- Syllabus

Real Numbers: $\mathbb{R} = \{\text{set of all real \#'s}\}$

working def^n : "numbers that can be expressed as a decimal"

Integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Rational Numbers: $\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \text{ are integers} \right\}$

Notation: $x \in \mathbb{R}$ **means that**

"x is an element of $\mathbb{R}$"

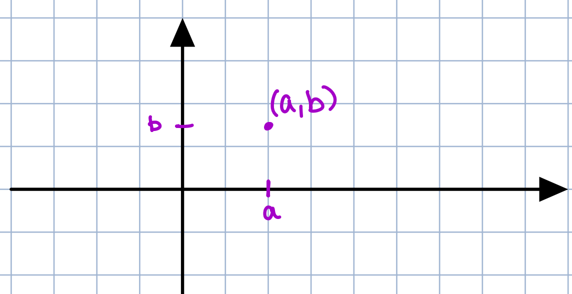
$x \in \mathbb{Q}$ **means that**

"x is an element of $\mathbb{Q}$"

Cartesian Plane:

$\mathbb{R}^2$ **pronounced** "R-two"

$= \{(a,b) \mid a, b \in \mathbb{R}\}$



= collection of all pts in the plane

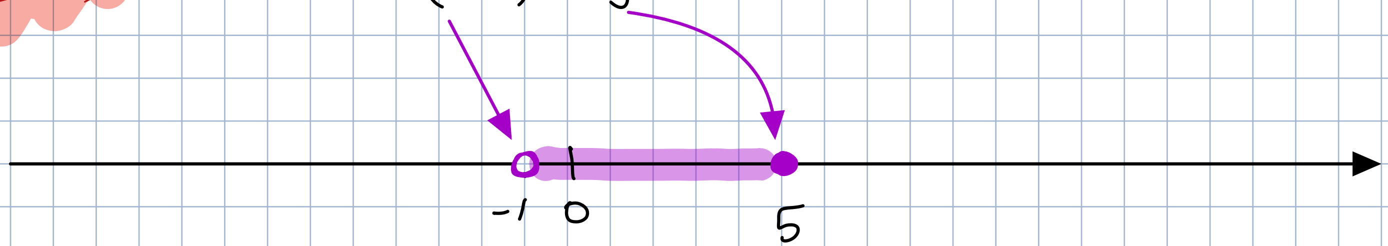
Intervals:

Let $a, b \in \mathbb{R}$ with $a<b$.

- $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$ "open"
- $[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ "closed"
- $(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
- $[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$

Visually: We often times express intervals using the real line.

Ex: Denote $(-1, 5]$ on real line:



Section 2: Functions

Def^n

A real-valued function f with domain D_f (or sometimes $\mathbf{Dom(f)}$) is an rule so that for every $x \in D_f$, there is exactly one y s.t. $f(x) = y$.

$\rightarrow$ Domain = "stuff that you're allowed to plug in"

Natural Domain: Set of all values for which a function is defined. (usually the one we mean)

For example, take $g(x) = \pi x^2$. This has nat. domain:

$$D_g = (-\infty, \infty)$$

What if we look at $A(r) = \pi r^2$? (area of circle)

Here, r = radius. So, r clearly can't be negative.

$r \geq 0 \longrightarrow$ So, in this specific case:

$$\mathrm{Dom}_A = [0, \infty)$$

List of Useful Functions:

- Polynomials:

$$\boxed{p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \cancel{x^{0}},}$$

where a_i's $\in \mathbb{R}$, $a_n \neq 0$

polynomial has degree $n \in \{0, 1, 2, \ldots\}$

Quick Examples:

$$p(x) = 2x^2 + 7$$

$$p(x) = 68x^{20} + x^3 + 3x^2 - 1$$

$$p(x) = \pi$$ (yes! constant functions are polynomials)

