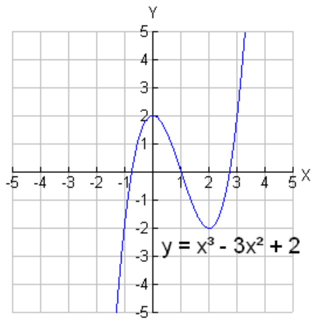
Revised Summary of Sections 15-18

This unit shows how to use analytical methods of calculus to sketch graphs and interpret them.

Polynomials functions are *differentiable* at all values of their domain (the real numbers). Recall that **critical numbers*** of a function are those $x = c$ where either $f'(c) = 0$ or DNE.



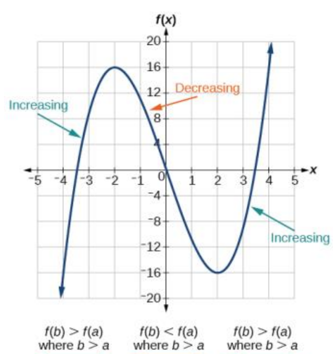
Polynomial *functions are ideal for investigating intervals of increase and decrease, extremes, concavity and points of inflection.*

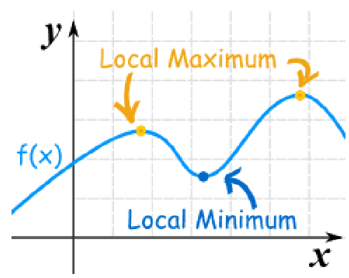
> * "Critical number" is a preferred term for those x where $f'(x) = 0$ or DNE. "Critical value" is a better term for $f(c)$. Yet, the c are often called "critical points", even though a "point" would technically be $(c, f(c))$.

Definitions

A function $f(x)$ is (strictly) **increasing** on an interval I if for each a, b in I, when $a < b$, $f(a) < f(b)$.

A function $f(x)$ is (strictly) **decreasing** on an interval I if for each a, b in I, when $a < b$, $f(a) > f(b)$.





On an interval where $f(x)$ is increasing, $f'(x) > 0$; on an interval where $f(x)$ is decreasing, $f'(x) < 0$.

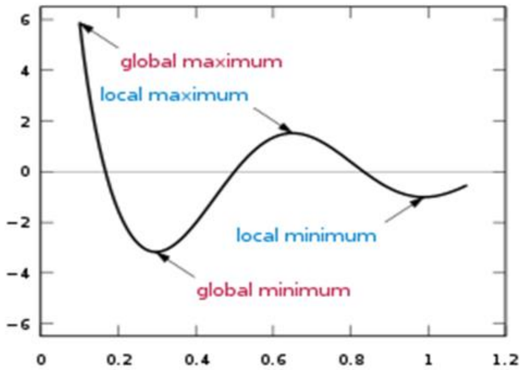
$f(x)$ has a **local (relative) maximum** at a if $f(x) \leq f(a)$ for all x in an arbitrarily small interval (called an ε-neighborhood) of a.

$f(x)$ has a **local (relative) minimum** at a if $f(x) \geq f(a)$ for all x in an an ε-neighborhood of a.

[Note: A *constant* function on I has both a local a max and a local min on that interval.]

Local maximum and minimum values of a function are called **local extremes** of the function.

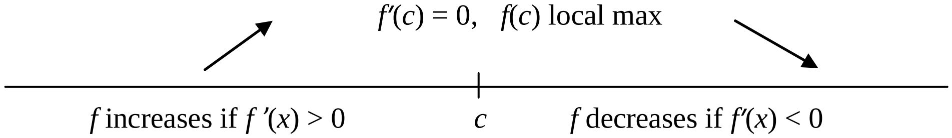
M is a **global (absolute) maximum** if for every x in I, $f(x) \leq M$; m is a **global (absolute) minimum** if for every x in I, $f(x) \geq m$.



