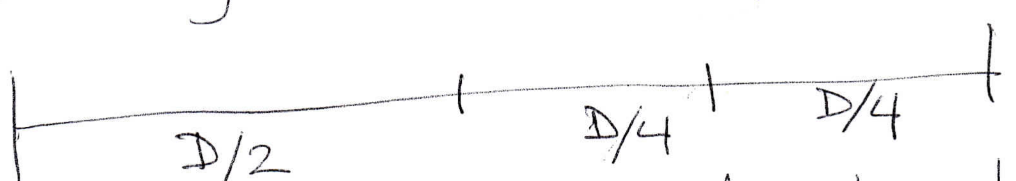
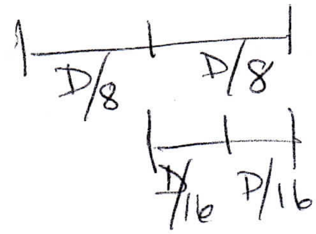
Chapter 6 Limits + Chapter 9 Continuity

There's a natural progression in the discussion of limits to one of "continuity" of a function at a point (and then, continuity on an interval)

First, a summary of limit. Intuitively, we see that the expression $\frac{1}{n}$, where n is a positive integer, gets smaller as n gets larger; starting at $n=1$, the sequence is $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ With the numerator stuck at 1, the value of $\frac{1}{n}$ goes to 0 as n goes to infinity. We agree on this notation: $\lim\limits_{n \to \infty} \frac{1}{n} = 0$.

Zeno's paradox describes a similar expression, $\frac{1}{2^n}$, which is the sequence of distances between the object moving toward its target, where we stop halfway, go again but stop halfway through the remaining difference, start again, stop, etc.





where D is beginning distance from the target.

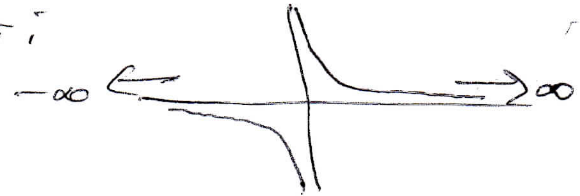
To simplify the expression, let $D = 1$. Then the remaining distance sequence is $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ that is, $\frac{1}{2^n}$. Eventually, the object reaches its target, so although we could conceivably measure the increasingly smaller intervals, we conclude these go to zero. $\lim\limits_{n \to \infty} \frac{1}{2^n} = 0$.

It turns out that the numerator could be <u>any</u> constant, and the limit as $n \to \infty$ of any of these expressions is zero:

$$\lim_{n \to \infty} \frac{k}{n} = 0, \quad \lim_{n \to \infty} \frac{k}{2n} = 0, \quad \lim_{n \to \infty} \frac{k}{n^2} = 0$$

$\lim\limits_{n \to \infty} \frac{k}{2^n} = 0$; you get the idea. When we do limits of rational expressions (or fcns), where the numerator might have an n (or x) as well, determining the limit as n (or x) goes to a number (finite) requires a graphical approach to comprehend the answer. But as $n \to \pm\infty$, there are a few shortcuts to get to limit. More on that soon.

We looked at $f(x) = \frac{1}{x}$ next, because it goes with $\frac{1}{n}$. The graph of this fcn is familiar:



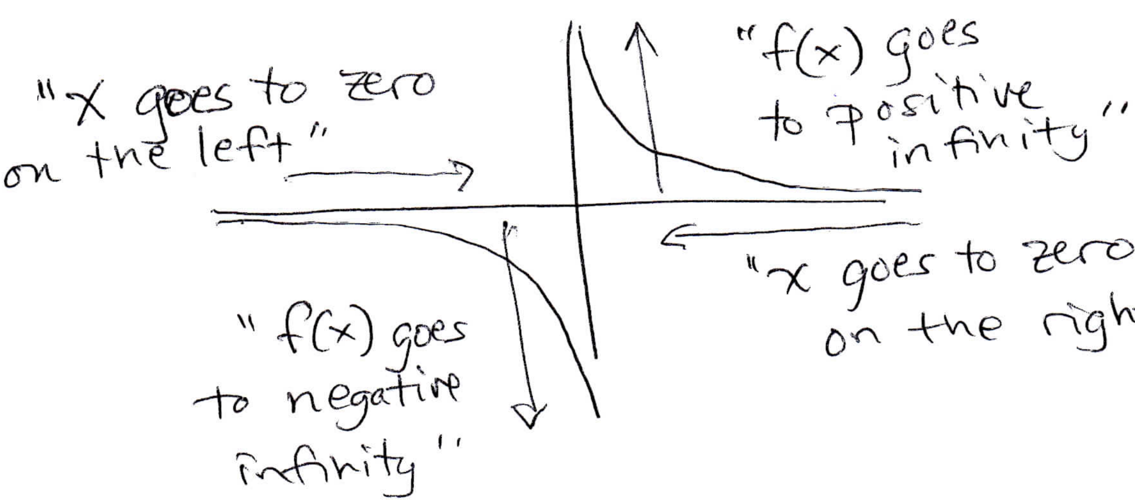
Notice several things.

