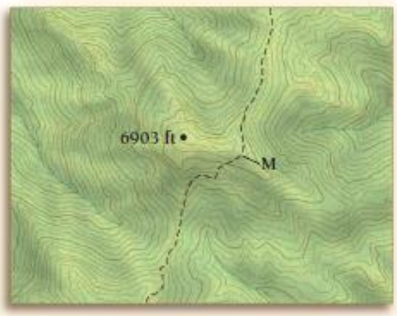
Constrained Optimization

In Section 6.3, we discussed a method for determining maximum and minimum values on a surface represented by a two-variable function $z = f(x, y)$. If restrictions are placed on the input variables x and y, we can determine the maximum and minimum values on the surface subject to the restrictions. This process is called **constrained optimization**.

Path Constraints: Lagrange Multipliers

Imagine that you are hiking up a mountain. If there are no constraints on your movement, you may seek out the mountain's summit—its "maximum point." The figure at the right shows a relief map of a mountaintop; its unconstrained maximum point occurs at the •, labeled with a spot elevation of 6903 ft. A hiking trail, marked as a black dashed line, bypasses the summit. If you were constrained to this hiking path, you could not reach the summit. You could, however, achieve a maximum elevation along the path. This constrained maximum point is approximated at M.



(*Source*: USGS maps at www.mytopo.com.)

In many applications modeled by two-variable functions, constraints on the input variables are necessary. If the input variables are related to one another by an equation, it is called a **constraint**.

Let's return to a problem we considered in Chapter 2: A hobby store has 20 ft of fencing to fence off a rectangular electric-train area in one corner of its display room. The two sides up against the wall require no fence. What dimensions of the rectangle will maximize the area?

We maximize the function

$$A = xy$$

subject to the condition, or *constraint*, $x + y = 20$. Note that A is a function of two variables.

When we solved this earlier, we first solved the constraint for y:

$$y = 20 - x.$$

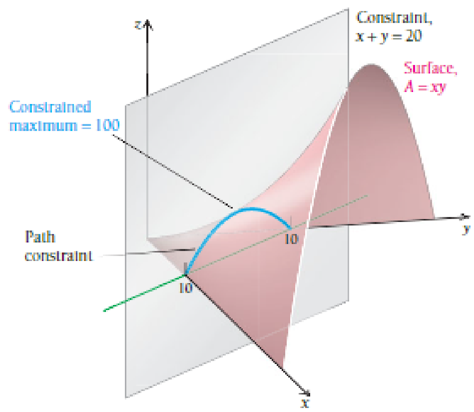
We then substituted $20 - x$ for y to obtain

$$A(x, y) = x(20 - x)$$
$$= 20x - x^2,$$

which is a function of one variable. Next, we found a maximum value using Maximum–Minimum Principle 1 (see Section 2.4). By itself, the function of two variables

$$A(x, y) = xy$$

has no maximum value. This can be checked using the D-test. With the constraint $x + y = 20$, however, the function does have a maximum. We see this in the following graph.



It may be quite difficult to solve a constraint for one variable. The method outlined below allows us to proceed without doing so.

The Method of Lagrange Multipliers

To find a maximum or minimum value of a function $f(x, y)$ subject to the constraint $g(x, y) = 0$:

1. Form a new function, called the **Lagrange function**:

   $$F(x, y, \lambda) = f(x, y) - \lambda g(x, y).$$

   The variable λ (lambda) is called a **Lagrange multiplier**.
2. Find the first partial derivatives F_x, F_y, and F_λ.
3. Solve the system

   $$F_x = 0, \quad F_y = 0, \quad \text{and} \quad F_\lambda = 0.$$

   Let (a, b, λ) represent a solution of this system. We normally must determine whether (a, b) yields a maximum or minimum of the function f. For the problems in this text, we will specify that a maximum or minimum exists.

The method of Lagrange multipliers can be extended to functions of three (or more) variables.

We can illustrate the method of Lagrange multipliers by resolving the electric-train area problem.

