

#### 4.4 Properties of Logarithms; Solving Exponential/Logarithmic Equations

**Fundamental Properties of Logarithms** (for any base  $b$ , and  $b \neq 1$ ):

1.  $\log_b 1 = 0$  because  $b^0 = 1$
2.  $\log_b b = 1$  because  $b^1 = b$
3.  $b^{\log_b a} = a$
4. ~~log<sub>b</sub>(b<sup>a</sup>) = a~~  $\log_b b^a = a$

EX: Evaluate.

$$\log_3 1 = 0$$

$$\log_6 6^x = x$$

$$\ln 1 = 0$$

$$\log_9 9^{2x} = 2x$$

$$\log_7 7 = 1$$

$$e^{\ln 5} = 5$$

$$\log_{10} 10 = 1$$

$$e^{2\ln 5} = 5^2 \text{ because } e^{2\ln 5} = e^{\ln 5^2}$$

$$10^{\log x} = x$$

$$3^{5\log_3 10} = 3^{\log_3 10^5} = 10^5$$

$$4^{\log_4(x+3)} = x+3$$

**Properties of Logarithms** (for positive real numbers  $M, N$ , and  $b \neq 1$ , and any real number  $p$ ):

1.  ~~$\log M + \log N = \log(MN)$~~
2.  $\log M - \log N = \log(M/N)$
3.  $\log(M^p) = p \log M$

EX: Write the single logarithm in a form with no logarithm of a product, quotient, root, or power, so that you remain with a sum, difference, and/or constant multiple of logarithms. Assume all variables are positive. Simplify answer accordingly.

$$\begin{aligned}
 \text{a. } \log\left(\frac{100(w+1)^5}{\sqrt[9]{r^4}}\right) &= \log 100 + \log(w+1)^5 - \log(r^{-4/9}) \\
 &= \log 100 + 5\log(w+1) - \frac{4}{9}\log r \\
 &= \boxed{2 + 5\log(w+1) - \frac{4}{9}\log r}
 \end{aligned}$$

$$\begin{aligned}
 \text{b. } \ln \sqrt[y^2]{x^5} &= \ln\left(\frac{x^5}{y^2}\right)^{1/2} = \frac{1}{2}\ln\frac{x^5}{y^2} = \frac{1}{2}(\ln x^5 - \ln y^2) \\
 &= \frac{1}{2}(5\ln x - 2\ln y) = \boxed{\frac{5}{2}\ln x - \ln y}
 \end{aligned}$$

EX: Write each statement as a single logarithm. Simplify answer accordingly.

$$\begin{aligned} \text{a. } \log_3 18 + 2\log_3 5 - \log_3 50 &= \log_3 18 + \log_3 5^2 - \log_3 50 \\ &= \log_3 \frac{(18)(25)}{(50)} = \log_3 \left(\frac{18}{2}\right) = \log_3 9 = \boxed{2} \end{aligned}$$

$$\begin{aligned} \text{b. } 5\log_4 x + 2\log_4 5 - \log_4 y - 3\log_4 z &= \log_4 x^5 + \log_4 5^2 - (\log_4 y + \log_4 z^3) \\ &= \boxed{\log_4 \left(\frac{25x^5}{yz^3}\right)} \end{aligned}$$

**Change-of-Base Formula** (for positive real numbers m, a, & b with  $a, b \neq 1$ ):  $\log_b m = \frac{\log_a m}{\log_a b}$

$$\log_4 30 = \frac{\log 30}{\log 4} \quad (\text{base 10})$$

$$\log_4 30 = \frac{\ln 30}{\ln 4} \quad (\text{base e})$$

EX: Simplify using the change-of-base formula.

a.  $(\log_4 7)(\log_7 64)$

$$\cancel{\log_4 7} \cdot \frac{\log_4 64}{\cancel{\log_4 7}} = \log_4 64 = \boxed{3}$$

b.  $(\log_3 e)(\ln 9)$

$$\cancel{\log_3 e} \cdot \frac{\log_3 9}{\cancel{\log_3 e}} = \boxed{12}$$

c.  $(\log_2 5)(\log_{25} 16^{-1})$

skip

## Exponential and Logarithmic Equations

### Solving Exponential Equations

1. Isolate the exponential expression on one side of the equation.
2. Take the log of both sides and "bring down the exponent."
3. Solve for the variable.