* **Any polynomial p(x) has a domain** *

$$D_p = (-\infty, \infty)$$

- Rational Functions:

$$\boxed{r(x) = \frac{p(x)}{q(x)}}$$, where $p(x), q(x)$ are polynomials

Quick Examples:

$$r(x) = \frac{2x^2 + 1}{8x^3 + 1}$$

$$r(x) = \frac{\pi}{3x^3 + 2x - 8}$$

$$r(x) = \frac{p(x)}{1}$$ (poly's are also rational fcn's. Kinda like how 5 is also a rational # i.e. $5 = 5/1$)

* Any rational function $r(x) = \frac{p(x)}{q(x)}$ *

has domain

$$D_r = \{x \in \mathbb{R} \mid q(x) \neq 0\}$$

- Radical Functions:

$$\boxed{f(x) = x^{1/m} = \sqrt[m]{x}} \quad m \in \{1, 2, 3, \ldots\}$$

The domain of these depends on m.

$\boxed{m = \text{even}}$ $\longrightarrow$ $f(x) = \sqrt{x}$

$f(x) = \sqrt[6]{x}$

* **The domain of $f(x) = x^{1/m}$ (if $m = \text{even}$)** *

is $D_f = [0, \infty)$

$\boxed{m = \text{odd}}$ $\longrightarrow$ $f(x) = x^{1/3} = \sqrt[3]{x}$

* **The domain of $f(x) = x^{1/m}$ (if $m = \text{odd}$)** *

is $D_f = (-\infty, \infty)$

Notation: Evaluation of f at a: $f(a)$

Let $f(x) = x^2 + 2$ $\quad a=5$

$\longrightarrow f(5) = 5^2 + 2$ easy!

How about evaluation at $a = x+5$?

$f(x+5) = (x+5)^2 + 2$ just as easy!

This is a composition!

Let $g(x) = 3x + \sqrt{x}$.

What is $f(g(x)) = (f \circ g)(x)$?

$= (g(x))^2 + 2$

$= (3x + \sqrt{x})^2 + 2$

What about $(g \circ f)(x)$?

$g(f(x)) = 3f(x) + \sqrt{f(x)}$

$= 3(x^2+2) + \sqrt{x^2+2}$

Ex: Find the domain

(a) $f(x) = \dfrac{2x^2+3}{x-1}$

(b) $g(x) = \sqrt{5-x}$

(c) $h(x) = \dfrac{x^2-4}{x+2}$

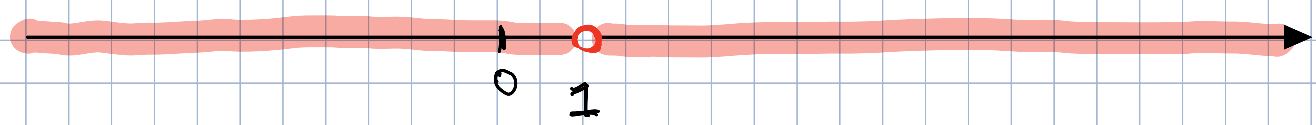
(a) f is rational! Find where denom $=0$

bad pts

Denominator : $x-1=0$

$x=1$

Domain : everything except $x=1$.

