) = 4 - 6x + 9x^2$, and the profit on 0 items is $-\$60$. Find the profit function.

7. An oak tree grows on the edge of a precipice. An acorn on the tree is perched exactly 400 feet above the ground. A squirrel causes the acorn to fall. The velocity of the acorn at time t seconds after it begins to fall is given by $v(t) = -32t$ ft./sec. (a) How far above the ground is the acorn, two seconds after it begins to fall? (b) How long will it take for the acorn to hit the ground?

8. The Little Shop of Humors finds that at a sales level of x comic books per day its marginal profit is $P'(x) = 1.30 + .06x - .0018x^2$ dollars per book. Also, the shop will lose \$95 per day if it sells no books. Find the profit function for the sales of x comic books per day.

Section 30 - Answers

1. (a) $2x + C$
   (b) $\frac{1}{2}x^2 + \frac{1}{4}x^4 + C$
   (c) $12x - \frac{3}{2}x^2 + C$
   (d) $6\sqrt{t} + C$
   (e) $\frac{3x^{5/3}}{5} + \frac{3x^{2/3}}{2} + C$
   (f) $9t^{-1} - 2\ln|t| + C$
   (g) $\frac{3}{5}x^{5/3} + \frac{1}{x} + C$
   (h) $\frac{2}{5}x^{\frac{5}{2}} - 2\sqrt{x} + \sqrt{6}x + C$
   (i) $\frac{2}{3}x^{3/2} + 3\ln|x| - e^x + C$
   (j) $\frac{3}{2}u^{2/3} - 2u + C$
   (k) $\frac{2}{7}x^{7/2} - \frac{2}{3}x^{3/2} + C$
   (l) $5\ln|x| + \frac{1}{3}x^6 + \frac{1}{2}x^4 + C$

2. (a) $x^3 - 5x^2 + 6x - 30$ (b) $\frac{1}{4}x^4 - \frac{5}{3}x^3 + 3x^2 - 30x + C$ (c) $\frac{1}{3}x^3 + 6x + D$ and $\frac{1}{2}x^2 - 5x + E$

3. (a) $f(x) = x + e^x + \ln|x| + 2$
   (b) $f(x) = \frac{2}{3}x^3 - e^x + 5x + 8$
   (c) $f(x) = x^3 + \frac{1}{2}x^2 - x - e^x + 4$

4. $2x^3 + \frac{5}{2}x^2 + 12$

5. Answer: (a) $C(x) = 5x + x^2 + \ln|x| + 494$ (b) $994 + \ln 20 \approx \$997$

6. $-60 + 4x - 3x^2 + 3x^3$

7. (a) 336 ft. (b) 5 secs.

8. $P(x) = 1.3x + .03x^2 - .006x^3 - 95$.

31 u-Substitution

By "an easy integral" we mean one where you can find the antiderivative by just looking at it, such as $\displaystyle\int (x^2 - 1)dx = \frac{x^3}{3} - x + C$. The "method of u-substitution" changes the way you view a hard integral so it can been "seen" as an easy integral.

Suppose we wish to integrate $\displaystyle\int \frac{(2x+2)}{(x^2+2x-4)^4}dx$. We know that we cannot simply integrate the numerator and integrate the denominator. So we look at the integrand more closely. Do you see in the integrand some function, and also see the derivative of that function being used as a multiplier? Yes. The function is $(x^2 + 2x - 4)$ and its derivative is $(2x + 2)$. Finding this condition is the key that tells you that this integral could possibly be solved by the method we call "u-substitution." In this method we rewrite the integrand which uses a variable of x, into an equivalent integrand that uses only the variable u.

We begin by letting u represent the function we found: $u = x^2 + 2x - 4$. It follows that the derivative $\frac{du}{dx} = 2x+2$. We then look at the $\frac{du}{dx}$ as though it were a fraction and get: $du = (2x+2)dx$. Now we are ready to rewrite the original integral by substituting in the equivalent expressions: u replaces $(x^2 + 2x - 4)$ and du replaces the entire expression $(2x + 2)dx$.

The integral $\displaystyle\int \frac{(2x+2)}{(x^2+2x-4)^4}dx$ becomes $\displaystyle\int \frac{1}{u^4}du$.

We know how to integrate $\displaystyle\int \frac{1}{u^4}du$ and do so: $\displaystyle\int \frac{1}{u^4}du = \int u^{-4}du = -\frac{1}{3}u^{-3} + C$.

Since our problem was given in terms of x, we need to give our answer in terms of x, so we substitute back for u and get the final answer of $-\frac{1}{3}(x^2 + 2x + 4)^{-3} + C$.

Check: $\displaystyle\frac{d}{dx}[-\tfrac{1}{3}(x^2 + 2x + 4)^{-3} + C] = (-\tfrac{1}{3})(-3)(x^2 + 2x + 4)^{-4}(2x + 2) = \frac{2x+2}{(x^2+2x+4)^4}$ ✓

Example 31.1. Find: $\displaystyle\int \sqrt{e^x + 5x}\,(e^x + 5)dx$

Solution: Let $u = e^x + 5x$. Then $\frac{du}{dx} = e^x + 5$, or $du = (e^x + 5)dx$.

$$\int \sqrt{e^x + 5x}\,(e^x + 5)dx = \int \sqrt{u}\,du = \int u^{\frac{1}{2}}du.$$

$$\int u^{\frac{1}{2}}du = \frac{2}{3}u^{\frac{3}{2}} + C$$

So, $\displaystyle\int \sqrt{e^x + 5x}\,(e^x + 5)dx = \frac{2}{3}(e^x + 5x)^{\frac{3}{2}} + C.$

Check: $\displaystyle\frac{d}{dx}[\tfrac{2}{3}(e^x + 5x)^{\frac{3}{2}} + C] = \tfrac{2}{3}\cdot\tfrac{3}{2}(e^x + 5x)^{\frac{1}{2}}(e^x + 5) = \sqrt{e^x + 5x}(e^5 + 5)$ ✓

Sometimes the derivative function is not so obvious. Remember that the derivative function must be used as a multiplier in the integrand, not a divisor or addend. When changing the integrand from the x variable to the u variable you must change all of the variables. You cannot have an integral with both x's and u's when you are finished. In other words, the substitution has to be complete: nothing left out and nothing left over. All of the pieces of u and du must fit into corresponding parts of $f(x)dx$.

Example 31.2. Find: $\displaystyle\int \frac{(1+\sqrt{x})^9}{2\sqrt{x}}dx$

Solution: Notice that $\displaystyle\int \frac{(1+\sqrt{x})^9}{2\sqrt{x}}dx = \int (1+\sqrt{x})^9 \cdot \frac{1}{2\sqrt{x}}dx.$

Let $u = 1+\sqrt{x} = 1 + x^{\frac{1}{2}}$. Then $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$, or $du = \frac{1}{2\sqrt{x}}dx$.

$\displaystyle\int \frac{(1+\sqrt{x})^9}{2\sqrt{x}}dx = \int u^9 du.$

$\displaystyle\int u^9 du = \frac{1}{10}u^{10} + C$

So, $\displaystyle\int \frac{(1+\sqrt{9})^9}{2\sqrt{x}}dx = \frac{1}{10}(1+\sqrt{x})^{10} + C.$

Check: $\displaystyle\frac{d}{dx}[\tfrac{1}{10}(1+\sqrt{x})^{10} + C] = \tfrac{1}{10} \cdot 10(1+\sqrt{x})^9(\tfrac{1}{2}x^{-\frac{1}{2}}) = \frac{(1+\sqrt{x})^9}{2\sqrt{x}}$ ✓

Example 31.3. Find: $\displaystyle\int \sqrt{1+x}\ dx.$

Solution: Let $u = 1+x$. Then $\frac{du}{dx} = 1$, or $du = dx$

$\displaystyle\int \sqrt{1+x}\ dx = \int \sqrt{u}\ du$

$\displaystyle\int \sqrt{u}\ du = \int u^{\frac{1}{2}} = \frac{2}{3}u^{\frac{3}{2}} + C$

So, $\displaystyle\int \sqrt{1+x}\ dx = \frac{2}{3}(1+x)^{3/2} + C.$

Check: $\displaystyle\frac{d}{dx}[\tfrac{2}{3}(1+x)^{\frac{3}{2}} + C] = \tfrac{2}{3} \cdot \tfrac{3}{2}(1+x)^{\frac{1}{2}} = \sqrt{1+x}$ ✓

Example 31.4. Find: $\displaystyle\int e^{x^3-3x}(x^2-1)dx$

Solution: Let $u = x^3 - 3x$. Then $\frac{du}{dx} = 3x^2 - 3$.

$\frac{du}{dx} = 3x^2 - 3 = 3(x^2-1)$, so $du = 3(x^2-1)dx$, or $\frac{1}{3}du = (x^2-1)dx$.

$\displaystyle\int e^{x^3-3x}(x^2-1)dx = \int e^u \cdot \frac{1}{3}du = \frac{1}{3}\int e^u\ du$

$\displaystyle\frac{1}{3}\int e^u\ du = \frac{1}{3}e^u + C$

So, $\displaystyle\int e^{x^3-3x}(x^2-1)dx = \frac{1}{3}e^{x^3-3x} + C$

Check: $\quad \frac{d}{dx}(\frac{1}{3}e^{x^3-3x} + C) = \frac{1}{3}e^{x^3-3x}(3x^2-3) = \frac{1}{3}e^{x^3-3x}3(x^2-1) = e^{x^3-3x}(x^2-1)$ ✓

In Example 31.4 our $\frac{du}{dx}$ was not identical to the (x^2-1) given in the integrand. Multiplying both sides of $du = 3(x^2-1)dx$ by $\frac{1}{3}$ took care of the situation easily. This algebraic manipulation is valid only for *multiplication (or division)* by a *constant*. You cannot multiply both sides by any expression involving the variable (in this case x). You cannot add, subtract, raise to a power or use any operation other than multiplication (or division).

Example 31.5. Find: $\displaystyle\int \frac{6x+12}{x^2+4x}dx$

Solution: Let $u = x^2+4x$. Then $\frac{du}{dx} = 2x+4$. So, $3\frac{du}{dx} = 6x+12$, or $3du = (6x+12)dx$

$$\int \frac{6x+12}{x^2+4x}dx = \int \frac{1}{u}\cdot 3du = 3\int \frac{1}{u}du$$

$$3\int \frac{1}{u}du = 3\ln|u| + C$$

$$\text{So, } \int \frac{6x+12}{x^2+4x}dx = 3\ln|x^2+4x| + C$$

Check: $\frac{d}{dx}(3\ln|x^2+4x|+C) = 3\cdot\dfrac{1}{x^2+4x}(2x-4) = \dfrac{6x+12}{x^2+4x}$ $\checkmark$

There is no rule for how to choose the u. Practice will make you better at seeing functions and derivatives. Here are a few tips, but they don't always hold.

1. Often it is wise to choose the most complicated expression for the u (but if that is $(g(x))^n$, choose u to be just the $g(x)$ part).
2. When you are dealing with polynomials, often the one with the higher degreer is the better choice for u.

There can be a certain amount of trial-and-error involved. Don't be discouraged. Practice. In the next example we follow tip 1 and blatantly disregard tip 2.

Example 31.6. Find: $\displaystyle\int x^3(x^2+1)^{\frac{3}{2}}dx$

Solution: Let $u = x^2+1$. Then $\frac{du}{dx} = 2x$, so $\frac{1}{2}\frac{du}{dx} = x$, or $\frac{1}{2}du = xdx$.

In the integrand we have an x^3, but our du only includes an x. We know that we have to substitute *completely* from x to u. Look at our definition of u. We let $u = x^2+1$. That tells us that $u-1 = x^2$. With that, we can now substitute.

$$\int x^3(x^2+1)^{\frac{3}{2}}dx = \int x^2(x^2+1)^{\frac{3}{2}}xdx = \int (u-1)u^{\frac{3}{2}}\frac{1}{2}du = \frac{1}{2}\int (u-1)u^{\frac{3}{2}}du$$

$$\frac{1}{2}\int (u-1)u^{\frac{3}{2}}du = \frac{1}{2}\int (u^{\frac{5}{2}} - u^{\frac{3}{2}})du = \frac{1}{2}\left(\frac{2}{7}u^{\frac{7}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right) + C = \frac{1}{7}u^{\frac{7}{2}} - \frac{1}{5}u^{\frac{5}{2}} + C$$

$$\text{So, } \int x^3(x^2+1)^{\frac{3}{2}}dx = \frac{1}{7}(x^2+1)^{\frac{7}{2}} - \frac{1}{5}(x^2+1)^{\frac{5}{2}} + C$$

Check: $\frac{d}{dx}[\frac{1}{7}(x^2+1)^{\frac{7}{2}} - \frac{1}{5}(x^2+1)^{\frac{5}{2}} + C] = \frac{1}{7}\cdot\frac{7}{2}(x^2+1)^{\frac{5}{2}}2x + \frac{1}{5}\cdot\frac{5}{2}(x^2+1)^{\frac{3}{2}}2x$
$= 2x\cdot\frac{1}{2}(x^2+1)^{\frac{3}{2}}((x^2+1)-1) = x(x^2+1)^{\frac{3}{2}}x^2 = x^3(x^2+1)^{\frac{3}{2}}$ $\checkmark$

Example 31.7. Find: $\displaystyle\int \frac{x}{x+1}dx$

Solution: Let $u = x+1$. Then $\frac{du}{dx} = 1$, or $du = dx$. Also, $u-1 = x$.

$$\int \frac{x}{x+1}dx = \int \frac{u-1}{u}du$$

$$\int \frac{u-1}{u}du = \int \left(1 - \frac{1}{u}\right)du = u - \ln|u| + C$$

So, $\displaystyle\int \frac{x}{x+1}dx = (x+1) - \ln|x+1| + C$

Check: $\frac{d}{dx}[(x+1) - \ln|x+1| + C] = 1 + 0 - \frac{1}{x+1} = \dfrac{(x+1)-1}{x+1} = \dfrac{x}{x+1} \quad \surd$

Example 31.8. Find: $\displaystyle\int x^2(2x+3)dx$

Solution: $\displaystyle\int x^2(2x+3)dx = \int (2x^3+3x^2)dx = \frac{1}{2}x^4 + x^3 + C$

Tip 3: When given a function to integrate, look at it before you attack it. Example 31.8 doesn't require a u-substitution. Don't make your life harder than it has to be.

If you look at the "checks" for the solutions in the examples (except Example 31.8) you will see that they all involve a chain rule. This is not a coincidence. Integration is anti-differentiation and the method of u-substitution is the "chain rule in reverse." Recall the chain rule:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x) \tag{31.1}$$

The derivative in the chain rule (right hand side of Statement 31.1) contains a function $g(x)$ and its derivative $g'(x)$ used as a multiplier. That is precisely what we were looking for in the integrands of our examples.