EX: Solve.

$$\text{a. } 3^{2x} = 81 \rightarrow \log_3 3^{2x} = \log_3 81 \rightarrow 2x \log_3 3 = 4 \rightarrow 2x = 4 \rightarrow x = 2$$

Check

$$3^{2(2)} = 3^4 = 81 \quad \checkmark$$

b.  $3^{2x} = 80 \rightarrow \log_3 3^{2x} = \log_3 80 \rightarrow 2x \log_3 3 = \log_3 80$

$$\rightarrow 2x = \log_3 80 \rightarrow \boxed{x = \frac{\log_3 80}{2}}$$

c.  $5(2^{9x}) - 3 = 37$

$$2^{9x} = \frac{40}{5} \rightarrow 2^{9x} = 8 \rightarrow \log_2 2^{9x} = \log_2 8$$

$$\rightarrow 9x \cancel{\log_2 2}^1 = 3 \rightarrow 9x = 3 \rightarrow \boxed{x = \frac{1}{3}}$$

d.  $7^x = 4^{2x-1}$  or  $\log_4 7^x = \log_4 4^{2x-1}$  ← Do this one.

$$\begin{aligned} x \log_4 7 &= (2x-1) \cancel{\log_4 4}^1 \\ x \log_4 7 &= 2x \cancel{\log_2 2}^1 - 1 \end{aligned}$$

$$\begin{aligned} x \log_4 7 - 2x &= -1 \\ x(\log_4 7 - 2) &= -1 \end{aligned}$$

$$\boxed{x = \frac{1}{2 - \log_4 7}}$$

e.  $-14 + 3e^{x-4} = 11$

$$3e^{x-4} = 25$$

$$\ln e^{x-4} = \ln \frac{25}{3}$$

$$\begin{aligned} e^{x-4} &= \frac{25}{3} \\ (x-4) \cancel{\ln e}^1 &= \ln \frac{25}{3} \\ x-4 &= \ln \frac{25}{3} \end{aligned}$$

$$\boxed{x = 4 + \ln \frac{25}{3}}$$

f.  $e^{2x} - 4e^x = 5$

~~skip~~

### Solving Logarithmic Equations

1. Write as a single log (or a single log on each side).
2. Write in exponential form or "exponentiate."
3. Solve for the variable. CHECK YOUR SOLUTIONS.

EX: Solve.

a.  $\ln x = -3$

$$\boxed{e^{-3} = x}$$

b.  $\log_x 625 = 4 \rightarrow x^4 = 625 \rightarrow x = 625^{\frac{1}{4}} = \boxed{5}$

c.  $\ln x + \ln(2x+1) = 0 \rightarrow \ln x(2x+1) = 0 \rightarrow e^0 = x(2x+1)$   
 $\rightarrow 1 = 2x^2 + x \rightarrow 0 = 2x^2 + x - 1$   
 $\rightarrow 0 = (2x-1)(x+1) \rightarrow \boxed{x = \frac{1}{2}}$   $\times$

d.  $\log_4 x - \log_4(x-1) = \frac{1}{2}$   
 $\rightarrow \log_4\left(\frac{x}{x-1}\right) = \frac{1}{2} \quad \boxed{\frac{x}{x-1} = 2} \quad \text{Check! (it works)}$   
 $\rightarrow \frac{x}{x-1} = 4^{\frac{1}{2}} \quad \boxed{x = 2x-2}$   
 $\boxed{2 = x}$

e.  $\log(x+12) - \log x = \log(x+2)$

$$\log\left(\frac{x+12}{x}\right) = \log(x+2)$$

so  $\frac{x+12}{x} = x+2$

$$x+12 = x^2 + 2x$$

$$0 = x^2 + x - 12$$

$$0 = (x-3)(x+4)$$

$$\boxed{x=3} \quad \times$$

f.  $\ln(x-1) + \ln(x-3) = 2 \ln x$

$$\ln(x-1)(x-3) = \ln x^2$$

so  $(x-1)(x-3) = x^2$

$$x^2 - 4x + 3 = x^2$$

$$-4x + 3 = 0$$

$$x = 3/4$$

Check!

$$\ln(3/4 - 1) + \ln(3/4 - 3)$$

STOP!