EXAMPLE 1 Find the maximum value of

$$A(x, y) = xy$$

subject to the constraint $x + y = 20$.

Solution Note first that $x + y = 20$ is equivalent to $x + y - 20 = 0$.

1. We form the Lagrange function F, given by

   $$F(x, y, \lambda) = xy - \lambda \cdot (x + y - 20).$$

2. We find the first partial derivatives:

$$F_x = y - \lambda,$$
$$F_y = x - \lambda,$$
$$F_\lambda = -(x + y - 20).$$

3. We set each derivative equal to 0 and solve the resulting system:

$$y - \lambda = 0, \quad (1)$$
$$x - \lambda = 0, \quad (2)$$
$$-(x + y - 20) = 0, \quad \text{or} \quad x + y - 20 = 0. \quad (3)$$

From equations (1) and (2), it follows that

$$x = y = \lambda.$$

Substituting x for y in equation (3), we get

$$x + x - 20 = 0$$
$$2x = 20$$
$$x = 10.$$

Thus, $y = x = 10$. The maximum value of A subject to the constraint occurs at $(10, 10)$ and is

$$A(10, 10) = 10 \cdot 10 = 100.$$

❮ Quick Check 1

EXAMPLE 2 Find the maximum value of

$$f(x, y) = 3xy$$

subject to the constraint

$$2x + y = 8.$$

Note: f might be interpreted, for example, as a production function with a budget constraint $2x + y = 8$.

Solution Note that first we express $2x + y = 8$ as $2x + y - 8 = 0$.

1. We form the Lagrange function F, given by

$$F(x, y, \lambda) = 3xy - \lambda(2x + y - 8).$$

2. We find the first partial derivatives:

$$F_x = 3y - 2\lambda,$$
$$F_y = 3x - \lambda,$$
$$F_\lambda = -(2x + y - 8).$$

3. We set each derivative equal to 0 and solve the resulting system:

$$3y - 2\lambda = 0, \quad (1)$$
$$3x - \lambda = 0, \quad (2)$$
$$-(2x + y - 8) = 0, \quad \text{or} \quad 2x + y - 8 = 0. \quad (3)$$

Solving equation (2) for λ, we get

$$\lambda = 3x.$$

Substituting in equation (1) for λ, we get

$$3y - 2 \cdot 3x = 0, \quad \text{or} \quad 3y = 6x, \quad \text{or} \quad y = 2x. \qquad (4)$$

Substituting $2x$ for y in equation (3), we get

$$2x + 2x - 8 = 0$$
$$4x = 8$$
$$x = 2.$$

Then, using equation (4), we have

$$y = 2 \cdot 2 = 4.$$

The maximum value of f subject to the constraint occurs at $(2, 4)$ and is

$$f(2, 4) = 3 \cdot 2 \cdot 4 = 24.$$

Quick Check 2

EXAMPLE 3 **Business: The Beverage-Can Problem.** The standard beverage can holds 12 fl. oz, or has a volume of 21.66 in^3. What dimensions yield the minimum surface area? Find the minimum surface area. (Assume that the shape of the can is a right circular cylinder.)

Solution We want to minimize the function s, given by

$$s(h, r) = 2\pi rh + 2\pi r^2$$

subject to the volume constraint

$$\pi r^2 h = 21.66,$$

or $$\pi r^2 h - 21.66 = 0.$$

Note that s does not have a minimum without the constraint.

1. We form the Lagrange function S, given by

$$S(h, r, \lambda) = 2\pi rh + 2\pi r^2 - \lambda(\pi r^2 h - 21.66).$$

2. We find the first partial derivatives:

$$\frac{\partial S}{\partial h} = 2\pi r - \lambda \pi r^2,$$

$$\frac{\partial S}{\partial r} = 2\pi h + 4\pi r - 2\lambda \pi rh,$$

$$\frac{\partial S}{\partial \lambda} = -(\pi r^2 h - 21.66).$$

3. We set each derivative equal to 0 and solve the resulting system:

$$2\pi r - \lambda \pi r^2 = 0, \qquad (1)$$
$$2\pi h + 4\pi r - 2\lambda \pi rh = 0, \qquad (2)$$
$$-(\pi r^2 h - 21.66) = 0, \quad \text{or} \quad \pi r^2 h - 21.66 = 0. \qquad (3)$$

