- Quiz monday (sections 6 & 9)

- Exam on Friday (09/24/21)
  → Includes material covered up to and including Sept. $20^{th}$.

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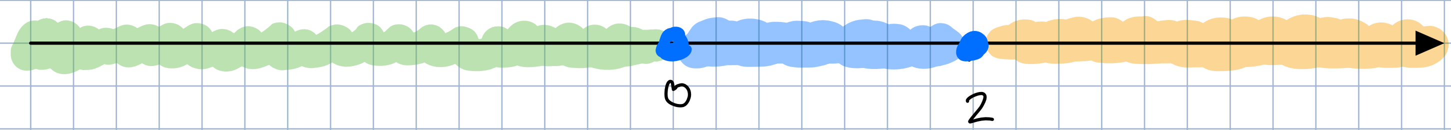
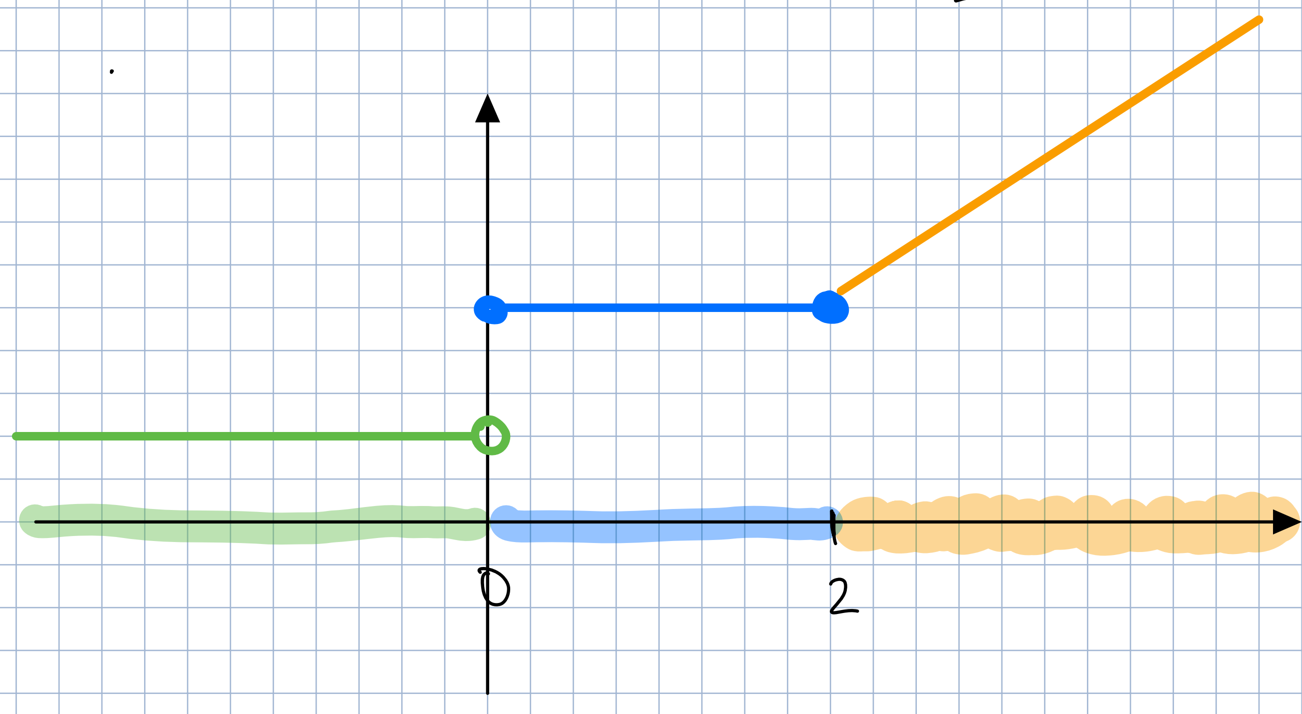
**Ex:** At which values of $x \in D_f$ is $f$ cts?

$$f(x) = \begin{cases} 1 & \text{if } x<0 \\ 4 & \text{if } 0 \leq x \leq 2 \\ 2x & \text{if } x>2 \end{cases}$$

Domain ⤵




<u>OBSERVE</u>: $f$ <u>must</u> be continuous on $(-\infty,0)\cup(0,2)\cup(2,\infty)$ because on those intervals, $f$ coincides with known continuous functions!