As $x \to \infty$, $\frac{1}{x} \to 0$; as $x \to -\infty$, $\frac{1}{x} \to 0$. The first goes to zero "from above" (positive values $\frac{1}{2}, \frac{1}{10}, \frac{1}{4000}, \ldots$). The second goes to zero "from below" (negative values $-\frac{1}{2}, -\frac{1}{10}, -\frac{1}{4000}, \ldots$)

Choices of x ~~are~~ random here, but increasing.

Hence, $\lim\limits_{x \to \infty} \frac{1}{x} = 0$ and $\lim\limits_{x \to -\infty} \frac{1}{x} = 0$.

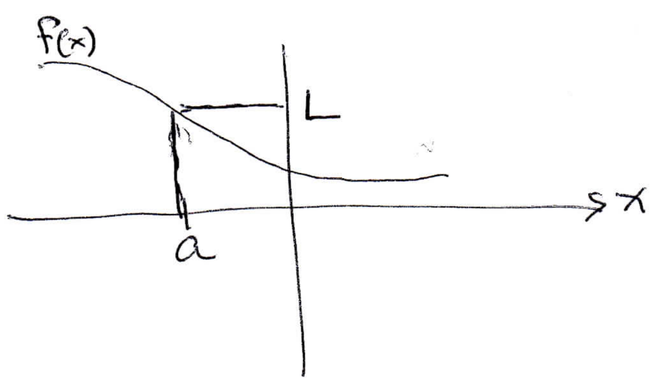
Now look at what's happening if we approach the y-axis ($x=0$) ~~along~~. As x gets closer to 0, $\frac{1}{x}$ does one of two things, depending on which side of $x = 0$ we're on.



$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x \to 0^+} \frac{1}{x} = \infty$$

The superscripts $-$ and $+$ here mean left and right approaches. By definition, these are called "left-hand limits" and "right-hand limit".

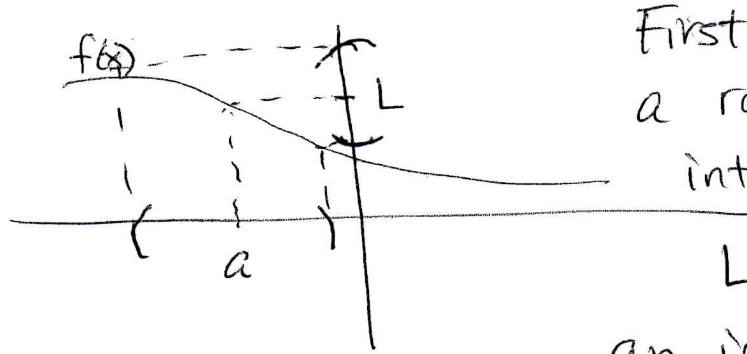
More on this soon! But we still have not defined the phrase "the limit of $f(x)$ as x approaches a" in any rigorous way. In fact, without the use of ε, δ and other notation, our definition will not be rigorous. But it can be intuitive enough to satisfy us with regard to derivative of $f(x)$. So —



Consider the fcn $f(x)$ as drawn (say it's part of a polynomial graph). We define limit of f at a as follows:

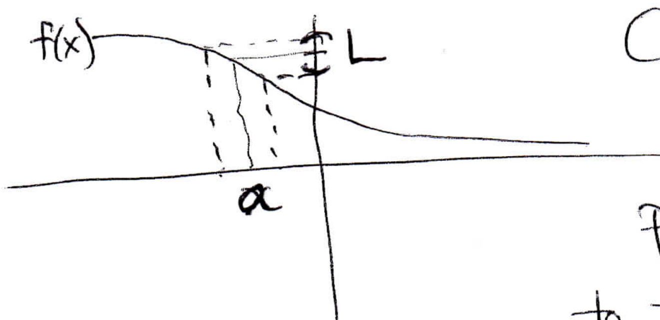
$\lim_{x \to a} f(x) = L$ because, for any small interval around L, we can find some interval around a such that all x in that interval go to (in the sense of the

fcn) the stated interval around L. What!?
Look again at the graph, this time with
the intervals shown by parentheses.



First we name a rather "big" interval around L, and seek an interval of a that maps x's to L's interval.

Let's make that interval much smaller, because when we say "any" interval of L we, in fact, mean an "arbitrarily small" one.



Can you see that the dotted lines traced from the parentheses around L to the graph and down to x will lie in some interval of a?

These intervals are called "ε and δ neighborhoods" of L and a, respectively. The rigorous definition of "The limit of f(x) as x approaches a is L" follows:

Def $\lim_{x \to a} f(x) = L$ if, for any $\varepsilon > 0$ there is a $\delta > 0$ such that if the distance of x from a is less than δ, then the distance of $f(x)$ from L is less than ε.

The fully notational way to say this is:

$\lim_{x \to a} f(x) = L$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

The conversational way to say this is:

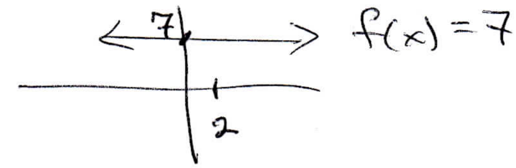
"L is the limit of $f(x)$ as x approaches a if, when you give me an ε, no matter how small, I can find a δ where any ~~f(x)~~ x within δ-distance of a has a fcn value f within ε-distance of L."

We're ready for properties of limits and examples of $\lim_{x \to a} f(x) = L$, where L is finite.

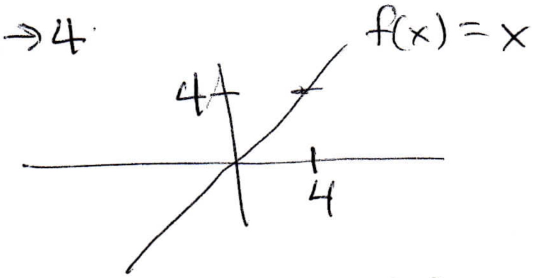
Properties   Example

1. $\lim\limits_{x \to a} c = c$ ex $\lim\limits_{x \to 2} 7 = 7$

Think of any constant fcn $f(x) = c$ and the property is clear.



2. $\lim\limits_{x \to a} x = a$ ex $\lim\limits_{x \to 4} x = 4$



3., 4. $\lim\limits_{x \to a} \left(f(x) \pm g(x)\right) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$

These are natural properties, which like 1 and 2 we won't have to prove to believe.

Ex $\lim\limits_{x \to 3} (x + 10) = \lim\limits_{x \to 3} x + \lim\limits_{x \to 3} 10 = 3 + 10$

$= 13$

Before looking at properties 5 and 6, here's an important fact about limits of polynomials at $x = a$:

Polynomials and limits: If $f(x)$ is a polynomial, then for any $a \in D_f$,
$\lim_{x \to a} f(x) = f(a)$.

Think of the graph where we developed the def. of limit. It was a portion of a polynomial (though it could have been a portion of a fcn with a similar smooth graph). Any $a \in D_f$ maps to $f(a)$ in the range. And, $\lim_{x \to a} f(x) = f(a)$. In other words, the limit of a poly as $x \to a$ is the value of the polynomial.

EX $f(x) = x^3 - 2x^2 + x + 5$

$$\lim_{x \to 1} (x^3 - 2x^2 + x + 5) = 1^3 - 2(1^2) + 1 + 5$$
$$= 5 \,//$$

Here, we employed properties 1, 2, 3 and 4.

It was natural to do so.