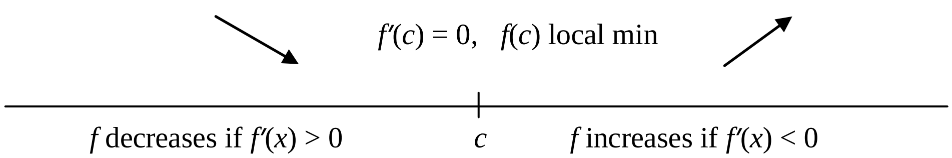
First Derivative Test

Visually, $f(c)$ is a local max if f is increasing when $x < c$ and decreasing when $x > c$. $f(c)$ is a local min if f is decreasing when $x < c$ and increasing when $x > c$.

Analytically, $f'(x)$ changes sign from positive to negative on either side of c when $f(c)$ is a local max.



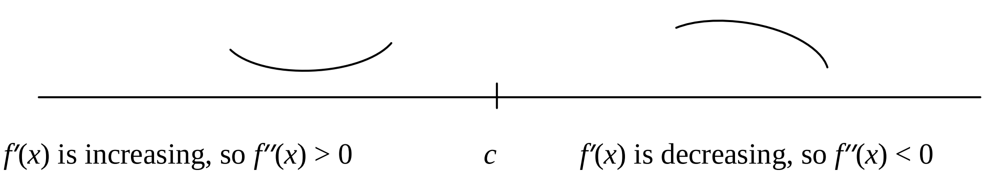
And $f'(x)$ changes sign from negative to positive on either side of c when $f(c)$ is a local min.



Thus, the FDT shows whether $f(c)$ is a local max or min by the sign of $f'(c)$ on either side of c.

Second Derivative Test

Intervals of the graph where the shape is roughly similar to a cup **concave up**.
Intervals of the graph where the shape is roughly similar to a frown **concave down**.

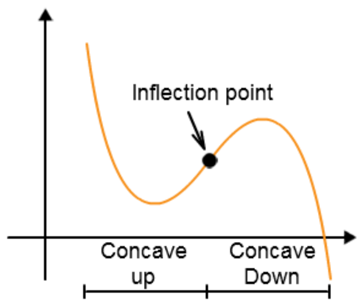


$f(x)$ is concave up on an interval where $f'(x)$ is increasing because the progress of the slope of the tangent is to increase. Thus, $f''(x) > 0$ on "concave up intervals."

$f(x)$ is concave down on an interval where $f'(x)$ is decreasing because the progress of the slope of the tangent is to decrease. Thus, $f''(x) < 0$ on "concave down intervals."

The SDT states that if $f'(c) = 0$ and $f''(c) > 0$, then $f(c)$ is a local min because the graph is concave up there. Likewise, if $f'(c) = 0$ and $f''(c) < 0$, then $f(c)$ is a local max because the graph is concave down there.

An **inflection point** is a point where the graph's concavity changes.



If $f''(c) = 0$, $f(c)$ *could* be an inflection point, but not necessarily. Note that $f(x) = x^4$ has both a first and second derivative = 0 at $x = 0$, has an $f(0)$ is a minimum of the function. (See examples below of basic functions and their critical and inflection points.)

To decide what the situation is in the case where $f''(x) = 0$, we do the following:

EITHER

1. Resort to the *first derivative test*, checking values on either side of c to see if $f'(c)$ changes sign.

   If it does, we have a local extreme at $x = c$.

   If it doesn't, we have an inflection point at $x = c$.

OR

2. Stay with the *second derivative test*, testing values either side of c to see if $f''(x)$ changes sign.

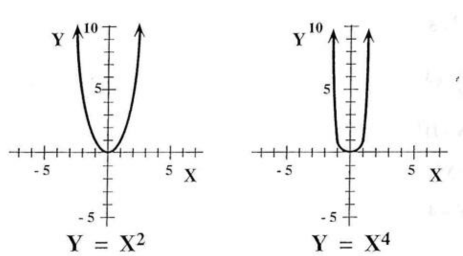
   If $f''(x)$ *changes sign* at c, then c is an *inflection* point, since concavity has changed.