What about $x=0$?

→ We now check the def$^n$ of continuity:

$\boxed{x=0}$ For continuity, we require that

① $\lim_{x\to 0} f(x)$ exists ✗

② $\lim_{x\to 0} f(x) = f(0)$

To this end, we check LH/RH limits:

$$\left.\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} 1 = 1 \\ \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} 4 = 4 \end{aligned}\right\} \text{limit DNE!}$$

So, $f(x)$ is discontinuous at $x=0$.

What about $x=2$?

→ We again check the def$^n$ of continuity.

The criteria are very similar; only "a" changes.

$\boxed{x=2}$ For continuity, we require that

① $\lim_{x \to 2} f(x)$ exists ✓

② $\lim_{x \to 2} f(x) = f(2)$ ✓

We again turn to LH/RH limits:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 4 = 4$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 2x = 4$$

limit exists and has a value of 4.

Lastly, we must verify that $f(2)$ also has the same value as the limit:

$$f(2) = 4 = \lim_{x \to 2} f(x)$$ ✓

So, $f(x)$ is continuous at $x = 2$.

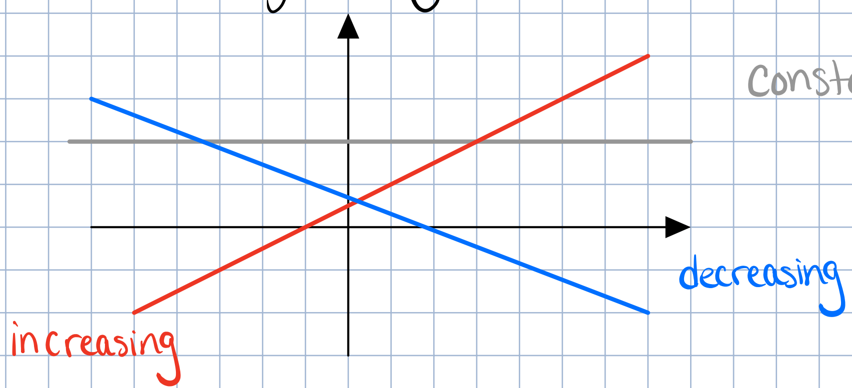
# Section 7: The Derivative

→ Study of how a function changes

Now, some functions are easy to analyze...

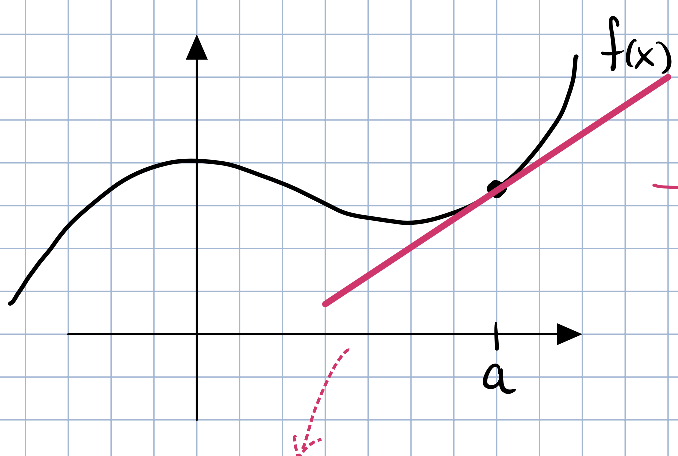
⇒ Lines are particularly easy to understand:

$y = mx + b$




We can easily determine whether a line corresponds to an increasing, decreasing, or constant function by just focusing on the "m"-value i.e. the slope!