$\ln(-1/4) \notin \ln(-2/4)$   
 are not defined,  
 hence this has  
 no solution

## Exponential and Logarithmic Equations

Solve the following equations:

1.  $8^t = -6$

$$\log_8(-6) = t$$

no solution

6.  $12 = \log(4t)$

$$10^{12} = 4t$$

$$\frac{10^{12}}{4} = t$$

2.  $17 = 3^x$

$$\log_3 17 = x$$

7.  $-1 = \log(x+3)$

$$10^{-1} = x+3$$

$$10^{-1} - 3 = x$$

$$x = \frac{1}{10} - 3$$

8.  $\log(x+3) = \frac{1}{2}$

$$10^{1/2} = x+3$$

$$x = \sqrt{10} - 3$$

3.  $4 = 15 - e^{x-8}$

$$-11 = -e^{x-8}$$

$$11 = e^{x-8}$$

$$\ln 11 = x-8$$

$$x = \ln 11 + 8$$

4.  $29 + 10^{t+12} = 74$

$$10^{t+12} = 45$$

$$\log 45 = t+12$$

$$t = \log 45 - 12$$

5.  $\ln(x-1) = 3$

$$e^3 = x-1$$

$$x = e^3 + 1$$

9.  $1 = \log_4 2 + \log_4(3+x)$

$$1 = \log_4 (2)(3+x)$$

$$4^1 = 6 + 2x$$

$$x = -11$$

10.  $\log(t+3) + \log(t) = 1$

$$\log(t+3)(t) = 1$$

$$10^1 = (t+3)(t)$$

$$10 = t^2 + 3t$$

$$0 = t^2 + 3t - 10$$

$$0 = (t-2)(t+5)$$

$$t = 21, \checkmark 5$$

$$11. \log_2(t+1) + \log_2(t-1) = 5$$

$$\log_2(t+1)(t-1) = 5$$

$$2^5 = (t+1)(t-1)$$

$$0 = t^2 - 1 < 32$$

$$0 = t^2 - 33$$

$$12. -2 = \log(2) - \log(3+x)$$

$$-2 = \log\left(\frac{2}{3+x}\right)$$

$$10^{-2} = \frac{2}{3+x} \rightarrow \frac{1}{100} = \frac{2}{3+x}$$

$$\rightarrow 3+x = 200 \rightarrow x = 197$$

$$13. \ln(4t) - \ln(3t) = 2$$

$$\ln \frac{4t}{3t} = \ln \frac{4}{3} = 2$$

~~means~~  $e^2 = \frac{4}{3}$

But it doesn't, so this is an untrue statement.

$$14. \log_2(t+1) - \log_2(t-1) = 3$$

$$\log_2\left(\frac{t+1}{t-1}\right) = 3$$

$$2^3 = \frac{t+1}{t-1}$$

$$8t - 8 = t + 1$$

$$7t = 9$$

$$\frac{1}{t} = 9/7$$

$$15. \log_3(4t) = 1 - \log_3(3t)$$

$$\log_3(4t) + \log_3(3t) = 1 \rightarrow \frac{3}{12} = t^2$$

$$\log_3(4t)(3t) = 1 \rightarrow \frac{1}{4} = t^2$$

$$\log_3 12t^2 = 1$$

$$3^1 = 12t^2$$

$$t = \frac{1}{2}$$

$$16. \log_2(-x) = 3 - \log_2(2-x)$$

$$\log_2(-x) + \log_2(2-x) = 3$$

$$\log_2(-x)(2-x) = 3$$

$$\log_2(x^2 - 2x) = 3$$

$$x^2 - 2x = 2^3 = 8$$

~~$$17. e^{2x} - e^x - 6 = 0$$~~

~~$$(e^x)^2 - e^x - 6 = 0$$~~

~~$$\text{let } m = e^x$$~~

~~$$m^2 - m - 6 = 0$$~~

~~$$\text{etc.}$$~~

~~$$18. e^x + 29 = 12e^{\frac{x}{2}}$$~~

~~$$x^2 - 2x - 8 = 0$$~~

~~$$(x-4)(x+2) = 0$$~~

~~$$x = 4, -2$$~~

~~Keep this~~

Keep this because  $\log(-(-2)) = \log 2$  ok