Section 31 - Exercises (answers follow)

1. Find:
   (a) $\int (3x+1)^5 dx.$
   (b) $\int (-t+1)^3 dt$
   (c) $\int \sqrt{4x-1} dx$
   (d) $\int (-4x^3+2x-1)(x^4-x^2+x)^4 dx$
   (e) $\int x(x^2+1)^3 dx$
   (f) $\int 4e^{2z} dz$
   (g) $\int \frac{2x^4}{x^5+1} dx$
   (h) $\int x^2 e^{x^3-4} dx$
   (i) $\int x(x-2)^5 dx$
   (j) $\int \frac{e^{2x}}{e^{2x}+5} dx$
   (k) $\int \frac{\ln x}{x} dx$
   (l) $\int \frac{4x}{\sqrt{x^2+9}} dx$
   (m) $\int 2x^3 \sqrt{x^2+9}\, dx$
   (n) $\int \frac{x}{(x+1)^2} dx$

2. Show that $\int e^{kt} dt = \frac{1}{k} e^{kt} + C$ for constant k. (This integral will be very useful in Sec. 35).

3. The rate of growth of the profit (in millions of dollars) in a business is $P'(x) = xe^{-x^2}$, where x represents time measured in years. The total profit in the fourth year that the business is in operation is \$8,000.
   (a) Find the profit function.
   (b) What happens to the total profit in the long run?

4. The rate of change of the unit price p (in dollars) of widgets is given by $p'(x) = \frac{-250x}{(1+x^2)^{3/2}}$ where x is the quantity demanded daily in units of a hundred. Find the demand function for these widgets if the quantity demanded daily is 300 items ($x = 3$) when the unit price is \$50/item.

5. The marginal cost of producing the xth roll of sausage is given by $10 - x^3/(x^2+1)^2$. The total cost to produce two rolls is \$700. Find the total cost function $C(x)$.

6. Show that $\displaystyle\int \frac{12x^5 + 15x^4 + 1}{(x+1)^2} dx = \frac{3x^5 + x}{x+1} + C$

Section 31 - Answers

1. (a) $\frac{1}{18}(3x+1)^6 + C$

   (b) $-\frac{1}{4}(-t+1)^4 + C$

   (c) $\frac{1}{6}(4x-1)^{3/2} + C$

   (d) $-\frac{1}{5}(x^4 - x^2 + x)^5 + C$

   (e) $\frac{1}{8}(x^2+1)^4 + C$

   (f) $2e^{2z} + C$

   (g) $\frac{2}{5}\ln|x^5+1| + C$

   (h) $\frac{1}{3}e^{x^3-4} + C$

   (i) $\frac{1}{7}(x-2)^7 + \frac{1}{3}(x-2)^6 + C$

   (j) $\frac{1}{2}\ln|e^{2x}+5| + C$

   (k) $\frac{1}{2}(\ln x)^2 + C$

   (l) $4\sqrt{x^2+9} + C$

   (m) $\frac{2}{5}(x^2+9)^{\frac{5}{2}} - 6(x^2+9)^{\frac{3}{2}} + C$

   (n) $\ln|x+1| + \frac{1}{x+1} + C$

2. Hint: Let $u = kt$.

3. (a) $P(x) = -\frac{1}{2}e^{-x^2} + .008 + \dfrac{1}{2e^{16}}$ (b) $\lim\limits_{x\to\infty} P(x) = .008 + \dfrac{1}{2e^{16}}$

4. $\dfrac{250}{\sqrt{1+x^2}} + 50 - 25\sqrt{10}.$

5. $C(x) = 10x - \frac{1}{2}\ln|x^2+1| - \frac{1}{2}(x^2+1)^{-1} + 680.1 + \frac{1}{2}\ln 5$

6. Hint: Differentiate the right side to get the integrand rather than integrate the left side. You are still showing that the two sides are equal.

32 Integration by Parts

The method of substitution is used to turn a hard integral into an easy one. By contrast, the method described here is used to turn a hard integral into the sum of a function and an easy integral. The "function" in the last sentence is part of the anti-derivative and the easy integral is the other part, hence the name "integration by parts". This will make more sense when we look at examples.

The method of substitution was justified by the chain rule. The method of Integration by Parts is justified by the product rule. Suppose u and v are both functions of x. Then to differentiate the product uv, you get: $\frac{d}{dx}(uv) = \frac{du}{dx}v + \frac{dv}{dx}u$.
Algebraically we can manipulate this to get: $u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$.

Now we integrate both sides with respect to x: $\int u\frac{dv}{dx}dx = \int \frac{d}{dx}(uv)dx - \int v\frac{du}{dx}dx$.
Writing dv for $\frac{dv}{dx}dx$ and du for $\frac{du}{dx}dx$ this shortens to

$$\int u\,dv = uv - \int v\,du. \tag{32.1}$$

Formula 32.1 SHOULD BE MEMORIZED. It is the formula for *integration by parts* (so called because uv is part of the answer, and, if you are lucky, $\int v\,du$ will be an easy integral.)

When using the method of substitution you are challenged to look at the integrand to try to find a function, and *its* derivative used as a multplier. We let the function be "u" and the derivative is "du."

When using the method of integration by parts you are looking in the integrand for a function and *any* derivative used as a multiplier. We let the function be "u" and the derivative be "dv."

In short, given a problem of computing $\int f(x)dx$ you want to get $f(x)dx$ into the form $u \cdot dv$.

Example 32.1. Find: $\int xe^x dx$.
Solution: Here we choose $u = x$ and $dv = e^x dx$. We check to be sure that the product $u \cdot dv$ matches the integrand exactly. It does. Notice that the dx in the integrand is accounted for by the dx in the assignment of $dv = e^x dx$. The dx of the original integrand will always be included in the dv assignment.

To apply the Integration by Parts (IBP) formula we need to figure the entities du and v. Since our $u = x$, our $du = dx$. To figure v we need to integrate dv. So, $v = \int e^x dx = e^x$. Now with $u = x$, $du = dx$, $v = e^x$ and $dv = e^x dx$, we can apply the rule for Integration by Parts...

$$\begin{aligned} \int udv &= uv - \int vdu \\ \int x \cdot e^x dx &= xe^x - \int e^x dx \\ \text{...and finish the problem} \qquad &= xe^x - e^x + C \end{aligned}$$

Example 32.2. Find: $\displaystyle\int x \ln x \, dx$.

Solution: Here we choose $u = \ln x$ and $dv = xdx$. We check to be sure that the product $u \cdot dv$ is equal to the integrand. It is $(x \ln x \, dx = (\ln x) x \, dx)$.

Since our $u = \ln x$, our $du = \frac{1}{x} dx$. Since $dv = xdx, v = \frac{1}{2}x^2$. Now we can apply the rule for Integration by Parts...

$$\begin{aligned}
\int udv &= uv - \int vdu \\
\int x \ln x dx &= (\ln x) \cdot \frac{1}{2}x^2 - \int \frac{1}{2}x^2 \cdot \frac{1}{x} dx \\
&= \frac{1}{2}x^2 \ln x - \int \frac{1}{2} x \, dx \\
&= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C
\end{aligned}$$

Choosing u and dv

By now you may be wondering how to choose a u and dv that will work. There is no magic way to know. There is a certain amount of trial-and-error involved, but with experience you can cut down on the "error" part considerably.

There are, however, a few hints:

<u>Hint 1:</u> Look back at Example 32.1, $\displaystyle\int xe^x \, dx$. Suppose we had chosen $u = e^x$ and $dv = xdx$. Then we would calculate $du = e^x dx$ and $v = \frac{1}{2}x^2$. When we apply the IBP formula, we would have $\displaystyle\int xe^x \, dx = e^x x - \int \frac{1}{2}x^2 e^x dx$. While this is correct, it isn't particularly useful. The new integral is harder to do than the original one! So, Hint 1 is that you want to choose your u and dv so that the integrand for the new integral, $v \cdot du$, is easily integrated.

<u>Hint 2:</u> Look back at Example 32.2, $\displaystyle\int x \ln x \, dx$. In view of Hint 1, we would likely choose $u = x$ and $dv = \ln x \, dx$, because then the du we need for the new integrand is very simple. However, the difficulty is in finding v. If $dv = \ln x dx$, what is v? We know the *derivative* for $y = \ln x$, but we don't know the *anti*-derivative. Without a v, we are stuck. So, Hint 2 is that you want to choose your dv so that finding v is doable.

These two hints can sometimes (as in Example 32.2) be in conflict with each other. But, they are only hints. There is no set rule. If you make a choice that doesn't work, be willing to try another. Sometimes a correct choice can be elusive. It is true though that the more you practice, the more you will see patterns of successful choices.

Since we have raised the question of an antiderivative for $y = \ln x$, we will look at this next. The antiderivative for this function can actually be found by using the method of Integration by Parts. The solution illustrates the clever use of letting dv be only dx.

Example 32.3. Find: $\int \ln x\, dx$

Solution: Let $u = \ln x$ and $dv = dx$. Clearly $u \cdot dv$ is equal to the integrand of the original problem.

We calculate $du = \frac{1}{x} dx$ and $v = x$.

Now we use the IBP formula and solve the problem:

$$\begin{aligned}
\int u\, dv &= uv - \int v\, du \\
\int \ln x\, dx &= (\ln x)x - \int dx \\
&= x \ln x - x + C
\end{aligned}$$

You don't have to memorize this integral, but it can be useful to know:

$$\int \ln x\, dx = x \ln x - x + C \tag{32.2}$$

Example 32.4. Find: $\int \frac{\ln x}{x^3} dx$

Solution: Choose $u = \ln x$ and $dv = \frac{1}{x^3} dx$. We see that $u \cdot dv$ is equal to the integrand. Note: We cannot choose $dv = x^3 dx$ because then $u \cdot dv = (\ln x)x^3 dx$. The $u \cdot dv$ MUST be multiplied to get the integrand of the original problem.

Since $u = \ln x, du = \frac{1}{x} dx$ and since $dv = \frac{1}{x^3} dx = x^{-3} dx$, we have $v = -\frac{1}{2} x^{-2}$

Now we use the IBP formula and solve the problem:

$$\begin{aligned}
\int u\, dv &= uv - \int v\, du \\
\int \frac{\ln x}{x^3} dx &= (\ln x)\left(-\frac{1}{2} x^{-2}\right) - \int -\frac{x^{-2}}{2} \cdot \frac{1}{x} dx \\
&= \frac{-\ln x}{2x^2} + \int \frac{1}{2} x^{-3} dx \\
&= \frac{-\ln x}{2x^2} + \frac{1}{2}\left(\frac{-x^{-2}}{2}\right) \\
&= \frac{-\ln x}{2x^2} - \frac{1}{4x^2} + C
\end{aligned}$$

The method of Integration by Parts always leaves you with a new integral to deal with. In the previous examples, the new integral has been easy. This is not always the case. Sometimes the new integral involves some algebraic maniputlation. Sometimes the solution to the new integral will require the use of a u-substitution. Sometimes the new integral will involve IBP itself to solve (this will, of course, lead you to yet another integral)!

This next example is rather difficult...until you see the solution.

Example 32.5. Find: $\displaystyle\int \frac{xe^x}{(x+1)^2}dx$

Solution: Choose $u = xe^x$ and $dv = \dfrac{1}{(x+1)^2}dx = (x+1)^{-2}dx$. So, $du = (e^x + xe^x)dx$ and $v = -(x+1)^{-1}$.

Now we use the IBP formula and solve the problem:

$$\int u\,dv = uv - \int v\,du$$

$$\begin{aligned}
\int \frac{xe^x}{(x+1)^2}dx &= xe^x(-(x+1)^{-1}) - \int -(x+1)^{-1}(e^x + xe^x)dx \\
&= \frac{-xe^x}{x+1} + \int \frac{e^x + xe^x}{x+1}dx \\
&= \frac{-xe^x}{x+1} + \int \frac{e^x(1+x)}{x+1}dx \\
&= \frac{-xe^x}{x+1} + \int e^x dx \\
&= \frac{-xe^x}{x+1} + e^x + C
\end{aligned}$$

Hint 3: While Integration by Parts is pretty slick, other methods are often easier and more straight-forward. When faced with an integral, consider other options before attempting Integration by Parts. IBP should be your last resort.

Example 32.6. The velocity of a dragster[54] t seconds after leaving the starting line is given by the function $v(t) = 100te^{-0.2t}$ ft./sec. What is the distance traveled by the car during the first ten seconds of its run?

Solution: We are given the velocity function $v(t)$. We want to know a distance. The position function $s(t) = \displaystyle\int v(t)dt$. At $t = 0$ the car isn't moving, so $s(0) = 0$ is our initial condition. We are looking for $s(10) - s(0) = s(10)$

$s(t) = \displaystyle\int 100te^{-0.2t}dt = 100\int te^{-0.2t}dt$. We will use IBP to integrate.

[54] A *drag race* is a car race where the car travels a straight course from the starting line to the finish line. A *dragster* is a car in a drag race.

Choose $u = t$ and $dv = e^{-0.2t}dt$. Then, $du = dt$ and $v = \dfrac{e^{-0.2t}}{-0.2} = -5e^{-0.2t}$.

$$\begin{aligned}
\int u\,dv &= uv - \int v\,du \\
\int te^{-0.2t}dt &= t(-5e^{-0.2t}) - \int (-5e^{-0.2t})dt \\
&= -5te^{-0.2t} + 5\int e^{-0.2t}dt \\
&= -5te^{-0.2t} + 5\left(\frac{e^{-0.2t}}{-0.2}\right) + C \\
&= -5te^{-0.2t} - 25e^{-0.2t} + C \\
&= -5e^{-0.2t}(t+5) + C
\end{aligned}$$

So, $s(t) = -500e^{-0.2t}(t+5) + C$ and $s(0) = 0 = -500e^{0}(5) + C$. So, $C = 2500$.

$s(t) = -500e^{-0.2t}(t+5) + 2500$

$s(10) = -500e^{-2}(15) + 2500 = \dfrac{-7500}{e^2} + 2500 \approx 1,485$ ft.

Section 32 - Exercises (answers follow)

1. Find:

   (a) $\int (x+1)e^x dx$

   (b) $\int 2xe^x dx$

   (c) $\int (x-3)e^{3x} dx$

   (d) $\int xe^{-x/5} dx$

   (e) $\int x^2 \ln x dx$

   (f) $\int x^3 \ln x dx$

   (g) $\int \frac{1-x}{3e^x} dx$

   (h) $\int \ln(2x) dx$

   (i) $\int \frac{\ln x}{x^2} dx$

   (j) $\int x^2 \ln(3x) dx$

   (k) $\int (2x+9)e^x dx$

   (l) $\int 2x^7 e^{x^4} dx$

2. Find the function f given that the slope of the tangent line to the graph of f at any point $(x, f(x))$ is xe^{-2x} and that the graph passes through the point $(0, 3)$.