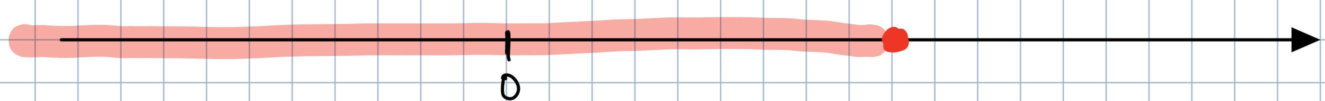


$D_f = (-\infty, 1) \cup (1, \infty)$

(b) g is radical! $g(x) = (5-x)^{1/2}$

Since the radical is even : $5-x \geq 0$

$\rightarrow 5 \geq x$



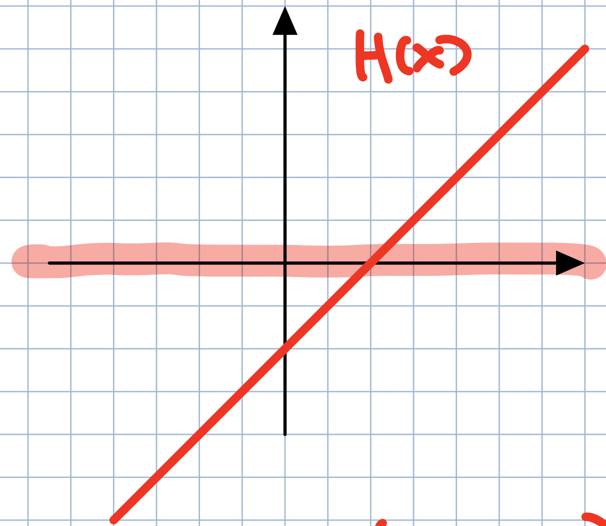
$D_g = (-\infty, 5]$

$$h(x) = \frac{x^2-4}{x+2}$$

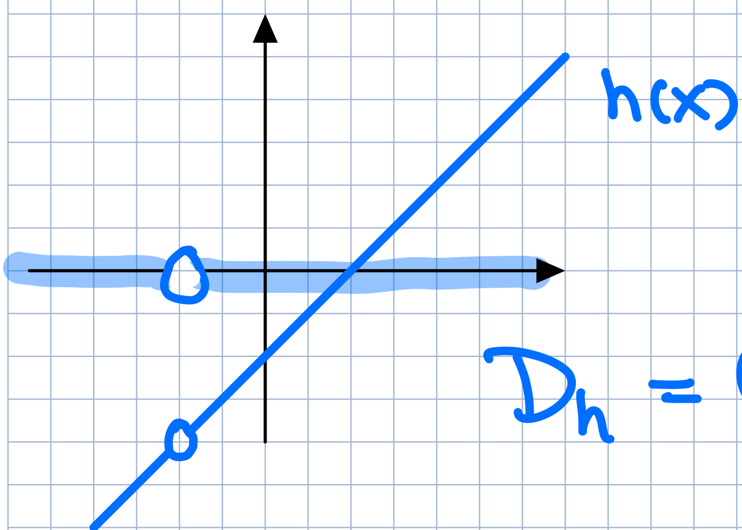
(c) $h(x)$ is a rational fcn!

<u>Note:</u> $h(x) = \dfrac{(x+2)(x-2)}{x+2}$

$$H(x) = (x-2)$$



$D_H = (-\infty, \infty)$



$D_h = (-\infty, -2) \cup (-2, \infty)$

Find domain: $h(x) = \dfrac{x^2-4}{x+2}$

<u>Note:</u>

$$h(x) = \frac{(x+2)(x-2)}{(x+2)}$$

$D_h = (-\infty, -2) \cup (-2, \infty)$

$$H(x) = (x-2)$$

$D_H = (-\infty, \infty)$

So, $h(x) = H(x)$ if $x \neq -2$

* Cancelling terms affects domain! *

Ways to Produce New Functions:

• Piecewise Function

→ Kinda like a "frankenstein" function

Defines a function using different "pieces" of other functions.

For example, consider

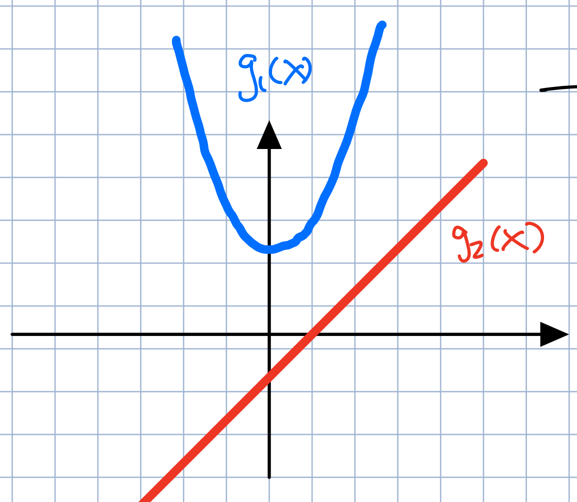
$$f(x) = \begin{cases} x^2+2, & \text{if } x \leq 0 \\ x-1, & \text{if } x > 0 \end{cases}$$

where to chop pieces that make f.

functions to be used/chopped-up.