Note that, since π is a constant, we can solve equation (1) for r:

$$\pi r(2 - \lambda r) = 0$$
$$\pi r = 0 \quad \text{or} \quad 2 - \lambda r = 0$$
$$r = 0 \quad \text{or} \quad r = \frac{2}{\lambda}. \qquad \text{We assume } \lambda \neq 0.$$

Since $r = 0$ cannot be a solution to the original problem, we continue by substituting $2/\lambda$ for r in equation (2):

$$2\pi h + 4\pi \cdot \frac{2}{\lambda} - 2\lambda\pi \cdot \frac{2}{\lambda} \cdot h = 0$$

$$2\pi h + \frac{8\pi}{\lambda} - 4\pi h = 0$$

$$\frac{8\pi}{\lambda} - 2\pi h = 0$$

$$-2\pi h = -\frac{8\pi}{\lambda},$$

so

$$h = \frac{4}{\lambda}.$$

Since $h = 4/\lambda$ and $r = 2/\lambda$, it follows that $h = 2r$. Substituting $2r$ for h in equation (3) yields

$$\pi r^2(2r) - 21.66 = 0$$

$$2\pi r^3 - 21.66 = 0$$

$$2\pi r^3 = 21.66$$

$$\pi r^3 = 10.83$$

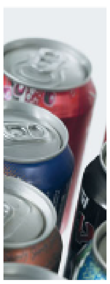
$$r^3 = \frac{10.83}{\pi}$$

$$r = \sqrt[3]{\frac{10.83}{\pi}} \approx 1.51 \text{ in.}$$

Thus, when $r = 1.51$ in., we have $h = 3.02$ in. The surface area is then a minimum and is approximately

$$2\pi(1.51)(3.02) + 2\pi(1.51)^2, \quad \text{or about } 42.98 \text{ in}^2.$$

《 Quick Check 3



The actual dimensions of a standard-sized 12-oz beverage can are $r = 1.25$ in. and $h = 4.875$ in. A natural question arising from the solution of Example 3 is, "Why don't beverage companies make cans using the dimensions found in that example?" To do this would mean an enormous cost for retooling. New can-making machines and new beverage-filling machines would have to be designed and purchased. Vending machines would no longer be the correct size. A partial response to the desire to save aluminum has been found in recycling and in manufacturing cans with bevelled edges. These cans require less aluminum. As a result of many engineering advances, the amount of aluminum required to make 1000 cans has been reduced over the years from 36.5 lb to 28.1 lb. Consumer preference is another very important factor affecting the shape of the can. Market research has shown that a can with the dimensions found in Example 3 is not as comfortable to hold and might not be accepted by consumers.

Closed and Bounded Regions: The Extreme-Value Theorem

In Examples 1, 2, and 3, all the constraints were given as equations. Constraints may also be stated as inequalities. If there are multiple constraints on the input variables x and y, these may be plotted on the xy-plane to form a *region of feasibility*, which contains the x and y values that satisfy all the constraints simultaneously. If the constraints form a closed and bounded region (*closed* meaning it includes the boundaries, and *bounded* meaning it has finite area, with no portions tending to infinity), then the Extreme-Value Theorem can be adapted for the two-variable function.

THEOREM **Extreme-Value Theorem for Two-Variable Functions**

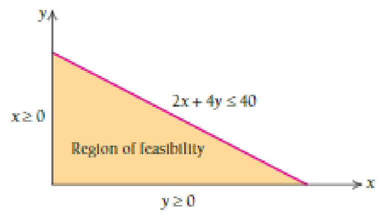
If $f(x, y)$ is continuous for all (x, y) within a region of feasibility that is closed and bounded, then f is guaranteed to have both an absolute maximum value and an absolute minimum value.