3. The rate of change of revenue (in dollars per item) from the sale of x items is $R'(x) = 10 + x^2e^{-x}$. Find the revenue function. Note: If there are no sales, there is no revenue.

Section 32 - Answers

1.

   (a) $xe^x + C$ (b) $2e^x(x-1) + C$ (c) $\frac{1}{3}xe^{3x} - \frac{10}{9}e^{3x} + C$

   (d) $-5(x+5)e^{-x/5} + C$ (e) $\frac{x^3}{3}\left(\ln x - \frac{1}{3}\right) + C$ (f) $\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$

   (g) $\frac{1}{3}xe^{-x} + C$ (h) $x\ln(2x) - x + C$ (i) $-\frac{1}{x}(\ln x + 1) + C$

   (j) $\frac{1}{3}x^3 \ln(3x) - \frac{1}{9}x^3 + C$ (k) $2xe^x + 7e^x + C$ (l) $\frac{1}{2}x^4e^{x^4} - \frac{1}{2}e^{x^4} + C$

2. $-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + \frac{13}{4}$

3. $R(x) = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + 10x + 2$

33 Definite Integrals

We continue our introduction to integral calculus. In a short course it's important to learn how to do the mechanics. But there is a reason for doing all this, a reason which will be explained in the next two sections. Integral calculus provides a powerful tool for solving real world problems.

Recall that when the domain of $f(x)$ is an interval, the indefinite integral $\int f(x)dx$ is a function (with an arbitrary constant C added) whose derivative is $f(x)$. We write:

$$\int f(x)dx = F(x) + C.$$

Now suppose that the closed interval $[a, b]$ is in the domain of $f(x)$. We define the *definite integral*, written $\int_a^b f(x)dx$, to be the number $F(b) - F(a)$. Note that a definite integral is a number, a constant. It is not a function.

Example 33.1. Evaluate the definite integral $\int_{-2}^{5} 2xdx$.

Solution: $\int 2xdx = x^2 + C$

So, $\int_{-2}^{5} 2xdx = (5^2 + C) - ((-2)^2 + C) = 25 + C - 4 - C = 21$

Special Notes about "+C"

1. Did you notice what happened to the "+C" during the calculation step in Example 33.1? The "$+C$" in $F(b)$ canceled with the "$+C$" in $F(a)$. This will always happen.[55] So, while there is no algebraic harm in adding "$+C$" to the antiderivative of a definite integral, it is a waste of time. We won't bother to do it.

2. We repeat that a definite integral *is a number*. It is a specific constant value. It would make no sense then to add "$+C$" to the final evaluation (answer) of a definite integral. It would be like answering the question, "How tall is the Empire State Building?" with "It is exactly 1,454 feet[56] plus or minus whatever number you want."

When writing out the solutions for definite integrals a notation convention is useful: $F(x)\Big|_a^b$ denotes $F(b) - F(a)$. We will use this in the following examples.

[55] $(F(b) + C) - (F(a) + C) = F(b) + C - F(a) - C = F(b) - F(a)$.

[56] This measurement includes the lightning rod on top.

Example 33.2. Evaluate $\displaystyle\int_1^4 \left(\sqrt{x} + \frac{2}{x} + 1\right) dx$

Solution:

$$\begin{aligned}
\int_1^4 \left(\sqrt{x} + \frac{2}{x} + 1\right) dx &= \left(\frac{2}{3}x^{\frac{3}{2}} + 2\ln|x| + x\right)\Bigg|_1^4 \\
&= \left[\frac{2}{3}(4)^{\frac{3}{2}} + 2\ln|4| + 4\right] - \left[\frac{2}{3}(1)^{\frac{3}{2}} + 2\ln|1| + 1\right] \\
&= \frac{2}{3}\cdot 8 + 2\ln 4 + 4 - \frac{2}{3} - 0 - 1 \\
&= \frac{23}{3} + 2\ln 4 \quad \leftarrow \text{This is a real number, } \approx 12.44
\end{aligned}$$

Limits of Integration

The numbers a and b are called the *limits of integration*; b is the *upper limit* and a is the *lower limit.* The passage from $F(x)\Big|_a^b$ to $F(b) - F(a)$ is called *plugging in the limits.*

Recall from the beginning of this section that we defined a definite integral to have limits of integration a and b IF the interval $[a, b]$ is in the domain of the integrand function. If $[a, b]$ is not in the domain of $f(x)$, then $\displaystyle\int_a^b f(x)dx$ makes no sense. In the next two examples we see what happens if the domain issue is ignored.

Example 33.3. Evaluate $\displaystyle\int_0^1 \frac{1}{x^2}dx$ Notice that zero is not in the domain of $f(x) = \frac{1}{x^2}$.

Solution:

$$\int_0^1 \frac{1}{x^2}dx = \int_0^1 x^{-2}dx = -x^{-1}\Big|_0^1 = \frac{-1}{x}\Big|_0^1 = \frac{-1}{1} - \frac{-1}{\text{uh-oh!}} \swarrow \text{We can't put the zero in the denominator!}$$

Example 33.4. Evaluate $\displaystyle\int_{-1}^1 \frac{1}{x^2}dx$ Notice that zero is not in the domain of $f(x) = \frac{1}{x^2}$.

Solution:

$$\int_{-1}^1 \frac{1}{x^2}dx = \int_{-1}^1 x^{-2}dx = -x^{-1}\Big|_{-1}^1 = \frac{-1}{x}\Big|_{-1}^1 = \frac{-1}{1} - \frac{-1}{-1} = -1 - 1 = -2 \quad \mathit{WRONG}!$$

Notice that in Example 33.3 we are alerted to the domain error when we go to plug in the lower limit. However, in Example 33.4 the arithmetic seems to work out! If we do not notice the domain difficulty, we will arrive at a totally bogus answer and not know it.[57]

While we are discussing cautions, here is another suggestion: Use Parentheses! Be sure that when you evaluate $F(b) - F(a)$ that you subtract the entire expression for $F(a)$ and not just the first term.

[57] A few years ago a definite integral with illegal limits of integration was put on the Calculus I final exam. Of the 800^+ students taking the test, fewer than a dozen students noticed the difficulty. – The question was not intended to trick the students; the instructors who wrote the exam hadn't caught the error either! FYI., the test was scored as though the limits were valid and the alert students were given extra credit (in addition to the time they saved by not having to plow through the antiderivative, evaluation and arithmetic). Lesson learned: Constant vigilance! It can happen!

Definite Integrals and u-Substitution:

When your integral needs a substitution (Section 31) it is often easier to change the limits of integration when you substitute. Below are two approaches to a substitution problem. The second approach is recommended.

Example 33.5. Evaluate: $\displaystyle\int_0^2 \frac{x}{x^2+3}dx$.

First Approach: Find the antiderivative using u-substitution. Re-convert the answer to an x-variable expression. Plug in the limits of integration and evaluate.

Find the antiderivative for indefinite integral $\displaystyle\int \frac{x}{x^2+3}dx$ and put the answer in terms of x:

Let $u = x^2+3$. Then $du = 2xdx$ and so $xdx = \frac{1}{2}du$.

$$\int \frac{x}{x^2+3}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln(x^2+3)$$

Now evaluate the definite integral with the original, x-valued, limits of integration:

$$\int_0^2 \frac{x}{x^2+3}dx = \frac{1}{2}\ln(x^2+3)\Big|_0^2 = \frac{1}{2}\ln 7 - \frac{1}{2}\ln 3) = \frac{1}{2}(\ln 7 - \ln 3) = \frac{1}{2}\ln\left(\frac{7}{3}\right)$$

Second Approach: Rewrite the original, x-valued limits of integration into their corresponding u-valued limits of integration. Find the antiderivative using a u-substitution. Plug the u-valued limits of integration into the u expression of the antiderivative.

Let $u = x^2+3$. Then $du = 2xdx$, and so $xdx = \frac{1}{2}du$. When $x = 0, u = 0^2+3 = 3$. When $x = 2$, $u = 2^2+3 = 7$. So, as x runs from 0 to 2, u runs correspondingly from 3 to 7. Thus:

$$\int_0^2 \frac{x}{x^2+3}dx = \frac{1}{2}\int_3^7 \frac{du}{u} = \frac{1}{2}\ln|u|\Big|_3^7 = \frac{1}{2}\ln 7 - \frac{1}{2}\ln 3 = \frac{1}{2}(\ln 7 - \ln 3) = \frac{1}{2}\ln\left(\frac{7}{3}\right)$$

In the second approach we changed the limits of integration when we moved from x to u, so it was not necessary to go back into x.

Using either method correctly, you will get the same result. You can evaluate the definite integral using an x variable or a u variable. But you MUST BE CONSISTENT. The limits of integration that are x values (original limits of integration) can only be substituted into the antiderivative expressed in x. The limits of integration that are u values (calculated from the x-valued limits) can only be substituted into the antiderivative expressed in u.

Definite Integral and Integration by Parts:

We illustrate with Example 32.1 in Section 32 (page 250).

Example 33.6.

$$\begin{aligned}\int_0^1 xe^x\,dx &= xe^x\Big|_0^1 - \int_0^1 e^x dx\\ &= (1\cdot e^1 - 0\cdot e^0) - e^x\Big|_0^1\\ &= (e-0)-(e^1-e^0)\\ &= e-e+1=1\end{aligned}$$

You could, of course, just find the entire antiderivative first, and then substitute in the limits of integration. This can be preferable because it is possible that some terms from the second integral will cancel with some terms of the uv expression (hence there is less evaluation and arithmetic).

$$\begin{aligned}\int_0^1 xe^x\,dx &= (xe^x - e^x)\Big|_0^1\\ &= (1\cdot e^1 - e^1) - (0\cdot e^0 - e^0)\\ &= e-e-0+1=1\end{aligned}$$

Some Rules for Definite Integrals:

Below are some rules for definite integrals. In the following sections we will see a geometric interpretation of these rules that will be very useful in applications. For now, the rules are just mechanical. The proofs for the rules below follow easily from the definition of definite integral: $\displaystyle\int_a^b f(x)dx = F(b)-F(a)$, and the rules that we already have for integration (see page 237). Proofs for the rules below are left as exercises.

1. $\displaystyle\int_a^a f(x)dx = 0.$ e.g. $\displaystyle\int_3^3 5x^2\,dx = 0$
2. $\displaystyle\int_a^b k\cdot f(x)dx = k\cdot\int_a^b f(x)dx$ for any constant k. e.g. $\displaystyle\int_1^7 5x^2\,dx = 5\int_1^7 x^2\,dx$
3. $\displaystyle\int_a^b [f(x)\pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$
   e.g. $\displaystyle\int_{-3}^{-1} (5x^2 - e^x)\,dx = \int_{-3}^{-1} 5x^2\,dx - \int_{-3}^{-1} e^x\,dx$
4. $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ for any c. e.g. $\displaystyle\int_2^6 5x^2dx = \int_0^6 5x^2dx + \int_2^0 5x^2dx$
5. $\displaystyle\int_a^b f(x)dx = -\int_b^a f(x)dx.$ e.g. $\displaystyle\int_2^5 5x^2dx = -\int_5^2 5x^2dx$

Note: These rules are to be interpreted intelligently. If any part of $[a,b]$ (or $[b,a]$ if $b<a$) is not in the domain of f we have no business writing $\displaystyle\int_a^b f(x)dx$. For Rule 3, it is not required that c fall between a and b, but it must be true that $[a,c]$ (or $[c,a]$) and $[c,b]$ (or $[b,c]$) are in the domain of f.

Section 33 - Exercises (answers follow)

1. Evaluate the integrals.

(a) $\int_{-1}^{1}(x^2+2)dx$ (b) $\int_{1}^{4} 2x^{-3/2}dx$ (c) $\int_{1}^{3}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)dx$

(d) $\int_{0}^{1} 6(4\sqrt{x}-3x\sqrt{x})dx$ (e) $\int_{-1}^{1} e^{x+1}dx$ (f) $\int_{1}^{2}\frac{x^2}{(x^3+1)^2}dx$

(g) $\int_{0}^{9} x\sqrt[3]{x^2+5}dx$ (h) $\int_{-1}^{0} 2x\sqrt{x+1}dx$ (i) $\int_{-2}^{2} xe^{-x}dx$

(j) $\int_{1}^{2}\frac{(\ln x)^2}{x}dx$ (k) $\int_{1}^{4}\ln x dx$ (l) $\int_{0}^{1} x^2e^{2x}dx$

(m) $\int_{-3}^{-1}\frac{6x^5+3x^2}{x^6}dx$ (n) $\int_{0}^{1}\frac{6x^2+2e^x}{e^x+x^3}dx$ (o) $\int_{1}^{4}\sqrt{x}\ln x\, dx$

(p) $\int_{0}^{3}\frac{e^x+1}{5}dx$ (q) $\int_{-2}^{1}\frac{3}{x-2}dx$ (r) $\int_{-1}^{1}(x^2+7)^2dx$

(s) $\int_{0}^{1} 2x(3x^2-1)^4dx$

2. Prove the five Rules for Definite Integrals on page 259.

Section 33 - Answers

1.

(a) $\frac{14}{3}$ (b) 2 (c) $\frac{8}{3}+\ln 3$ (d) $\frac{44}{5}$ (e) e^2-1

(f) $\frac{7}{54}$ (g) $\frac{3}{8}(86^{4/3}-5^{4/3})$ (h) $-\frac{8}{15}$ (i) $-3e^{-2}-e^2$ (j) $\frac{1}{3}(\ln 2)^3$

(k) $-3+4\ln 4$ (l) $\frac{e^2-1}{4}$ (m) $\frac{26}{27}-6\ln 3$ (n) $2\ln(e+1)$ (o) $\frac{16}{3}\ln 4-\frac{28}{9}$

(p) $\frac{1}{5}e^3+\frac{2}{5}$ (q) $-3\ln 4$ (r) $107\frac{11}{15}$ (s) $\frac{33}{15}$

2. Rule 1: $\int_{a}^{a} f(x)dx = F(a)-F(a)=0$

Rule 2: Integration Rule #1 (see page 237) tells us that $\int kf(x)dx = k\int f(x)dx = k\cdot F(x)$.

So, $\int_{a}^{b} kf(x)dx = [k\cdot F(x)]\Big|_{a}^{b} = k\cdot F(b)-k\cdot F(a) = k\cdot[F(b)-F(a)] = k\int_{a}^{b} f(x)dx$.

The proof for Rule 3 is similar to the proof for Rule 2, except that it uses Integration Rule 2 on page 237.

The proofs for Rules 4 and 5 are straightforward algebra manipulations like the proof for Rule 1 above.