But what about more complicated functions?




called "the tangent line" to f @ $x=a$.

→ We attempt to understand f by attaching a line (cuz they're nice) that is tangent (so they're in the same direction) to f at $x=a$

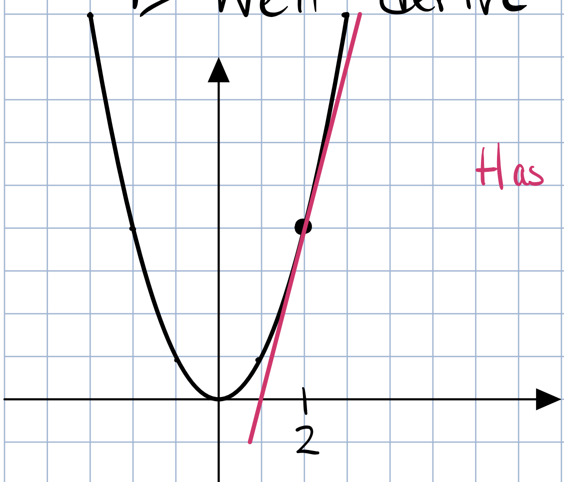
We then arrive at the following observation:

**Key Observation:** the slope of the tangent line tells us exactly how $f$ changes (though only @ $x=a$)

**Ex:** Consider $y(x) = x^2+1$

Find slope of tangent line at $a=2$

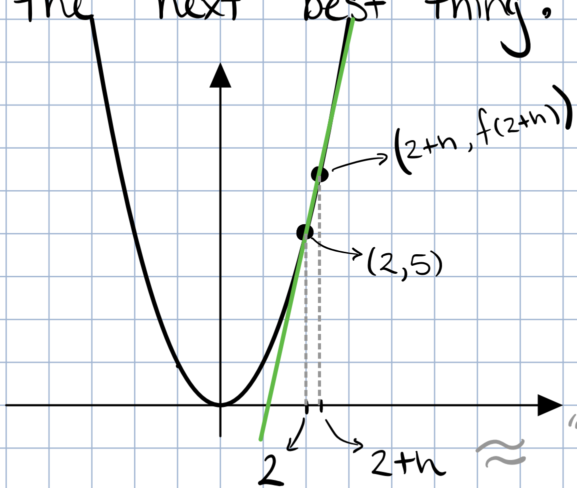
$\rightarrow$ We'll derive the expression momentarily.




Has eq$^n$ $y-y_0 = m(x-x_0)$

But what's $m$??

Since it's hard to find $m$ directly, we're going to do the next best thing: approximate!




We're going to take a look at a secant line (hits our function twice) rather than a tangent line!

$\approx$ "2 plus a little bit more"

Though the secant lines are a little more complicated, the slope is easier to determine:

$$m_h = \frac{\Delta y}{\Delta x} = \frac{f(2+h) - f(2)}{(\cancel{2}+h) - \cancel{2}}$$

Slope of secant line i.e. approximation

$$= \frac{f(2+h) - 5}{h}$$

We now make another crucial observation:

**Key Observation:** As $h \to 0$, the secant line looks more and more like the tangent line

In particular:

$$\text{Slope of tangent line} = \lim_{h \to 0} \text{slope of secant line}$$

$$m = \lim_{h \to 0} \frac{f(2+h) - 5}{h}$$

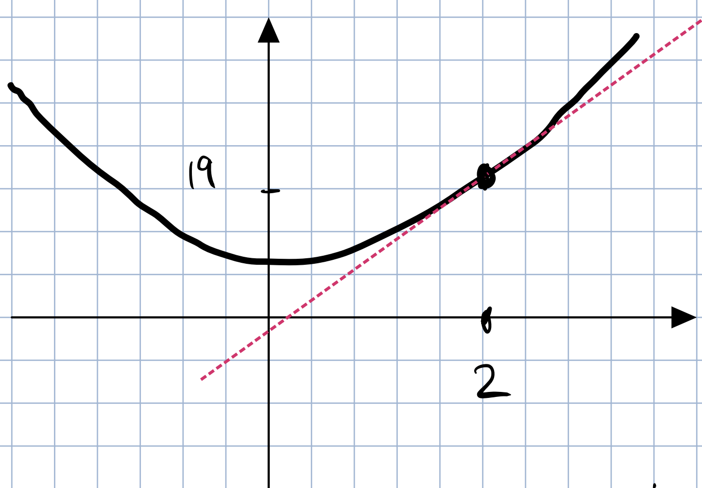
Here, $m$ is called the derivative of $f$ @ $x = 2$

More generally:

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

this called the def$^n$ of the derivative at a point $x = a$.

**Ex:** Find the eq$^n$ of the tangent line to $f(x) = x^2 + 7x + 1$ @ $(2, 19)$




Before we do any calculus, notice that our line must pass through $(2, 19)$.

So, our eq$^n$ will look like:

$$y - 19 = m(x - 2).$$

$\rightarrow$ All that's left is to find

$$m = ???$$

From our discussion above,

$$m = \text{slope of tangent line} = f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{\left[(2+h)^2 + 7(2+h) + 1\right] - 19}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4} + 4h + h^2 + \cancel{14} + 7h + \cancel{1} - \cancel{19}}{h}$$

$$= \lim_{h \to 0} \frac{11h + h^2}{h}$$

$$= \lim_{h \to 0} 11 + h = 11.$$

So, our tangent line is:

$$y - 19 = 11(x-2)$$

or

$$y = 11x - 3$$

**Warning:** By def$^n$, a derivative is a limit! So, derivatives may or may not exist.

**Ex:** Find $f'(0)$ where

$$f(x) = \begin{cases} -2, & \text{if } x<0 \\ 1, & \text{if } x\geq 0 \end{cases}$$

Notice that we're asking for the deriv. @ $a=0$.

$$\rightarrow f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)-1}{h}$$

Since $f$ does diff. things on either side of 0, we must use LH/RH limits:

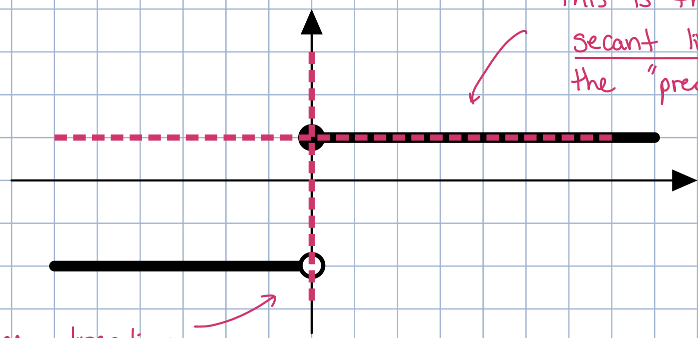
$$\lim_{h\to 0^-} \frac{f(h)-1}{h} = \lim_{h\to 0^-} \frac{-2-1}{h} \longrightarrow \frac{-3}{0^-} = \infty$$

$$\lim_{h\to 0^+} \frac{f(h)-1}{h} = \lim_{h\to 0^+} \frac{1-1}{h} \longrightarrow \frac{0}{0^+} = 0$$

Thus, $f'(0)$ DNE! (the LH/RH limits disagree)

→ Side note: This fact is easily verified by looking at the graph of $f(x)$:

this is the trending behavior of the secant line obtained from RHS. Notice the "predicted" slope is 0!



Whereas this is the trending behavior of the secant line obtained from LHS. Note the slope is infinite!