Critical points may occur at a vertex, along a boundary, or in the interior. Therefore, all these parts of a region must be checked for critical points.

EXAMPLE 4 **Business: Maximizing Revenue.** Kim likes to create stylish tee shirts, one style with a script x on the front and another with a script y on the front. She sells them to her math students as a fundraiser for her favorite charity. Kim determines that her weekly revenue is modeled by the two-variable function

$$R(x, y) = -x^2 - xy - y^2 + 20x + 22y - 25,$$

where x is the number of x-shirts sold, and y is the number of y-shirts sold. Kim spends 2 hr working on each x-shirt and 4 hr working on each y-shirt, and she works no more than 40 hr per week on this project. How many of each style should she produce in order to maximize her weekly revenue? Assume $x \geq 0$ and $y \geq 0$; in other words, she cannot produce negative quantities of the tee shirts.



Solution The number of hours Kim works per week is a constraint: $2x + 4y \leq 40$. We write the inequality with a less-than-or-equal-to sign since she may not work the full 40 hr. Along with the constraints $x \geq 0$ and $y \geq 0$, this constraint allows us to sketch the region of feasibility. This is a closed and bounded region. Since the revenue function R is continuous for all x and all y, the Extreme-Value Theorem guarantees an absolute minimum and an absolute maximum point. In this example, we are interested in the absolute maximum (revenue).

We determine that the three vertex points of the region are $(0, 0, -25)$, $(20, 0, -25)$, and $(0, 10, 95)$. These are all critical points.

Next, we check the interior of the region. We find the partial derivatives of R with respect to x and with respect to y:

$$R_x = -2x - y + 20,$$
$$R_y = -x - 2y + 22.$$

Setting these expressions equal to 0, we solve the system for x and y:

$$-2x - y + 20 = 0,$$
$$-x - 2y + 22 = 0;$$

or $\quad 2x + y = 20,$

$x + 2y = 22.$ After simplification

The system is solved when $x = 6$ and $y = 8$. However, this point is outside the region of feasibility; Kim would have to work $2(6) + 4(8) = 44$ hr, which is not allowed under the given constraint. Therefore, this solution must be ignored. (We address this issue at the end of this example.)

The boundaries of the region must also be checked for possible critical points:

- To check along the y-axis, we substitute $x = 0$ into the revenue function:

$$R(0, y) = -y^2 + 22y - 25.$$

The derivative is $R_y = -2y + 22$. Setting this expression equal to 0, we obtain $y = 11$. However, this is outside the region of feasibility and is ignored.

- To check along the x-axis, we substitute $y = 0$ into the revenue function:

$$R(x, 0) = -x^2 + 20x - 25.$$

The derivative is $R_x = -2x + 20$. Setting this expression equal to 0, we get $x = 10$. This is a feasible solution, and thus is a critical value. The critical point is $(10, 0, 75)$.

- To check along the line $2x + 4y = 40$, we use the method of Lagrange multipliers to determine possible critical values. The constraint is written as $2x + 4y - 40 = 0$, and the Lagrange function is formed:

$$L(x, y, \lambda) = -x^2 - xy - y^2 + 20x + 22y - 25 - \lambda(2x + 4y - 40).$$

Its first partial derivatives are as follows:

$$L_x = -2x - y + 20 - 2\lambda,$$
$$L_y = -x - 2y + 22 - 4\lambda,$$
$$L_\lambda = -2x - 4y + 40.$$

We set each partial derivative equal to 0:

$$-2x - y + 20 - 2\lambda = 0 \quad (1)$$
$$-x - 2y + 22 - 4\lambda = 0 \quad (2)$$
$$-2x - 4y + 40 = 0 \quad (3)$$

We solve equations (1) and (2) for λ:

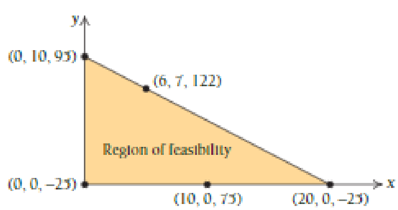
$$\lambda = -x - \tfrac{1}{2}y + 10 \quad \text{and} \quad \lambda = -\tfrac{1}{4}x - \tfrac{1}{2}y + \tfrac{11}{2}.$$