34 Definite Integral and Area . Fundamental Theorem of Calculus

In this section we give a geometric interpretation of the definite integral, leading to the Fundamental Theorem of Calculus. This is deep material; don't be misled by its simplicity!

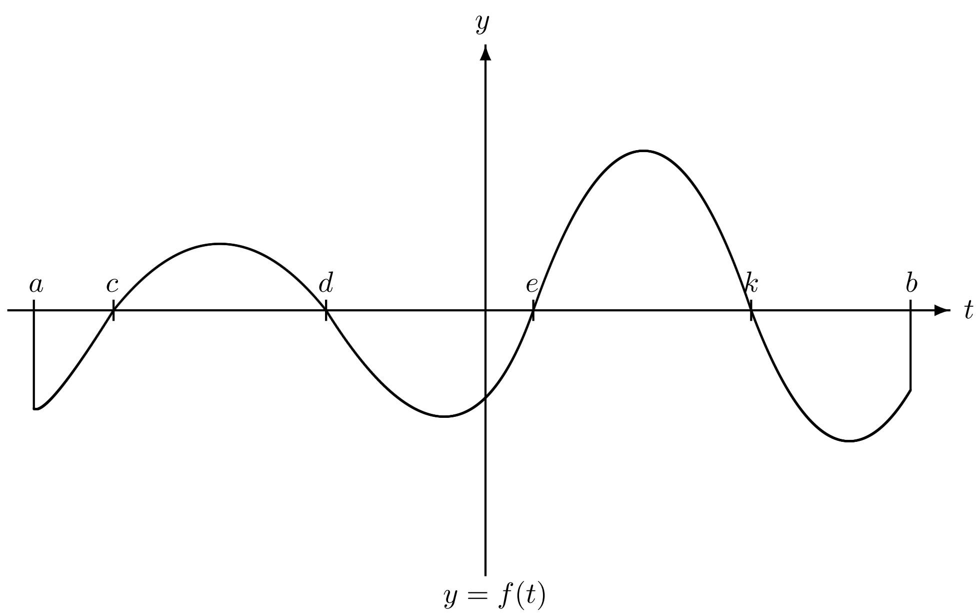
We saw that

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F(x)$ is any antiderivative of $f(x)$. The letter x has no real significance here since the right hand side only involves the letters F, a and b. It will be convenient here to call the horizontal axis the t-axis instead of the x-axis and hence to write $\int_a^b f(t)dt$ rather than $\int_a^b f(x)dx$.

We saw in Section 30 that if you have one antiderivative for f (whose domain is an interval) then all other antiderivatives can be obtained from that one by adding constants (Theorem 30.1). A sharp reader might ask: how do I know that there is any antiderivative for f at all? You will see here [58].

Look at the graph of f below. We have chosen to speak of the t-axis rather than the x-axis. You are looking at a continuous function $f(t)$ whose domain includes a closed interval $[a, b]$.



Consider the graph of f. Vertical lines $t = a$ and $t = b$, the t-axis, and the graph of f carve out five regions of the plane each having finite area. We will call $A^+(f)$ the sum of the two areas of the regions above the t-axis and call $A^-(f)$ the sum of the areas of the three regions below the t-axis. Both $A^+(f)$ and $A^-(f)$ are ≥ 0 because area is never negative. The number $A^+(f)$ can only be 0 if there are no regions above the t-axis; similarly for $A^-(f)$.

The *signed area of f on* $[a, b]$ is defined to be the number $S(f, [a, b]) = A^+(f) - A^-(f)$. Don't let the notation bother you. The *signed area* is simply the difference (subtraction) of the areas above the t-axis and the areas below the t-axis. Signed area could be positve or negative depending on

[58]Technically, what you see here will work for continuous functions $f(t)$ (and even for piecewise-continuous functions) but you will not be dealing with anything else, so we will avoid too much technical discussion of that kind.

whether there is more area in the regions above the t-axis or more area in the regions below the t-axis. For our graph, the signed area of f over interval $[a, b]$ is the sum of the areas on intervals $[c, d]$ and $[e, k]$, minus the sum of the areas on the intervals $[a, c]$, $[d, e]$ and $[k, b]$. An equivalent way of thinking of this is that signed area counts as positive the areas above the t-axis and counts as negative the areas below the t-axis and sums these positive and negative values.

Example 34.1. Use the graph above to find $S(f, [c, k])$, the signed area of f on interval $[c, k]$.
Solution: $S(f, [c, k])=$ area of region on $[c, d]$ minus area of region on $[d, e]$ plus area of region on $[e, k]$.

Example 34.2. Is $S(f[d, k])$ positive or negative? What about $S(f, [a, e])$?
Solution: We can "eyeball" from the graph that $S(f[d, k]) > 0$ because there is more area above the t-axis than below. Similarly, on interval $[a, e]$ there is more total area below the t-axis, so $S(f, [a, e]) < 0$.

Observations:

1. Signed area is zero only when the area above the horizontal axis is equal to the area below the horizontal axis, (i.e., when $A^+(f) = A^-(f)$).
2. Signed area equals actual area only when the area below the horizontal axis is zero, (i.e., when $A^-(f) = 0$).

Now that we have a good understanding of signed area, we create a function $G(x)$ as follows: $G(x) = S(f, [a, x])$ where $a \leq x \leq b$. This is a function of x whose domain is $[a, b]$. The number $G(x)$ is the signed area of f on $[a, x]$. It changes as the point x on the t-axis changes. For example, $G(e) = S(f, [a, e])$, the signed area we looked at in Example 34.2, and $G(a) = S(f[a, a]) = 0$.

We are now ready to make a connection between signed area and the definite integral. This connection is the Fundamental Theorem of Calculus.

Theorem 34.1. Fundamental Theorem of Calculus *$G'(x) = f(x)$.*

The FTC says that $G(x)$ is an antiderivative for $f(x)$; in other words, it says that $G(x) = \int f(x)dx$. Now we saw in Section 30 that any two antiderivatives differ by a constant, so we have a practical way of computing the signed area: namely, find ANY antiderivative $F(x)$ of $f(x)$; the signed area will then be $F(b) - F(a)$.

We'll see in the next section that the answers to a variety of useful problems turn out to be signed areas, so having such a neat way to evaluate them is powerful.

Proof of the Fundamental Theorem of Calculus. We are to show that $\lim_{h \to 0} \frac{G(x+h) - G(x)}{h} = f(x)$. When $h > 0$ (with $x + h \leq b$)

$$\begin{aligned} G(x+h) - G(x) &= S(f, [a, x+h]) - S(f, [a, x]) \\ &= S(f, [x, x+h]) \\ &= \text{the signed area of } \ f \ \text{ over } \ [x, x+h]. \end{aligned}$$

f is assumed to be continuous so if h is sufficiently small the value of f on $[x, x+h]$ does not change very much, and the smaller h is the less $f(x)$ changes on $[x, x+h]$. So $S(f, [x, x+h])$ is approximately the signed area of the rectangle of height $|f(x)|$ and width h (counted positively if $f(x) > 0$ and counted negatively if $f(x) < 0$). That is, $S(f, [x, x+h])$ is approximately $f(x)h$. So $\dfrac{G(x+h) - G(x)}{h}$ is approximately $f(x)$ and $\lim_{h \to 0} \dfrac{G(x+h) - G(x)}{h} = f(x)$.

$\square$

Finding Signed Area

> The Fundamental Theorem of Calculus tells us that for a continuous function f,
> $\int_a^b f(t)dt$ is the *signed* area of f on $[a, b]$.

Note in the boxed statement above that we are assuming that $a \le b$. We have discussed signed area in terms of area above the x-axis netted with area below the x-axis. While the measurement of area is independent of the way in which we look at the domain interval, the integral in the box is not. The Rules for Definite Integrals on page 259 still apply. For example, $\int_b^a f(x)dx = -\int_a^b f(x)dx =$ *minus* the signed area on $[a, b]$.

Example 34.3. Find the signed area of the function $f(x) = \frac{1}{2}x - 1$ on the interval $[-2, 8]$.
Solution: The FTC tells us that the signed area is the definite integral $\int_{-2}^{8} \left(\frac{1}{2}x - 1\right) dx$.

$$\int_{-2}^{8} \left(\frac{1}{2}x - 1\right) dx = \left(\frac{1}{4}x^2 - x\right)\Bigg|_{-2}^{8} = \left(\frac{1}{4} \cdot 64 - 8\right) - \left(\frac{1}{4} \cdot 4 + 2\right) = (16 - 8) - (1 + 2) = 5$$

The signed area is 5.

Finding Actual Area

Suppose we want to find the actual area between a graph and the x-axis. Look back at the graph of $y = f(t)$ on page 261. If we want to find the area between f and the horizontal axis on the interval $[c, d]$ we could simply evaluate $\int_c^d f(t)\,dt$ because the area and the signed area are the same (both are positive). If we want to find the area between f and the horizontal axis on the interval $[d, e]$, we could use the negative of the signed area over that interval, i.e., we could evaluate $-\int_d^e f(t)\,dt$.

Suppose we want to find the *area* between f and the horizontal axis on the interval $[c, e]$? Does it make sense to simply add the two areas found above: $\int_c^d f(t)\,dt - \int_d^e f(t)\,dt$? It does, ...and that is how we find actual area.

Example 34.4. Find the area between the graph of $f(x) = \frac{1}{2}x - 1$ on the interval $[-2, 8]$.
Solution: First we need to find the intervals in $[-2, 8]$ where f is negative and the intervals where f is positive. Since f is continuous, this means that we need to find its roots and then check the

intervals between roots to see if f is positive or negative on each of those intervals.

$f(x) = \frac{1}{2}x - 1 = 0 \Longrightarrow \frac{1}{2}x = 1 \Longrightarrow x = 2$. So, 2 is the only root of f.

$f(x) = \frac{1}{2}x - 1$ is negative on the interval $[-2, 2]$ and f is positive on the interval $[2, 8]$.

So, the area between f and the x-axis is:

$$-\int_{-2}^{2}\left(\frac{1}{2}x - 1\right)dx + \int_{2}^{8}\left(\frac{1}{2}x - 1\right)dx$$

$$-\left(\frac{1}{4}x^2 - x\right)\Big|_{-2}^{2} + \left(\frac{1}{4}x^2 - x\right)\Big|_{2}^{8}$$

$$-\left[\left(\frac{1}{4}\cdot 4 - 2\right) - \left(\frac{1}{4}\cdot 4 + 2\right)\right] + \left(\frac{1}{4}\cdot 64 - 8\right) - \left(\frac{1}{4}\cdot 4 - 2\right)$$

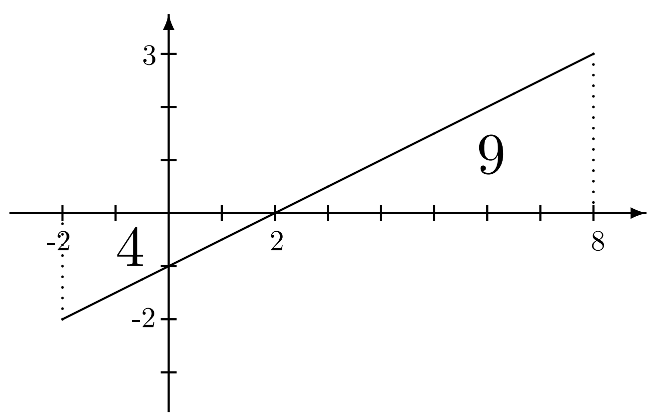
$$-\Big[(1-2) - (1+2)\Big] + (16 - 8) - (1 - 2)$$

$$-[-1-3] + 8 + 1$$

$$-(-4) + 9 = 13$$

So, the area between the graph of $f(x) = \frac{1}{2}x - 1$ and the x-axis on $[-2, 8]$ is 13.

Below is a graph of $f(x) = \frac{1}{2}x - 1$ on the interval $[-2, 8]$. We can use elementary geometry to find the areas of the two triangles formed. The area of the smaller triangle, A_1, is $\frac{1}{2}\cdot 4 \cdot 2 = 4$. The area of the larger triangle, A_2, is $\frac{1}{2}\cdot 6 \cdot 3 = 9$.



$f(x) = \frac{1}{2}x - 1$

Compare the graph to the answer in Example 34.3. In this example we found that the signed area, $\int_{-2}^{8}\left(\frac{1}{2}x - 1\right)dx = 5$. This makes sense since there is an area of 9 above the x-axis and an area of 4 below the x-axis. Remember, signed area counts area below the axis as negative and area above the axis as positive.

Now compare the graph to the answer in Example 34.4. The total area is 13. Look closely in the example at how we arrived at the answer 13. We had two integrals. The integral on $[-2, 2]$

corresponds to the small triangle region. Since the integral was negative (-4) we negated it to get the area 4. The integral on $[2, 8]$ corresponds to the large triangle. Since this is area above the x-axis the integral value (9) was equal to the area.

Example 34.5. Find the value of b such that the area between the graph of $f(x) = \frac{1}{x}$ and the x-axis on the interval $[1, b]$ is 1.
Solution: Since f is positive for all values of x in $[1, b]$, the area between the graph and the x-axis is equal to the definite integral $\displaystyle\int_1^b \frac{dx}{x}$. We evaluate the integral:
$\displaystyle\int_1^b \frac{1}{x}dx = \ln|x|\Big|_1^b = \ln b - \ln 1 = \ln b.$ We want $\ln b = 1$, so $b = e$.