   If $f''(x) > 0$ on *both sides* then $x = c$ is a local min.

   If $f''(x) < 0$ on *both* sides, then $x = c$ is a local max.

The following examples are illustrative because the functions are so simple to inspect through a sketch.

Example 1 $\qquad f(x) = x^2$ (parabola) $\qquad f(x) = x^4$ (quartic)



Analyzing the parabola is easy:

FDT: $f'(x) = 2x = 0$ at $x = 0$. $f'(-1) = -2 < 0$; f is decreasing left of zero. $f'(1) = 2 > 0$; f is increasing right of zero.

$f(0)$ is a local min.

SDT: $f''(x) = 2 > 0$ for all x, so f is concave up everywhere. Thus, $f(0)$ is a local min, as the graph shows.

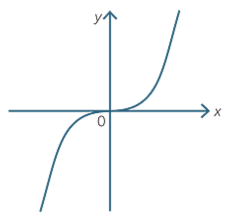
Now the quartic:

$f'(x) = 4x^3 = 0$ at $x = 0$; $f''(x) = 12x^2 = 0$ at $x = 0$, also. What kind of critical point, then, is $c = 0$?
There are two ways to find out:

FDT: Checking values into $f'(x)$ on either side of 0, $f'(-1) = 4(-1)^3 = -4$ and $f'(1) = 4(1)^3 = 4$. Because f' changes sign, negative to positive, $c = 0$ is a local min.

SDT: Checking values of $f''(x)$ on either side of 0, $f''(-1) = 12(-1)^2 = 12$ and $f''(1) = 12(1)^2 = 12$. No change in sign, f'' is positive on either side, so the function is concave up, and $c = 0$ is a local min, as the graph shows.

Example 2 $f(x) = x^3$ (cubic)

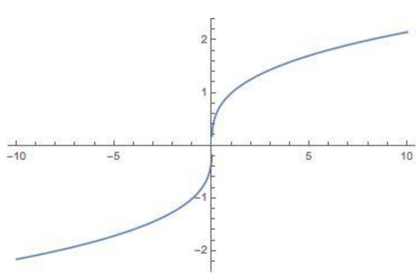


$f'(x) = 3x^2 = 0$ at $x = 0$; $f''(x) = 6x = 0$ at $x = 0$ also. So, what kind of critical point is $c = 0$? There are two ways to find out:

FDT: Checking values of $f'(x)$ on either side of 0, $f'(-1) = 3(-1)^2 = 3$, and $f'(1) = 3(1)^2 = 3$. Thus, the function is increasing on either side of $c = 0$, so c is an inflection point, as the graph shows.

SDT: Checking values of $f''(x)$ on either side of 0, $f''(-1) = 6(-1) = -6 < 0$. $f''(1) = 6(1) = 6 > 0$. The change in sign indicates the graph is concave down to the left of $x = 0$ and concave up to the right of it, at $x = 0$ is an inflection.

Example 3 $f(x) = x^{1/3}$ (cube root)



$$f'(x) = \frac{1}{3x^{2/3}}$$

We see that $f'(0)$ does not exist (division by zero). In the DNE sense, $c = 0$ is a critical value of the function. (You see from the graph that the tangent line to the function at $x = 0$ is a vertical line.)

FDT: The function is increasing everywhere, as is easily seen in the graph; algebraically, $f'(x)$ is *positive* everywhere it is defined, as $x^{2/3}$ is the square of a cube root. Check $f'(-1)$ and $f'(1)$.

Thus, by the first derivative test, the function is everywhere increasing. There is no local max or min.

Is $x = 0$ an inflection point? $$f''(x) = -\frac{2}{9x^{5/3}}$$

SDT: $f''(x)$ DNE at $x = 0$, for the same reason (division by zero). Checking values of $f''(x)$ on either side of 0:

$f''(-1) = -\frac{2}{9(-1)^{5/3}} = \frac{2}{9} > 0$ (concave up).

$f''(1) = -\frac{2}{9} < 0$ (concave down). $x = 0$ is a point of inflection, as the graph shows.