Equating the right-hand sides of these two equations gives us a single equation in terms of x and y. Note that the $-\frac{1}{2}y$ terms cancel (sum to zero):

$$-x - \tfrac{1}{2}y + 10 = -\tfrac{1}{4}x - \tfrac{1}{2}y + \tfrac{11}{2}$$
$$-\tfrac{3}{4}x = -\tfrac{9}{2}$$
$$x = 6.$$

We now substitute $x = 6$ into the constraint, $2x + 4y = 40$, to determine y:

$$2(6) + 4y = 40$$
$$12 + 4y = 40$$
$$4y = 28$$
$$y = 7.$$



This is a feasible solution. Therefore, $x = 6$ and $y = 7$ yield a critical point: $(6, 7, 122)$. In the graph to the right, all critical points (with their revenue values) are plotted on the region of feasibility. Therefore, Kim should produce 6 of the x-shirts and 7 of the y-shirts to maximize her weekly revenue at \$122. If there were no constraints, the maximum weekly revenue would occur at $x = 6$ and $y = 8$, for a total of \$123. Kim might think that working an extra 4 hr for one more dollar of revenue is not worth it.

Quick Check 4

Section Summary

- If input variables x and y for a function $f(x, y)$ are related by another equation, that equation is a *constraint*.
- *Constrained optimization* is a method of determining maximum and minimum points on a surface represented by $z = f(x, y)$, subject to given restrictions (constraints) on the input variables x and y.
- The *method of Lagrange multipliers* allows us to find a maximum or minimum value of a function $f(x, y)$ subject to the constraint $g(x, y) = 0$.
- If the constraints are inequalities, the set of points that satisfy all the constraints simultaneously is called the *region of feasibility*.
- If the region of feasibility is closed and bounded and the surface $z = f(x, y)$ is continuous over the region, then the *Extreme-Value Theorem* guarantees that f will have both an absolute maximum and an absolute minimum value.
- Critical points may be located at vertices, along a boundary, or in the interior of a region of feasibility.

EXERCISE SET 6.5

Find the maximum value of f subject to the given constraint.

1. $f(x, y) = xy;\ \ 3x + y = 10$

2. $f(x, y) = 2xy;\ \ 4x + y = 16$

3. $f(x, y) = 4 - x^2 - y^2;\ \ x + 2y = 10$

4. $f(x, y) = 3 - x^2 - y^2;\ \ x + 6y = 37$

Find the minimum value of f subject to the given constraint.

5. $f(x, y) = x^2 + y^2;\ \ 2x + y = 10$

6. $f(x, y) = x^2 + y^2;\ \ x + 4y = 17$

7. $f(x, y) = 2y^2 - 6x^2;\ \ 2x + y = 4$

8. $f(x, y) = 2x^2 + y^2 - xy;\ \ x + y = 8$

9. $f(x, y, z) = x^2 + y^2 + z^2;\ \ y + 2x - z = 3$

10. $f(x, y, z) = x^2 + y^2 + z^2;\ \ x + y + z = 2$

Use the method of Lagrange multipliers to solve each of the following.

11. Of all numbers whose sum is 50, find the two that have the maximum product.

12. Of all numbers whose sum is 70, find the two that have the maximum product.

13. Of all numbers whose difference is 6, find the two that have the minimum product.

14. Of all numbers whose difference is 4, find the two that have the minimum product.

15. Of all points (x, y, z) that satisfy $x + 2y + 3z = 13$, find the one that minimizes

$$(x - 1)^2 + (y - 1)^2 + (z - 1)^2.$$

16. Of all points (x, y, z) that satisfy $3x + 4y + 2z = 52$, find the one that minimizes

$$(x - 1)^2 + (y - 4)^2 + (z - 2)^2.$$

APPLICATIONS

Business and Economics

17. Maximizing typing area. A standard piece of printer paper has a perimeter of 39 in. Find the dimensions of the paper that will give the most area. What is that area? Does standard $8\frac{1}{2} \times 11$ in. paper have maximum area?