Example 34.6. Given $f(x) = x^2 - 1$ on the interval $[-1, 2]$, find: (a) the signed area, and (b) the actual area between the graph and the x-axis.
Solution (a) $\displaystyle\int_{-1}^{2}(x^2-1)dx = \left(\frac{1}{3}x^3 - x\right)\Big|_{-1}^{2} = \left(\frac{8}{3} - 2\right) - \left(\frac{-1}{3} + 1\right) = \frac{2}{3} - \frac{2}{3} = 0$
Solution (b) $x^2 - 1 = 0$ at $x = \pm 1$. $f < 0$ on $[-1, 1]$ and $f > 0$ on $[1, 2]$. The area, then, is:

$$-\int_{-1}^{1}(x^2-1)dx + \int_{1}^{2}(x^2-1)dx = -\left[\left(\frac{1}{3}x^3 - x\right)\Big|_{-1}^{1}\right] + \left(\frac{1}{3}x^3 - x\right)\Big|_{1}^{2}$$
$$= -\Big[\left(\tfrac{1}{3} - 1\right) - \left(-\tfrac{1}{3} + 1\right)\Big] + \left(\tfrac{8}{3} - 2\right) - \left(\tfrac{1}{3} - 1\right) = \tfrac{8}{3}.$$

Finding the Area Between Two Graphs

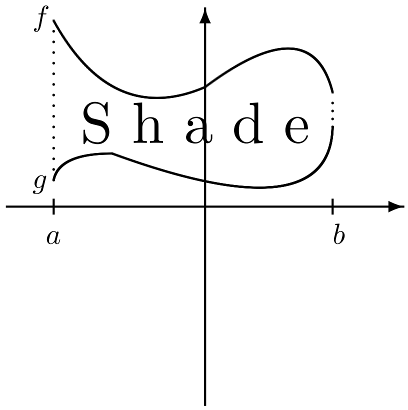
An immediate consequence of the Fundamental Theorem of Calculus is the following corollary:

Corollary 34.2. *If $f(x) \geq g(x)$ for all x in $[a, b]$, and both functions are continuous, then the area between the graph of f and the graph of g over $[a, b]$ is $\displaystyle\int_a^b (f(x) - g(x))dx$.* □

The proof for this corollary is easily done algebraically (just think of $f - g$ as a non-negative function). A graphing approach, however, is convincing and instructive:

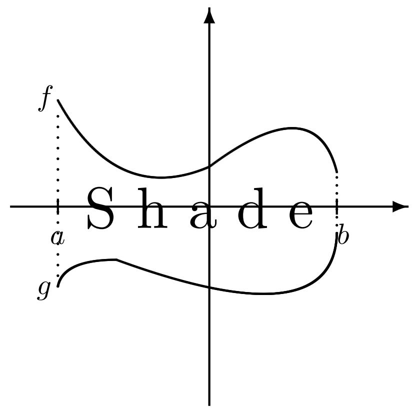
We are given that both f and g are continuous on the interval $[a, b]$. We are also given that $f \geq g$. From a graphing standpoint, this means that the graph of f will always be above (or coinciding) with the graph of g. There are three cases[59]: (1) both f and g are above the x-axis, (2) f is above the x-axis but g is below the x-axis, and (3) both f and g are below the x-axis. Look at each illustration below and figure out how you would use integrals to get the shaded area. In each case, the simplified result will be $\displaystyle\int_a^b (f(x) - g(x))dx$.

[59] Actually, there are many cases since neither f nor g is required to be strictly above or below the x-axis. However, the argument still holds. Interval $[a, b]$ can be broken into sub-intervals, each of which must meet one of the three cases.



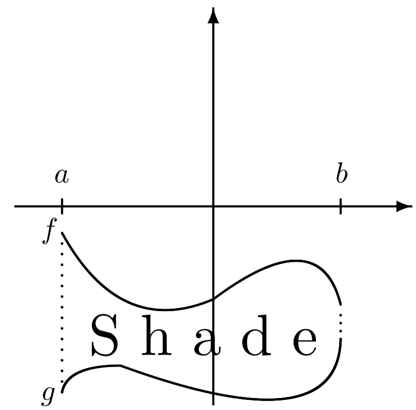
Area under f minus Area under g:

$$\int f - \int g = \int (f-g)$$



Area under f + Area above g:

$$\int f + -\int g = \int (f-g)$$



Area above g minus Area above f:

$$-\int g - -\int f = \int (f-g)$$

This calculation for the area between two graphs is consistent with our previous work finding the area between a graph and the x-axis. In this case we simply have $g(x) = 0$ (or $f(x) = 0$ if the graph is below the x-axis).

In order to use Corollary 34.2 to find the area between two curves, f and g, one must find out whether $f \geq g$ or $g \geq f$ on the desired interval $[a, b]$. It is also possible that the graphs cross each other and so neither graph is consistently above the other. If the graphs cross at some $x = c$ where $a < c < b$, then it is necessary to use two integrals to find the total area. One integral would find the area on the interval $[a, c]$ and the other would find the area on the interval $[c, b]$.

Example 34.7. Find the area between the graphs of $f(x) = 3x^3 + x^2 + 4x + 1$ and $g(x) = 2x^3 + x^2 + 2x + 1$ on the interval $[1, 4]$.

Solution: We first check to see if these graphs intersect somewhere in the interval $[1, 4]$ by setting them equal to each other and solving:

$3x^3 + x^2 + 4x + 1 = 2x^3 + x^2 + 2x + 1 \Longrightarrow x^3 + 2x = 0 \Longrightarrow x(x^2 + 2) = 0 \Longrightarrow x = 0.$

So, the graphs do not intersect over the interval $[1, 4]$. Since $f(1) = 9$ and $g(1) = 6$, we know that $f > g$ on $[1, 4]$

Thus, the area between the curves is $\displaystyle\int_1^4 [(3x^3 + x^2 + 4x + 1) - (2x^3 + x^2 + 2x + 1)]dx = \int_1^4 (x^3 + 2x)dx = \left(\frac{1}{4}x^4 + x^2\right)\Bigg|_1^4 = \left((64 + 16) - \left(\frac{1}{4} + 1\right)\right) = 80 - \frac{5}{4} = 78\frac{3}{4}$.

Suppose the interval given in Example 34.7 was $[-2, 4]$ instead of $[1, 4]$? We will use the known information from Example 34.7 to solve Example 34.8.

Example 34.8. Find the area between the graphs of $f(x) = 3x^3 + x^2 + 4x + 1$ and $g(x) = 2x^3 + x^2 + 2x + 1$ on the interval $[-2, 4]$.

Solution: We know that the graphs intersect at $x = 0$, so we need to break our interval into two pieces, $[-2, 0]$ and $[0, 4]$. We know that $f > g$ on $[0, 4]$. Since $f(-1) = -5$ and $g(-1) = -2$, we know $g > f$ on interval $[-2, 0]$.[60]

Thus, the area between the graphs on $[-2, 4]$ is the sum $\displaystyle\int_{-2}^{0}(g - f)dx + \int_{0}^{4}(f - g)dx$.

$$\int_{-2}^{0}(-x^3 - 2x)dx + \int_{0}^{4}(x^3 + 2x)dx = \left(-\frac{1}{4}x^4 - x^2\right)\Bigg|_{-2}^{0} + \left(\frac{1}{4}x^4 + x^2\right)\Bigg|_{0}^{4}$$

$$= [0 - (-4 - 4)] + [(64 + 16) - 0] = 88.$$

In the next example we are not specifically given an interval.

Example 34.9. Find the area *enclosed* by the graphs of $f(x) = x^3$ and $g(x) = \sqrt{x}$.

Solution: We look for where the graphs intersect:

$x^3 = \sqrt{x} \Longrightarrow x^6 = x \Longrightarrow x^6 - x = 0 \Longrightarrow x(x^5 - 1) = 0 \Longrightarrow x = 0$ or $x = 1$.

Since f and g intersect at $x = 0$ and at $x = 1$, the region that they enclose is on this interval.

For x values in $(0, 1)$, $x^3 < \sqrt{x}$, so we have $g > f$.

The area is $\displaystyle\int_{0}^{1}(x^{\frac{1}{2}} - x^3)dx = \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}x^4\right)\Bigg|_{0}^{1} = \left(\frac{2}{3} - \frac{1}{4}\right) - 0 = \frac{5}{12}$.

[60] Note that we cannot simply assume that f and g switch positions. It is possible for two graphs to "touch" but not "cross" each other.

Section 34 - Exercises (answers follow)

1. Find the signed area between the x-axis and $f(x)$ over the given interval.

   (a) $f(x) = -x^2$; $[-1, 2]$

   (b) $f(x) = 3 - x^2$; $[0, 1]$

   (c) $f(x) = 2x + 1$; $[9, 10]$

   (d) $f(x) = x^3$; $[-2, 4]$

   (e) $f(x) = e^x$; $[-5, 1]$

   (f) $f(x) = \frac{1}{x}$; $[1, e^2]$

2. Evaluate the definite integral $\displaystyle\int_{-2}^{4} |x|\, dx$ in two ways: (a) Draw the graph of $y = |x|$ on the interval $[-2, 4]$ and find the appropriate area geometrically, and (b) integrate and evaluate.

3. Sketch a graph of $f(x) = x + 3$. Use geometry to calculate the area between f and the x-axis on the interval $[1, 4]$. Then calculate the area using integration. Your answers should be the same.

4. Find the area between the graph of $f(x) = x^3 - 4x$ and the x-axis on the interval $[-1, 2]$.

5. Find the area between the graphs of $f(x) = x^2 - 4x + 4$ and $g(x) = x^2$ on the interval $[0, 3]$.

6. In each of the following find the area enclosed by the curves.

   (a) $y = 1 - x^2$ and the x axis.

   (b) $x = -3$, $x = 1$, $y = x^2 + 2$, $y = 0$

   (c) $y = \sqrt{x}$ and the lines $y = 2 - x$ and $y = 0$.

   (d) $f(x) = 2 - x^2$, $g(x) = x$

   (e) $f(x) = x^2$ and $g(x) = x^3$

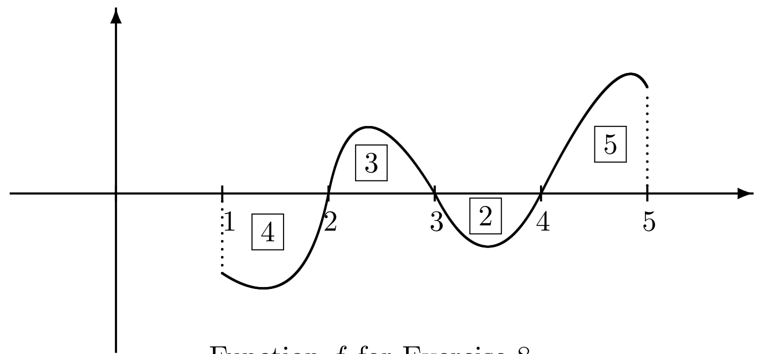
   (f) $y = x^3 + x^2 - x + 2$, $y = x^2 + x + 2$

   (g) $f(x) = \sqrt{x}$ and $g(x) = x^2$

   (h) $f(x) = 8x$, $g(x) = x$ and $h(x) = \dfrac{8}{x^2}$ (It will be helpful here to sketch the graphs so you can see the enclosed region.)

7. The area A under a *parabolic arch* can be calculated by $A = \frac{2}{3}bh$, where b and h are the base and height of the arch. Sketch a graph of parabola $f(x) = -x^2 + 8x - 7$. Then find the area enclosed by the arch and the x-axis two ways: (a) using the formula and (b) using calculus.

8. Function f is defined on the interval $[1,5]$. Its graph is below. The numbers in the boxes represent the AREAs between the graph and the x-axis on their respective unit intervals. For example, the area between the graph and the x-axis in the interval $[1,2]$ is four. Use this information to evaluate the six integrals. (The areas are not to scale.)

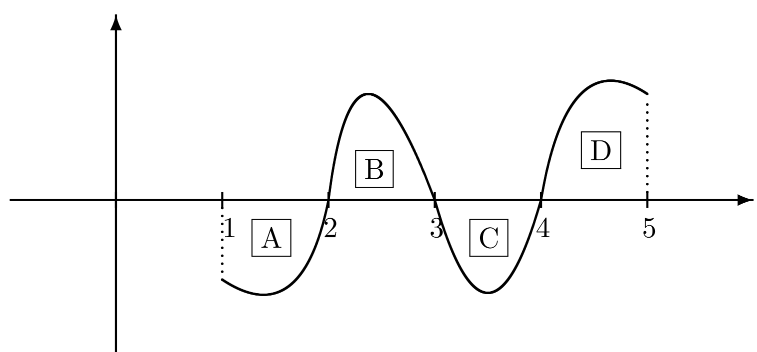


Function f for Exercise 8

$$\int_1^5 f(x)dx \qquad \int_1^4 |f(x)|dx \qquad \int_1^3 f(x)dx \qquad \int_3^1 f(x)dx \qquad \int_2^2 f(x)dx \qquad \left|\int_3^5 f(x)dx\right|$$

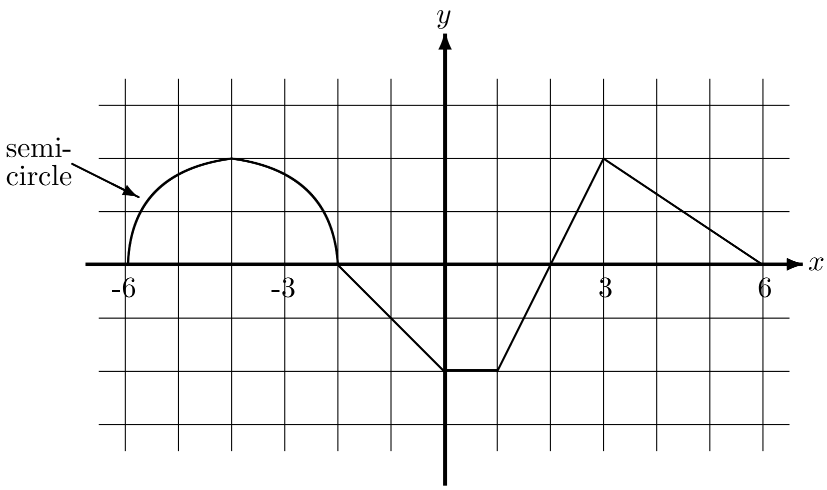
9. Consider the graph of g below. The graph is not drawn to a consistent scale but the x-intercepts are valid as marked. Suppose that the following integral values all apply to the function g: $\int_2^3 g(x)dx = 5 \qquad \int_4^2 g(x)dx = -2 \qquad \int_3^5 g(x)dx = 4 \qquad \int_1^5 g(x)dx = 0$.

Use the integral information to find the values for the areas A, B, C, and D between the graph of g and the x-axis on their respective intervals.



Function g for Exercise 9

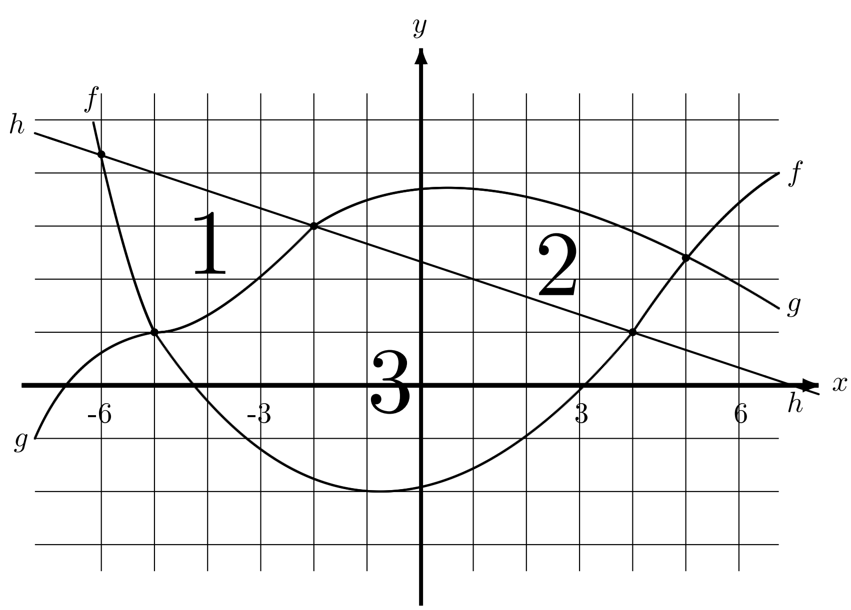
10. Use the graph of f given below, along with basic geometry area formulas, to evaluate the integrals that follow.