**Ex:** Let $f(x) = \frac{-1}{x^2}$. Find $f'(1)$.

By defn: $$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{\left[ \frac{-1}{(1+h)^2} - (-1) \right]}{h}$$

$$= \lim_{h \to 0} \frac{\left[ \frac{-1}{(1+h)^2} + \frac{(1+h)^2}{(1+h)^2} \right]}{\left(\frac{h}{1}\right)}$$

$$= \lim_{h \to 0} \left[ \frac{-1 + (1+h)^2}{(1+h)^2} \right] \left( \frac{1}{h} \right)$$

$$= \lim_{h \to 0} \left[ \frac{\cancel{-1} + \cancel{1} + 2h + h^2}{(1+h)^2} \right] \left( \frac{1}{h} \right)$$

$$= \lim_{h \to 0} \frac{2h + h^2}{h(1+h)^2}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2+h)}{\cancel{h}(1+h)^2} = \lim_{h \to 0} \frac{(2+h)}{(1+h)^2}$$

$$= 2$$

So, $f'(1) = 2$

→ this is the slope of tangent line to

$f(x) = \frac{-1}{x^2}$ @ $x = 1$.

# More Derivative Examples:

**Ex:** Let $f(x) = \frac{-1}{x^2}$. Find $f'(1)$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{-1}{(1+h)^2} - (-1)}{h}$$

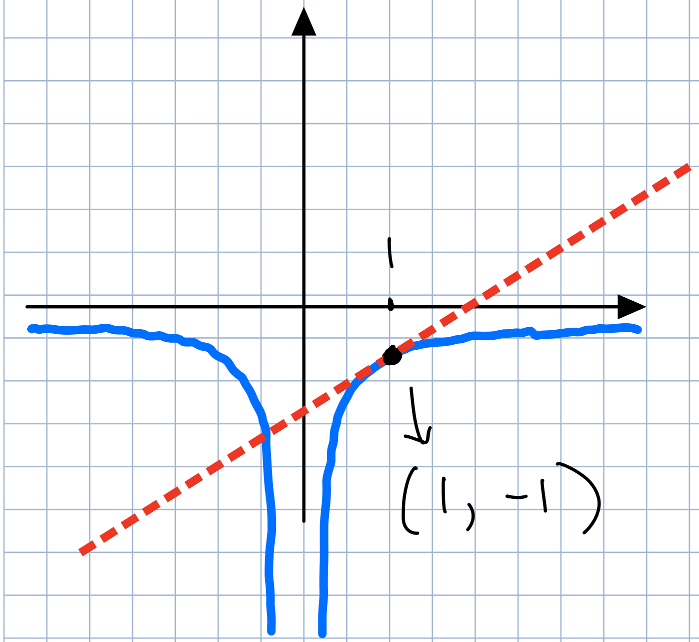
$$= \lim_{h \to 0} \frac{\frac{-1}{(1+h)^2} + 1}{h}$$

$$= \lim_{h \to 0} \frac{\frac{-1 + (1+h)^2}{(1+h)^2}}{h}$$

$$= \lim_{h \to 0} \frac{-1 + 1 + 2h + h^2}{(1+h)^2 h}$$

$$= \lim_{h \to 0} \frac{2h + h^2}{(1+h)^2 h} = \lim_{h \to 0} \frac{2 + h}{(1+h)^2}$$

$$= \frac{2}{1} = 2$$




**Tangent line:**

$$y - y_1 = m(x - x_1)$$

$$y + 1 = 2(x - 1)$$

**Ex:** Let $g(x) = x^3$. Find $g'(2)$

$$g'(2) = \lim_{h \to 0} \frac{g(2+h) - g(2)}{h}$$

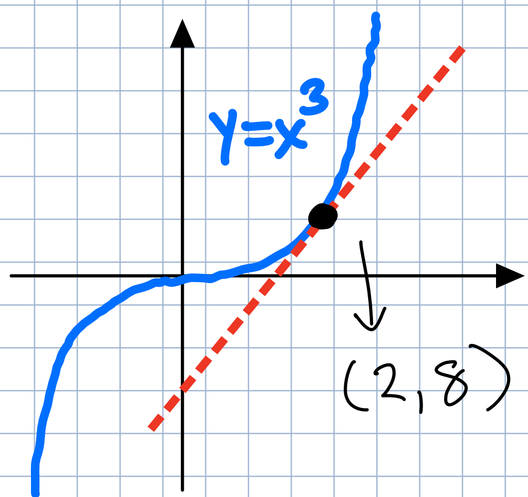
$$= \lim_{h \to 0} \frac{(2+h)^3 - 8}{h}$$