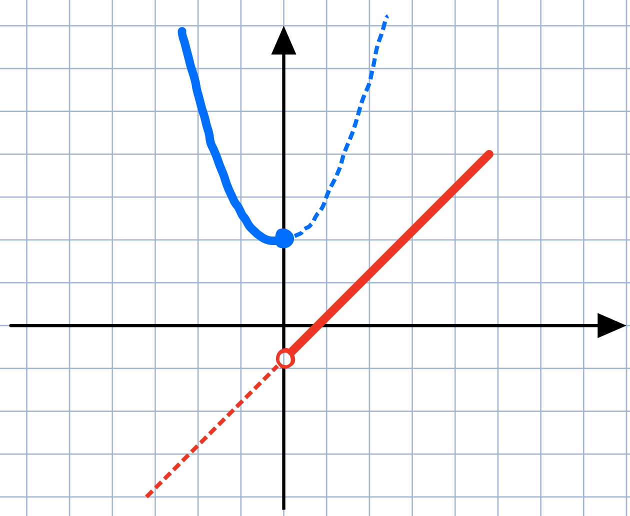
Let's take a look at the individual functions:

$g_1(x) = x^2+2$ $\qquad$ $g_2(x) = x-1$



→ functions we're going to use to make $f(x)$.

What does f(x) look like?



Ex: A car company charges \$270 to rent a car. The first 300 miles are free. If more than 300 miles are driven, then the company charges an additional ¢55 per mile.

Want: Cost function based on mileage.

$$C(m) = \begin{cases} 270, & \text{if } m \le 300 \\ 270 + (m-300)(0.55), & \text{if } m > 300 \end{cases}$$

- **More ways: Basic Algebraic Operations** $(+,-,\times,\div)$

Let $f(x) = \dfrac{x}{x^2+3}$, $g(x) = \sqrt{x}$

(a) $(f \pm g)(x) = f(x) \pm g(x)$

$$= \frac{x}{x^2+3} \pm \sqrt{x}$$

(b) $(f \cdot g)(x) = \dfrac{x\sqrt{x}}{x^2+3} = \dfrac{x^{3/2}}{x^2+3}$

(c) $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \left(\dfrac{x}{x^2+3}\right)\dfrac{1}{\sqrt{x}}$

$$= \frac{\sqrt{x}}{x^2+3}$$

(d) $(f \circ g)(x) = f(g(x))$

$$= \frac{g(x)}{g(x)^2+3} = \frac{\sqrt{x}}{(\sqrt{x})^2+3}$$

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Note: $y = x^2 \geq 0$ and $y = \sqrt{x} \geq 0$.
So their compositions gotta also be ≥ 0!

Common Mistake:

$$\sqrt{x}^{\,2} = x$$ Needs to be ≥ 0.

Instead:

$$\sqrt{x^2} = |x| = \sqrt{x}^{\,2}$$

$$= \frac{\sqrt{x}}{|x| + 3}$$

Exponents Rules:

Let $a \in \mathbb{R}$.

1. $a^0 = 1$, $a \neq 0$

2. $a^{-n} = \dfrac{1}{a^n}$, $a \neq 0$

3. $a^{m/n} = \sqrt[n]{a^m}$, if $n =$ even, $a \geq 0$
$= \left(\sqrt[n]{a}\right)^m$

4. $a^m \cdot a^n = a^{m+n}$, where defined

5. $\dfrac{a^m}{a^n} = a^{m-n}$, where defined

6. $(a^m)^n = a^{mn}$, where defined