Function f for Exercise 10

$$\int_{-6}^{-2} f(x)dx \qquad \int_{-4}^{0} f(x)dx \qquad \int_{0}^{3} |f(x)|dx \qquad \int_{1}^{6} f(x)dx \qquad \left|\int_{0}^{3} f(x)dx\right| \qquad \int_{6}^{2} f(x)dx$$

11. Consider the graphs of f, g and h below. Their intersections create three enclosed regions (numbered 1, 2 and 3). For each region, write an expression using integrals in terms of f, g, and h, that gives the area of that region. Your expressions should contain no absolute value signs.



Exercise 11

Section 34 - Answers

1. (a) -3 (b) $\frac{8}{3}$ (c) 20 (d) 60 (e) $e - e^{-5}$ (f) 2
2. 10
3. $16\frac{1}{2}$
4. $5\frac{3}{4}$
5. 10
6. (a) $\frac{4}{3}$ (b) $17\frac{1}{3}$ (c) $\frac{7}{6}$ (d) $4\frac{1}{2}$ (e) $\frac{1}{12}$ (f) 2 (g) $\frac{1}{3}$ (h) 6
7. 36
8. 2, 9, -1, 1, 0, 3
9. A=9, B=5, C=3, D=7
10. 2π, $\pi - 2$, 4, 3, 2, -4,
11. Region 1 area $= \displaystyle\int_{-6}^{-5} [h(x) - f(x)]dx + \int_{-5}^{-2} [h(x) - g(x)]dx$

    Region 2 area $= \displaystyle\int_{-2}^{4} [g(x) - h(x)]dx + \int_{4}^{5} [g(x) - f(x)]dx$

    Region 3 area $= \displaystyle\int_{-5}^{-2} [g(x) - f(x)]dx + \int_{-2}^{4} [h(x) - f(x)]dx$

35 Interpretation of the Definite Integral as a Limit of Sums

In Section 7 we interpreted the derivative as the slope of a tangent to a graph. This was intended to help your understanding, but the really important interpretation of the derivative as the rate of change came in Section 8. Similarly, in Section 34 we interpreted the definite integral as signed area between a graph and the horizontal axis. Here we give (at least the beginnings of) a more important interpretation of the definite integral.

Again, f is a continuous function whose domain includes $[a, b]$. Fix a (large) positive integer N and write $\Delta x = \frac{1}{N}(b - a)$. We are "partitioning" the closed interval $[a, b]$ into N equal pieces each of length Δx. Name the partition points $a = x_0, x_1, \cdots, x_{N-1}, x_N = b$. Then for any i, $x_i - x_{i-1} = \Delta x$.

If $f(x_1) \geq 0$ then the number $f(x_1)\Delta x$ is the area of the rectangle of height $f(x_1)$ and width Δx. If $f(x_1) < 0$ then $f(x_1)\Delta x = -$ (that area). Similarly, for $f(x_2)\Delta x, f(x_3)\Delta x, \cdots, f(x_N)\Delta x$. So the sum

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_N)\Delta x \tag{35.1}$$

is an approximation to the signed area $\displaystyle\int_a^b f(x)dx$. The larger N is the better the approximation. People abbreviate [61] sum 35.1 as $\displaystyle\sum_{i=1}^{N} f(x_i)\Delta x$. This sum is called a *Riemann sum* for $\displaystyle\int_a^b f(x)dx$.

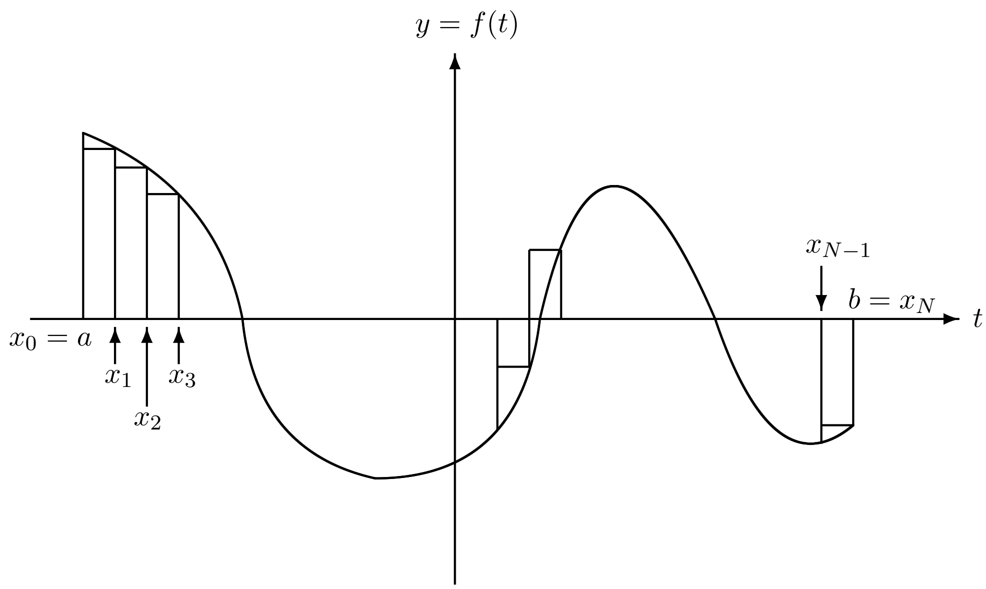


Illustration of *Riemann Sum*
Using Intervals of Width $\Delta x = \frac{b-a}{N}$ and Right-hand Endpoints

[61] $\sum$ means "sum", $\displaystyle\sum_{i=1}^{N}$ means "sum up letting i take all integer values from 1 through N inclusive, $f(x_i)\Delta x$ is the i^{th} signed area, and so $\displaystyle\sum_{i=1}^{N} f(x_i)\Delta x$ is the sum of all the signed areas.

Again, as our number of partitions N gets larger and larger, the Riemann sum gets closer and closer to the signed area of f over interval $[a, b]$. By now you know that "larger and larger" and "closer and closer" leads to the word "limit." Indeed we have the powerful theorem:

Theorem 35.1. *When f is continuous on the closed interval $[a, b]$ then*

$$\int_a^b f(x)dx = \lim_{N\to\infty}\left(\sum_{i=1}^{N} f(x_i)\Delta x\right).$$

We discuss two applications of this.

Average Value of f on $[a, b]$

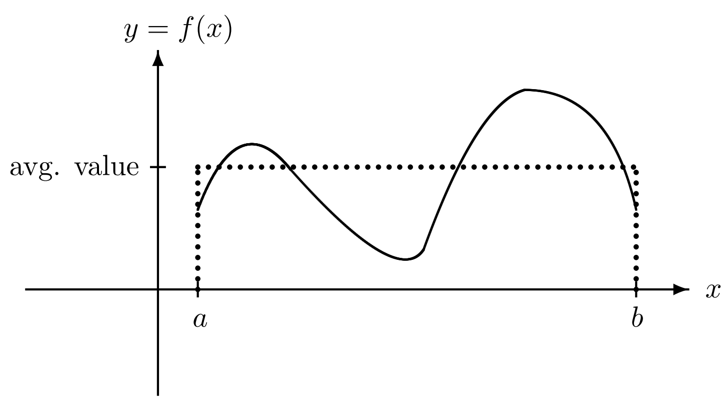
Suppose we were interested in knowing the average y-value of a function f over the interval $[a, b]$. We can't add up all of the y-values and divide by infinity. So, again, we will incorporate a limit. Specifically we will incorporate the limit in Theorem 35.1.

The average of the N y-values $f(x_1), \cdots, f(x_N)$ is

$$\begin{aligned}\frac{1}{N}(f(x_1) + \cdots + f(x_N)) &= \frac{1}{N\Delta x}(f(x_1)\Delta x + \cdots + f(x_N)\Delta x)\\ &= \frac{1}{b-a}(f(x_1)\Delta x + \cdots + f(x_N)\Delta x)\end{aligned}$$

which is an approximation to $\frac{1}{b-a}\int_a^b f(x)dx$. This motivates our defining *the average value of $f(x)$ on $[a, b]$* to be $\frac{1}{b-a}\int_a^b f(x)dx$.

Another way to think of this is to think of the graph of some function f as the side view of a wave of water in a tank. If you wanted to know the average depth of the water, you would wait until the water was still. The high wave crests would settle into the wave valleys and you would be looking at a horizontal line. The height of that line would be average depth of the water. Notice in the picture below that the average value is not simply $\frac{f(b)+f(a)}{2}$.



Average Value of f over $[a, b]$

The illustration above is consistent with our study of area in Section 34. If we think of the waves settling into level water at the average value, the area of the rectangle (formed by the dotted lines and the x-axis) must be the same as the area between f and the x-axis because they both represent the same amount of water. We know from rectangle geometry that the height of the rectangle is the area divided by the width, or $\frac{\text{Area}}{b-a}$. We also know that the area is $\int_a^b f(x)dx$. So, this gives us the height of the rectangle, which is the average value of the function, to be $\frac{1}{b-a}\int_a^b f(x)dx$.

While this illustration speaks of area, the same argument could be given for signed area by simply rescaling the y-axis.

Example 35.1. Find the average value of the function $f(x) = x^2$ on the interval $[0, 2]$.
Solution: Average Value is $\frac{1}{b-a}\int_a^b f(x)dx$, so for our problem we get:

$$\text{Average Value} = \frac{1}{2-0}\int_0^2 x^2\,dx = \frac{1}{2}\left(\frac{1}{3}x^3\Big|_0^2\right) = \frac{1}{2}\left(\frac{8}{3} - 0\right) = \frac{4}{3}.$$

Example 35.2. Refer to the dragster in Example 32.6 on page 253. What is the average velocity of the car during the first ten seconds of its run?
Solution: The average value is $\frac{1}{10-0}\int_0^{10} v(t)dt$. We found earlier that $\int v(t)dt$ is
$s(t) = -500e^{-0.2t}(t+5) + 2500$.

Again using the calculations done previously, the average velocity over the first ten seconds is $\frac{1}{10}(s(10) - s(0)) \approx 148.5$ ft/sec.

Does it make sense in the last example that the average velocity of the car is the total distance traveled divided by the time? While not generally true, it is true here because the car is always traveling in the same direction (so the velocity is never negative). So, here we do get average velocity is $\frac{s(10)-s(0)}{10}$. Average velocity is NOT $\frac{v(10)+v(0)}{2}$.

Present Value of an Investment:

This would be a good time to reread Section 5, especially the part entitled "Compounding Continuously." There it is explained that, with continuous compounding, \$$P$ invested at rate r (6% means $r = .06$) for t years will become \$$Pe^{rt}$ at the end of t years. More generally, t doesn't have to be a whole number. If you invest \$$P$ at time 0 at the rate r with continuous compounding then at time t you will own \$$Pe^{rt}$.

At the end of the "Compounding Continuously" section we expressed the relationship between *present value* and *future value* of investments (Formulas 5.4 on page 53). In these formulas, and in all of Section 5, we equated "present value" with "principal." We will now look at a more complex financial scenario, where "principal" (a fixed amount of money invested up-front) is not involved. However, we will still keep the concept of *present value*. If $P(t)$ represents the value of the account at time t, then $P(0)$ (when $t = 0$) is the present value, and $P(0) = P(t)e^{-rt}$. The present value of an account is the value of an account now that will be worth $P(t)$, t years from now. This is consistent with our study in Section 5, but is more general in scope.

Suppose you inherit a hotel. A manager takes care of income and expenses: your only task is to handle the profits (= income minus expenses) which flow into your bank account continuously (daily in real life but as usual we smooth things out - see Section 8). Your ownership starts at time $t = 0$. Let's give the name $F(t)$ to the amount of money that flows into your account between time 0 and time t. Then F is a function of t. We'll assume it's a differentiable function with derivative $f(t)$. This function $f(t)$ is called the *continuous money flow*. We interpret $f(t)$ as the rate of change of your profit per unit time [62] because

$$f(t) = F'(t) = \lim_{h \to 0} \frac{F(t+h) - F(t)}{h}.$$

and the fraction $\frac{F(t+h)-F(t)}{h}$ measures the amount of income received over the time interval $[t, t+h]$, divided by length of that time interval $h > 0$. In real life you often know a formula for $f(t)$ explicitly, or one which is a good enough approximation. Then the function F, the anti-derivative of f, is given by:

$$F(T) = \int_0^T f(t)dt$$

$F(T)$ is the *total money flow* over the time interval $[0, T]$. At time T you would accumulate this number of dollars if the bank account pays no interest.

More realistically, we will now assume your account pays interest compounded continuously at the rate r. But remember: money only starts earning interest once it reaches the account, so not only is the amount of money in the account changing all the time but the amount on which interest is being calculated is changing all the time. We ask: Under this set-up how much money will be in your account at the future time T? With integral calculus we can handle this complication.

The way to handle it is to approximate the situation and then see what happens as the approximation gets better and better. Let N be a large integer, let $\Delta t = T/N$, let $t_i = i \cdot \Delta t$ where $i = 0, 1, 2, \cdots, N$. During the time interval $[t_0, t_1]$ you take in $F(t_1) - F(t_0)$ dollars; during the time interval $[t_1, t_2]$ you take in $F(t_2) - F(t_1)$ more dollars; and so on. It is easiest to avoid repetition and talk about $[t_{i-1}, t_i]$ for a general $i = 1, 2, \ldots, N$. During the time interval $[t_{i-1}, t_i]$ the number of dollars you take in is

$$F(t_i) - F(t_{i-1}) = \left(\frac{F(t_i) - F(t_{i-1})}{\Delta t} \right) \cdot \Delta t$$

If N is large (so that Δt is small) then $\frac{F(t_i)-F(t_{i-1})}{\Delta t}$ is approximately $f(t_i)$ so approximately $f(t_i)\Delta t$ dollars arrive in the account at approximately time t_i.

Money that arrives in the account at approximately time t_i will earn interest only during the time interval from t_i to T. This is $T - t_i$ units of time. So the future value (the value at time T) of the money that arrived at approximately time t_i is $f(t_i)e^{r(T-t_i)}\Delta t$.