$$= \lim_{h \to 0} \frac{(4 + 4h + h^2)(2+h) - 8}{h}$$

$$= \lim_{h \to 0} \frac{8 + 8h + 2h^2 + 4h + 4h^2 + h^3 - 8}{h}$$

$$= \lim_{h\to 0} \frac{12h + 6h^2 + h^3}{h}$$

$$= \lim_{h\to 0} 12 + 6h + h^2 = 12$$




**Tangent line:**

$$y - y_1 = m(x - x_1)$$

$$y - 8 = 12(x - 2)$$

**Ex:** Let $f(x) = \sqrt[3]{x}$. Find $f'(0)$.

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt[3]{h} - 0}{h} = \lim_{h\to 0} \frac{h^{1/3}}{h}$$

$$= \lim_{h \to 0} h^{-2/3}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt[3]{h^2}}$$

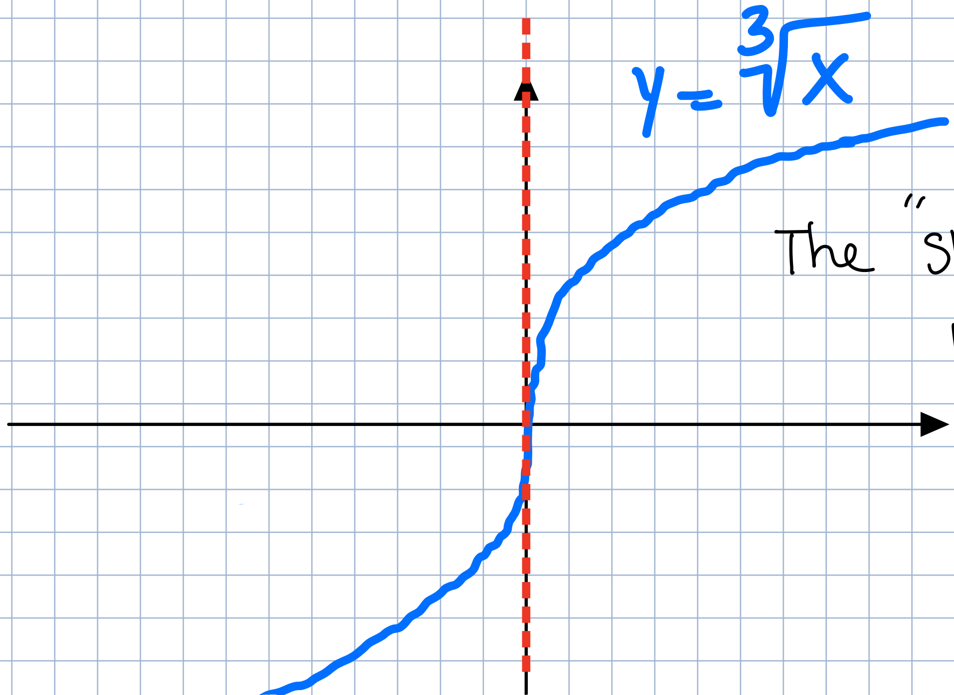
LH:

$$\lim_{h \to 0^-} \frac{1}{\sqrt[3]{h^2}} \rightarrow \frac{1}{0^+} = \infty$$

RH:

$$\lim_{h \to 0^+} \frac{1}{\sqrt[3]{h^2}} \rightarrow \frac{1}{0^+} = \infty$$

Thus, $f'(0)$ DNE




The "slope of the tangent line" doesn't make sense!