The total amount of money which will be in the account at time T is obtained by adding all these terms, one for each i, to get $\displaystyle\sum_{i=1}^{N} f(t_i)e^{r(T-t_i)}\Delta t$. The larger N is, the closer the approximation

[62] "Unit time" might be one year or one day or one second (dollars per year or per day or per second). Assume a unit of time has been picked for this discussion.

gets to the actual amount expected in your account at time T, so, by Theorem 35.1, the precise *future value* of your account at time T is

$$P(T) = \int_0^T f(t)e^{r(T-t)}dt = e^{rT}\int_0^T f(t)e^{-rt}dt$$

(We can pull e^{rT} outside the integral because it is constant with respect to the variable t.) This number $e^{rT}\int_0^T f(t)e^{-rt}dt$ is also called the *accumulated amount of money flow* over the time interval $[0, T]$. It is what you will have at time T if you let the profits from your hotel flow into your interest bearing account where they start earning interest on arrival.

So how much is your inheritance worth the day you inherit it? Let's suppose the effective life of the hotel is T units of time. You ask, "For how much should I sell the hotel now (at time 0) so that if I invest the proceeds of the sale in that same bank account I'll end up with the same amount as the accumulated money flow?" You get the answer from $P(0) = P(t)e^{-rt}$, seen earlier. The present value is e^{-rT} times the future value at time T. This is $e^{-rT} \cdot e^{rT}\int_0^T f(t)e^{-rt}dt$ which simplifies to just $\int_0^T f(t)e^{-rt}dt$. If you sell the hotel now (at time $t = 0$) for this price and invest the proceeds in the account immediately you will end up with the same amount of money at time T as if you had kept the hotel and let the money flow in gradually. This quantity $\int_0^T f(t)e^{-rt}dt$ is the *present value* of the income flow. If you can sell it now for more, you make a profit. If you sell for less you lose. Once you know the function $f(t)$ you can evaluate this integral and get the actual worth of what you inherited.

Example 35.3. Suppose you have the hotel described above and you know that the hotel will generate \$10,000 per year, with income arriving at a constant rate throughout the year. You also know that the hotel is viable for only nine years. You have a bank account where you can invest at an annual interest rate of 7% compounded continuously. You are offered \$70,000 for the hotel today. Should you sell? How much money would you earn over nine years if you do sell? if you don't sell?

Solution: To answer the first question, we need to find the present value of the hotel and compare it to the offering price of \$70,000. Our income flow is the constant function $f(t) = 10,000$. The interest rate is .07 and the length of time of the investment is $T = 9$ years. So, the present value of the hotel is: $\int_0^9 10,000e^{-.07t}dt$. We evaluate this:[63]

$10,000 \cdot \left.\dfrac{e^{-.07t}}{-.07}\right|_0^9 = \dfrac{1,000,000}{-7}\left(e^{-.63} - e^0\right) = \dfrac{1,000,000}{7}\left(1 - \frac{1}{e^{.63}}\right) \approx \$66,772.60$. You should sell your hotel for \$70,000.

To answer the second question, we are looking for the future value of principal \$70,000, invested

[63]It is useful to remember the integral from Section 31: $\int e^{kx}dx = \frac{1}{k}e^{kx} + C$ for any real number $k \neq 0$.

at an interest rate of 7% compounded continuously for nine years. We know from Section 5 that this is calulated by the formula: Pe^{rt}.

$70,000e^{(.07)(9)} = 70,000e^{.63} =\approx \$131,432.74.$

To answer the third question we can use the formula for future value of money flow: $e^{rT}\int_0^T f(t)e^{-rt}dt = e^{(.07)(9)}\int_0^9 10,000e^{-.07t}dt.$

Of course we already know the value of the integral to be \$ 66,772.60, so this calculation becomes $e^{.63}(66,772.60) \approx \$123,372.94.$

If you look carefully at Example 35.3 you can get some clearer understanding of *present value* for money flow situations. The present value of \$66,772.60 that was calculated from the money flow scenario (no money up front; money comes in steadily over nine years) represents the amount of money that *would* be needed up front (i.e., money in the form of principal) in order to arrive at a future value of \$123,372.94, the money that would be gotten by keeping the hotel.

Finally, we take a moment to compare the formulas for future value. For a money flow situation we have $P(T) = e^{rT}\int_0^T f(t)e^{-rt}dt$. When we have principal $P(0)$ (up-front money...money all available now at $t = 0$) we have $P(T) = P(0)e^{rT}$. Both of these formulas calculate $P(T)$, the value of the investment at time T. Both of these formulas have multipliers of e^{rT}. They differ in that one has an integral expression and the other has $P(0)$. Both of these quantities, $\int_0^T f(t)e^{-rt}dt$ and $P(0)$, then take on the same role. That is the role of *present value*.

Example 35.4. Sam has an investment that will produce money at the rate of (300+2t) dollars per year for 8 years. (a) What will be the value of his account in 8 years if the money is invested at an annual 5% interest rate? (b) Ross uses the same bank that Sam does, so he gets the same interest rate. How much money should Ross deposit today if he wants in eight years to have the same account balance as Sam?

Solution (a) We are looking for the future value of Sam's money flow investment:

$F(8) = e^{(.05)(8)}\int(300 + 2t)e^{-.05t}dt$. Using Integration by Parts and evaluating the integral[64] we get $F(8) \approx e^{0.4}(1759.19) \approx \$2,624.41.$

Solution (b) Ross needs to know the present value of Sam's investment. This is the integral part of the calculation in part (a). Ross needs \$1,759.19

[64] Details are left to the reader

Section 35 - Exercises (answers follow)

1. Find the average value of each function on the given interval.

   (a) $f(x) = x^3$ over $[0, 2]$

   (b) $f(x) = x^2 - 3$; $[1, 7]$

   (c) $f(x) = x^3 - x$ over $[0, 2]$

   (d) $f(x) = \sqrt{x+1}$; $[1, 2]$

   (e) $f(x) = e^{-x}$ over $[0, 2]$

2. The price of a new electronic device over its first five years on the market is given by the function $P(t) = -4t^3 + 20t + 400$. What is the average price of a new device over its first 5 years in existence?

3. The rate of consumption of hamburgers in Wyoming (in millions of hamburgers per year) since 1960 is given approximately by the function $H(t) = \frac{1}{10}t + \frac{12}{5}$ where $t = 0$ corresponds to 1960. Determine the average number of hamburgers per year eaten in Wyoming during the ten years 1970 to 1980.[65]

4. Find the average value of each of the following functions over the interval $[1, 5]$:

   (a) Function f from Section 34, exercise 8.

   (b) Function g from Section 34, exercise 9.

   (c) Function f from Section 34, exercise 10.

5. A package of frozen blueberries is taken from a freezer at $-5°C$ into a room at $20°C$. At time t, the temperature of the blueberries is increasing at the rate of $(10e^{-0.4t})°C$ per hour. Find the temperature of the blueberries after 15 minutes ($\frac{1}{4}$ hour).

6. The "Can You Dig It" backhoe company has installed a new assembly line for their latest model of energy efficient machines. They expect to produce backhoes at the rate of $30\sqrt{t}$ machines/week at the end of t weeks. (a) How many backhoes do they expect to produce during the first 36 weeks of production? (b) What is the average number of backhoes they expect to produce each week during this 36 week time period?

7. The marginal cost (\$) to produce x million paper clips is $C'(x) = 12x + 20$. Find the cost of increasing production from 5 million clips to 10 million clips.

8. The value of an investment fund share has changed at the rate of $\dfrac{dV}{dt} = 3\sqrt{t}$ dollars/year where t represents the number of years since the fund was created. One year after the fund was created, Basil bought one share of the fund for \$20. Eight years after that, he sold it. (a) How much was Basil's share worth when he sold it? (b) What was the average value of one share of the fund over the eight years that Basil held his share?

[65] By the way, don't go quoting these statistics, or those for any other problem. I just make most of them up.

9. A particle moves back and forth along a straight line with a velocity of $v(t) = 1 - t^2$ miles/hour. (a) How far is the particle from its original position 2 hours later? (b) What is the total distance traveled by the particle during the first two hours?

10. A study of the births and deaths of bears in Jellystone Park determined that the bear population of the park will grow at the rate of $45\sqrt{t} + 10$ bears/year t years from now. (a) At what rate will the bear population be growing one year from now? Put units of measure on your answer. (b) How many *more* bears will there be in the park nine years from now?

11. Each of the functions represents the rate of flow of money in dollars per year. Assume a 10-year period at 12% compounded continuously and find each of the following: (1) the present value (2) the accumulated amount at $t = 10$.

    Note: Answers for these next problems may vary slightly due to rounding. Do not round too much early in the process of evaluation. By all means, use a calculator for these exercises, although on a test you will not be required to do complex arithmetic evaluations.

    (a) $f(t) = 40,000 + 2000t$

    (b) $f(t) = 5000e^{.01t}$

    (c) $f(t) = 20,000e^{.05t}$

12. Money is transferred continuously into an account at the constant rate of \$4,000 per year. The account earns interest at the annual rate of 7 percent compounded continuously. How much will be in the account at the end of 6 years?

13. A real estate investment is expected to produce a uniform continuous rate of money flow of \$5000 per year for 10 years. Find the present value at each of the following rates, compounded continuously. (a) 12% (b) 10% (c) 15%.

14. An investment is expected to generate income at the rate of $f(t) = 250,000$ dollars per year for the next six years. Find the present value of this investment if the prevailing interest rate is 7% per year compounded continuously.

15. Kathleen has inherited some money from her great great aunt Zelda. For the next ten years, \$4,000 per year will flow at a constant rate into a special account. The account earns an annual interest of 6% compounded continuously. How much *interest* will Kathleen earn from the account during the first six *months*?

16. A benefactor of the university wishes to start an endowment fund on the condition that he remain anonymous until the fund reaches \$200,000. As sole contributor to the fund, he will donate \$10,000 per year at a continuous uniform rate. The money will be invested in an account that offers 5% interest, compounded continuously. In how many years will the donor's identity become known?

17. Your grandfather owns a bank. He is excited about your attending college and has already decided on a graduation gift for you!

    During the four years that you are at school he is setting aside \$1,200 annually at a constant rate. the money will be invested in an account that compounds continuously.

    Just for fun (and perhaps motivation?) Granddad will calculate your total gift using an interest rate that is equaal to your GPA (grade point average) at graduation. (So, if you graduate with a GPA of 4.0 your grandfather will use an interest rate of 4% for each of the four years. If you graduate with a 2.7 GPA, your interest rate will be 2.7% for each of the four years, etc.)

    You are interested in calculating the amount of your gift, so you set up an appropriate expresssion and simplify. You let G be your GPA. Show that the amount of your gift can be figured by the expression $\dfrac{120,000}{G}\left(e^{\frac{G}{25}}-1\right)$.

18. Congratulations! You have won second prize in a beauty contest. You are offered a choice between two prizes. Which will you take if you know that you can invest your winnings in a savings account that pays 4% compounded continuously?

    Prize A: \$60 per year, paid at a continuous rate, for 25 years!

    Prize B: \$1,300 cash right now!

Section 35 - Answers

1. (a) 2, (b) 16, (c) 1, (d) $2\sqrt{3}-\frac{4}{3}\sqrt{2}$, (e) $\dfrac{1-e^{-2}}{2}$
2. \$325
3. 3.9 million
4. (a) $\frac{1}{2}$, (b) 0, (c) $\frac{8}{15}$
5. $-25e^{-.1}+20\approx -2.62°C$
6. (a) 4,320 (b) 120
7. \$550
8. (a) \$72 (b) \$42.20
9. (a) $\frac{2}{3}$ miles (b) 2 miles
10. (a) 55 bears/year (b) 900 more bears
11. Note: Answers for these next problems may vary slightly due to rounding.
    (a) (1) $\$279,793$ (2) $\$928,945$
    (b) (1) $\$30,324$ (2) $\$100,679$
    (c) (1) $\$143,833$ (2) $\$477,542$

12. \$29,826

13. (a) \$29,117 (b) \$31,606 (c) \$25,896

14. \$1,224,833

15. $\dfrac{2000}{-.03}(1-e^{.03})-2000 \approx \$30.30$

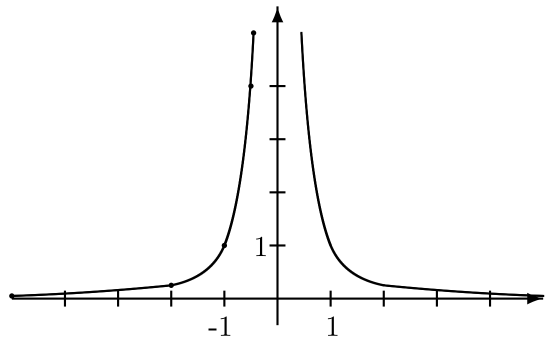
16. $\frac{\ln 2}{.05} = 20\ln 2 \approx 13.86$ years

17. Hint: Your interest rate will be $\frac{G}{100}$

18. Prize B. The present value of prize A is only \$948.18.

36 Improper Integrals

Consider the graph of $f(x) = \dfrac{1}{x^2}$ below.



$f(x) = \frac{1}{x^2}$

What is the area of the region between the graph and the x-axis on the interval $[1, 2]$? From what we learned in Section 34, this is an easy integral:

$$\int_1^2 \frac{dx}{x^2} = \int_1^2 x^{-2}dx = \left.\frac{x^{-1}}{-1}\right|_1^2 = -\left.\frac{1}{x}\right|_1^2 = -\frac{1}{2} + 1 = \frac{1}{2}$$

What is the area of the region between the graph and the x-axis on the interval $[1, 3]$?

$$\int_1^3 \frac{dx}{x^2} = \int_1^3 x^{-2}dx = \left.\frac{x^{-1}}{-1}\right|_1^3 = -\left.\frac{1}{x}\right|_1^3 = -\frac{1}{3} + 1 = \frac{2}{3}$$

What is the area of the region between the graph and the x-axis on the interval $[1, 4]$?

$$\int_1^4 \frac{dx}{x^2} = \int_1^4 x^{-2}dx = \left.\frac{x^{-1}}{-1}\right|_1^4 = -\left.\frac{1}{x}\right|_1^4 = -\frac{1}{4} + 1 = \frac{3}{4}$$

You are perhaps seeing a pattern here. What would the area be on the interval $[1, 10]$?

$$\int_1^{10} \frac{dx}{x^2} = \int_1^{10} x^{-2}dx = \left.\frac{x^{-1}}{-1}\right|_1^{10} = -\left.\frac{1}{x}\right|_1^{10} = -\frac{1}{10} + 1 = \frac{9}{10}$$

Suppose we choose interval $[1, b]$ for any real number $b \geq 1$? You can already guess that the area is $\dfrac{b-1}{b}$, but we can write it out:

$$\int_1^b \frac{dx}{x^2} = \int_1^b x^{-2}dx = \left.\frac{x^{-1}}{-1}\right|_1^b = -\left.\frac{1}{x}\right|_1^b = -\frac{1}{b} + 1 = -\frac{1}{b} + \frac{b}{b} = \frac{b-1}{b}$$

Suppose we look at larger and larger b values. Geometrically, we are looking at more and more area as the graph continues to the right and approaches the x-axis asymptote. As b gets larger and larger, the area under the curve on the interval $[1, b]$ gets closer and closer to 1. We can use a limit to say that as $b \to \infty$, the area, approaches 1: $\displaystyle\lim_{b\to\infty} \frac{b-1}{b} = 1$. This means that the area of the infinitely long region over $[1, \infty)$ is in fact finite. We express this with the integral $\displaystyle\int_1^\infty \frac{1}{x^2}dx = 1$.

An integral with an infinite limit of integration is called an *improper integral.* We cannot plug ∞ into our antiderivative as we do with definite integrals. We accomplish the evaluation, as we have done many times when dealing with infinity, by using a limit.[66]

$$\int_a^{\infty} f(x)\ dx = \lim_{b\to\infty}\int_a^b f(x)\ dx = \lim_{b\to\infty}[F(b) - F(a)] \tag{36.1}$$

In our discussion of the area between the graph of $f(x) = \frac{1}{x^2}$ and the x-axis we rightly used an integral because f is strictly positive. The improper integral expression (36.1) above, however, applies to integrals of any continuous functions. When f is not strictly positive on the interval $[a, \infty)$ the improper integral will yield the *signed area*, consistent with our study of Section 34.

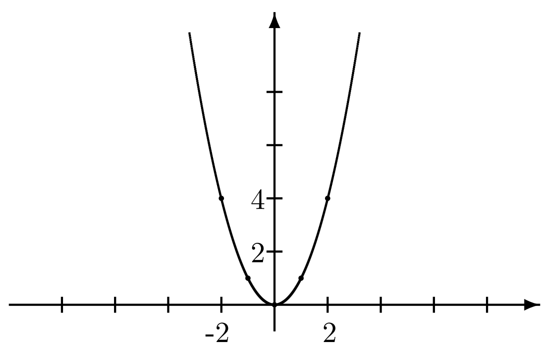
Look back at the graph of $f(x) = \frac{1}{x^2}$. We could have worked on the left half (where $x < 0$) of the graph. The symmetry of the graph convinces us that the area between the graph and the x-axis on the interval $(-\infty, -1]$ is 1. A left side improper integral and its limit definition is:

$$\int_{-\infty}^{b} f(x)\ dx = \lim_{a\to-\infty}\int_a^b f(x)\ dx = \lim_{a\to-\infty}[F(b) - F(a)] \tag{36.2}$$

We know that when we evaluate a definite integral we always get a number. Is this true of improper integrals? Consider the function $f(x) = x^2$ on the interval $[2, \infty)$.

$$\int_2^{\infty} x^2 dx = \lim_{b\to\infty}\int_2^b x^2 dx = \lim_{b\to\infty}\left(\frac{1}{3}x^3\Big|_2^b\right) = \lim_{b\to\infty}\left(\frac{b^3}{3} - \frac{8}{3}\right) = \infty.$$

If we look at a graph of $f(x) = x^2$ we wouldn't expect $\int_2^{\infty} x^2 dx$ to be finite.



$f(x) = x^2$

An improper integral whose limit is finite is said to *converge* to its finite limit. Otherwise, the improper integral is said to *diverge.*

$\int_1^{\infty} \frac{1}{x^2} dx$ converges to 1. $\quad \int_2^{\infty} x^2 dx$ diverges.

Divergent improper integrals are not useful but convergent ones are.

[66]If you need to review limits to infinity, look back at Section 20.

Refer to the graph of $fx) = \frac{1}{x}$ on page 20. Do you think that $\int_5^\infty \frac{1}{x} dx$ is convergent or divergent? Let's evaluate the integral and see.

$$\int_5^\infty \frac{1}{x}\ dx = \lim_{b\to\infty} \int_5^b \frac{1}{x}\ dx = \lim_{b\to\infty} \left(\ln|x| \Big|_5^b \right) = \lim_{b\to\infty} (\ln|b| - \ln|5|) = \infty.$$

From the calculation, we see that this integral diverges. This conclusion is not immediately evident simply by looking at the graph of $f(x) = \frac{1}{x}$. The area between the x-axis and the curve, on the interval $[5, \infty)$ is not finite. In contrast, the area between the x-axis and the curve of $y = \frac{1}{x^2}$ is finite. In both cases, we were adding smaller and smaller areas to our sum as $x \to \infty$, but in the case of $f(x) = \frac{1}{x}$, the areas were not small *enough*.

Example 36.1. Determine the convergence or divergence of $\int_{-\infty}^0 e^{2x} dx$.

Solution: $\int_{-\infty}^0 e^{2x} dx = \lim_{a\to-\infty} \int_a^0 e^{2x} dx = \lim_{a\to-\infty} \left(\frac{1}{2} e^{2x} \Big|_a^0 \right) = \lim_{a\to-\infty} \left(\frac{1}{2}e^0 - \frac{1}{2}e^{2a} \right) = \frac{1}{2} - 0 = \frac{1}{2}.$

The integral converges to $\frac{1}{2}$.

Example 36.2. Determine the convergence or divergence of $\int_5^\infty \frac{1}{\sqrt{x+3}} dx$.

Solution: $\int_5^\infty \frac{1}{\sqrt{x+3}} dx = \lim_{b\to\infty} \int_5^b (x+3)^{-\frac{1}{2}}\ dx = \lim_{b\to\infty} \left(2(x+3)^{\frac{1}{2}} \Big|_5^b \right)$

$= \lim_{b\to\infty} \left(2\sqrt{b+3} - 2\sqrt{8} \right) = \infty.$ The integral diverges.

Suppose we are interested in having both limits of integration be infinite: $\int_{-\infty}^\infty f(x)dx$? We handle this by splitting this very improper integral into two improper integrals. For the integral $\int_{-\infty}^\infty f(x)dx$ to make sense, it must be true that f is continuous on its domain, and the domain of f is $\mathbb{R}$. So, we can choose any constant value c in $\mathbb{R}$ and rewrite the integral as:

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx \text{ for any value } c \text{ in } \mathbb{R}. \tag{36.3}$$

It is often helpful to choose $c = 0$ because calculations with zero (adding zero, multiplying by zero, raising to zero power, etc.) are usually easier than with other numbers. However, you can use any real value for c and you will get the same result.

The integral $\int_{-\infty}^\infty f(x)dx$ converges ONLY if BOTH of the improper integrals on the right side of equation (36.3) converge. In this case, we say that $\int_{-\infty}^\infty f(x)dx$ converges to the sum of the two integrals. If either one (or both) of the two integrals on the right side diverges then $\int_{-\infty}^\infty f(x)dx$ diverges.

Example 36.3. Determine the convergence or divergence of $\displaystyle\int_{-\infty}^{\infty} x dx$.
Solution: While it really makes for sense to use $c = 0$, we will use $c = 3$ in this solution in order to demonstrate that any real number c will work. By definition, $\displaystyle\int_{-\infty}^{\infty} x dx = \int_{-\infty}^{3} x dx + \int_{3}^{\infty} x dx$.

We first evaluate $\displaystyle\int_{-\infty}^{3} x dx$.

$$\int_{-\infty}^{3} x dx = \lim_{a \to -\infty} \int_{a}^{3} x dx = \lim_{a \to -\infty} \left(\frac{1}{2}x^2 \Big|_a^3 \right) = \lim_{a \to -\infty} \left(\frac{9}{2} - \frac{a^2}{2} \right) = -\infty.$$

Since this integral diverges, we don't even have to evaluate $\displaystyle\int_{3}^{\infty} x dx$. We can already conclude that $\displaystyle\int_{-\infty}^{\infty} x dx$ must diverge.

It is interesting to note that had we evaluated $\displaystyle\int_{3}^{\infty} x dx$ in the previous example we would have gotten $+\infty$. We *cannot* claim that $-\infty + \infty = 0$ and therefore the integral $\displaystyle\int_{-\infty}^{\infty} x dx$ converges to zero. It doesn't work that way. If EITHER one of the integrals on the right side of equation (36.3) diverges then the integral $\displaystyle\int_{-\infty}^{\infty} x dx$ diverges.

Example 36.4. Determine the convergence or divergence of $\displaystyle\int_{-\infty}^{\infty} x e^{-x^2} dx$

Solution: $\displaystyle\int_{-\infty}^{\infty} x e^{-x^2} dx = \int_{-\infty}^{0} x e^{-x^2} dx + \int_{0}^{\infty} x e^{-x^2} dx$.

In order to evaluate these integrals, we need to find the antiderivative $\displaystyle\int x e^{-x^2} dx$. We use a u-substitution. Let $u = -x^2$. Then $du = -2x dx$, or $-\frac{1}{2} du = x dx$. So, $\displaystyle\int x e^{-x^2} dx = -\frac{1}{2} e^u du = -\frac{1}{2} e^u = -\frac{1}{2} e^{-x^2} + C$. Now we return to the improper integrals.

$$\int_{-\infty}^{0} x e^{-x^2} dx = \lim_{a \to -\infty} \int_{a}^{0} x e^{-x^2} dx = \lim_{a \to -\infty} \left(-\frac{1}{2} e^{-x^2} \Big|_a^0 \right) = \lim_{a \to -\infty} \left(-\frac{1}{2} e^0 - -\frac{1}{2e^{a^2}} \right) = -\frac{1}{2} + 0 = -\frac{1}{2}$$

$$\int_{0}^{\infty} x e^{-x^2} dx = \lim_{b \to \infty} \int_{0}^{b} x e^{-x^2} dx = \lim_{b \to \infty} \left(-\frac{1}{2} e^{-x^2} \Big|_0^b \right) = \lim_{b \to \infty} \left(-\frac{1}{2e^{b^2}} - -\frac{1}{2} e^0 \right) = 0 - -\frac{1}{2} = \frac{1}{2}$$

Since both integrals converge, we conclude that $\displaystyle\int_{-\infty}^{\infty} x e^{-x^2} dx$ converges to their sum: $-\dfrac{1}{2} + \dfrac{1}{2} = 0$.

The integral in 36.4 converged to zero. Does this mean that the area between the graph of $f(x) = xe^{-x^2}$ and the x-axis is zero? No. It does mean that the *signed area* between f and the x-axis is zero. If we want to know the area between f and the x-axis on $(-\infty, \infty)$ we would need to know where f is positive and where f is negative.

Example 36.5. Find the area between the graph of $f(x) = xe^{-x^2}$ and the x-axis on $(-\infty, \infty)$.
Solution: Since e^{-x^2} is always positive, we know that f is negative when $x < 0$ and f is positive when $x > 0$. So, the area between the graph of f and the x-axis is given by:

$$A = -\int_{-\infty}^{0} xe^{-x^2}dx + \int_{0}^{\infty} xe^{-x^2}dx = -- \frac{1}{2} + \frac{1}{2} = 1.$$

Capital value:

The discussion headed Present Value in Section 35 has a natural extension here. Returning to the example of your inherited hotel, it might be that there is no obvious time T after which the hotel is worthless. Perhaps you have instructed the manager to maintain it well and add improvements. Thus in figuring out its value, the right "value" for T is ∞. The present value of the hotel in Section 35 was $\int_0^T f(t)e^{-rt}dt$. The *capital value* of your hotel is the present value of ALL future income, namely $\int_0^\infty f(t)e^{-rt}dt = \lim_{T\to\infty} \int_0^T f(t)e^{-rt}dt$.

Section 36 - Exercises (answers follow)

1. Rewrite each improper integral as an appropriate limit. Determine whether the integral converges or diverges. Find the value of each that converges.

   (a) $\displaystyle\int_{-1}^{\infty} e^{-5x}dx$

   (b) $\displaystyle\int_{e}^{\infty} \frac{1}{x}dx$

   (c) $\displaystyle\int_{-\infty}^{0} e^{\frac{x}{2}}dx$

   (d) $\displaystyle\int_{-\infty}^{-2} \left(e^x - \frac{1}{x^2}\right)dx$

   (e) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+2}dx$

2. Determine whether the improper integrals converge or diverge. Find the value of each that converges.

   (a) $\displaystyle\int_{1}^{\infty} xdx$

   (b) $\displaystyle\int_{1}^{\infty} \frac{1}{x^3}dx$

   (c) $\displaystyle\int_{1}^{\infty} \frac{1}{\sqrt{3x}}dx$

   (d) $\displaystyle\int_{-\infty}^{0} \frac{1}{(x-2)^3}dx$

   (e) $\displaystyle\int_{0}^{\infty} 4e^{-4x}dx$

   (f) $\displaystyle\int_{1}^{\infty} e^{1-x}dx$

   (g) $\displaystyle\int_{2}^{\infty} \frac{1}{x\ln x}dx$

   (h) $\displaystyle\int_{-\infty}^{\infty} e^{x}dx$

3. Find the area between the graph of the given function and the x-axis over the given interval, if possible.

   (a) $f(x) = e^{-x}$ for $[0, \infty)$

   (b) $f(x) = \dfrac{1}{(x+1)^2}$ for $(-\infty, 0]$

4. Let $f(x) = \dfrac{e^{(x-2)}}{1+e^{(x-2)}}$. Investigate the integrals: (a) $\displaystyle\int_0^{+\infty} f(x)dx$ (b) $\displaystyle\int_{-\infty}^{0} f(x)dx$.

5. Calculate the capital value of the assets described in Problems 11 (all parts), 12, 13(all parts) and 14 of Section 35.

Section 36 - Answers

1. (a) $\displaystyle\lim_{b\to\infty}\int_{-1}^{b} e^{-5x}dx = \frac{1}{5}e^{5}$

   (b) $\displaystyle\lim_{b\to\infty}\int_{e}^{b} \frac{1}{x}dx$ diverges

   (c) $\displaystyle\lim_{a\to-\infty}\int_{a}^{0} e^{\frac{x}{2}}dx = 2$

   (d) $\displaystyle\lim_{a\to-\infty}\int_{a}^{-2} \left(e^{x} - \frac{1}{x^2}\right)dx = \frac{1}{e^2} - \frac{1}{2}$

   (e) $\displaystyle\lim_{a\to-\infty}\int_{a}^{C} \frac{x}{x^2+2}dx + \lim_{b\to\infty}\int_{C}^{b} \frac{x}{x^2+2}dx$ diverges Note: "C" represents any constant value you wish.

2. (a) diverges

   (b) converges to $\frac{1}{2}$

   (c) diverges

   (d) converves to $-\frac{1}{8}$

   (e) converges to 1

   (f) converges to 1

   (g) diverges

   (h) diverges

3. (a) 1

   (b) impossible (The function is discontinuous at $x = -1$)

4. (a) diverges (b) converges to $\ln\left(1+e^{-2}\right)$

5. Prob.11 (a) $472,222 (b) $45,455 (c) $285,714
   Prob.12 $57,143
   Prob.13 (a) $41,677 (b) $50,000 (c) $33,333
   Prob.14 $3,571,429

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