18. Maximizing room area. A carpenter is building a rectangular room with a fixed perimeter of 80 ft. What are the dimensions of the largest room that can be built? What is its area?

19. Minimizing surface area. An oil drum of standard size has a volume of 200 gal, or 27 ft^3. What dimensions yield the minimum surface area? Find the minimum surface area.

Do these drums appear to be made in such a way as to minimize surface area?

20. Juice-can problem. A standard-sized juice can has a volume of 99 in^3. What dimensions yield the minimum surface area? Find the minimum surface area.

21. Maximizing total sales. The total sales, S, of a one-product firm are given by

$$S(L, M) = ML - L^2,$$

where M is the cost of materials and L is the cost of labor. Find the maximum value of this function subject to the budget constraint

$$M + L = 90.$$

22. Maximizing total sales. The total sales, S, of a one-product firm are given by

$$S(L, M) = ML - L^2,$$

where M is the cost of materials and L is the cost of labor. Find the maximum value of this function subject to the budget constraint

$$M + L = 70.$$

23. Minimizing construction costs. A company is planning to construct a warehouse whose interior volume is to be 252,000 ft^3. Construction costs per square foot are estimated to be as follows:

Walls:	\$3.00
Floor:	\$4.00
Ceiling:	\$3.00

a) The total cost of the building is a function $C(x, y, z)$, where x is the length, y is the width, and z is the height. Find a formula for $C(x, y, z)$.

b) What dimensions of the building will minimize the total cost? What is the minimum cost?

24. Minimizing the costs of container construction. A container company is going to construct a shipping crate of volume 12 ft^3 with a square bottom and top. The cost of the top and the sides is \$2 per square foot, and the cost for the bottom is \$3 per square foot. What dimensions will minimize the cost of the crate?

25. Minimizing total cost. Each unit of a product can be made on either machine A or machine B. The nature of the machines makes their cost functions differ:

Machine A: $C(x) = 10 + \dfrac{x^2}{6}$,

Machine B: $C(y) = 200 + \dfrac{y^3}{9}$.

Total cost is given by $C(x, y) = C(x) + C(y)$. How many units should be made on each machine in order to minimize total costs if $x + y = 10{,}100$ units are required?

In Exercises 26–29, find the absolute maximum and minimum values of each function, subject to the given constraints.

26. $f(x, y) = x^2 + y^2 - 2x - 2y$; $x \ge 0, y \ge 0, x \le 4$, and $y \le 3$

27. $g(x, y) = x^2 + 2y^2$; $-1 \le x \le 1$ and $-1 \le y \le 2$

28. $h(x, y) = x^2 + y^2 - 4x - 2y + 1$; $x \ge 0, y \ge 0$, and $x + 2y \le 5$

29. $k(x, y) = -x^2 - y^2 + 4x + 4y$; $0 \le x \le 3, y \ge 0$, and $x + y \le 6$

30. Business: maximizing profits with constraints. A manufacturer of decorative end tables produces two models, basic and large. Its weekly profit function is modeled by

$$P(x, y) = -x^2 - 2y^2 - xy + 140x + 210y - 4300,$$

where x is the number of basic models sold each week and y is the number of large models sold each week. The warehouse can hold at most 90 tables. Assume that x and y must be nonnegative. How many of each model of end table should be produced to maximize the weekly profit, and what will the maximum profit be?

31. Business: maximizing profits with constraints. A farmer has 300 acres on which to plant two crops, celery and lettuce. Each acre of celery costs \$250 to plant and tend, and each acre of lettuce costs \$300 to plant and tend. The farmer has \$81,000 available to cover these costs.

a) Suppose the farmer makes a profit of \$45 per acre of celery and \$50 per acre of lettuce. Write the profit function, determine how many acres of celery and lettuce he should plant to maximize profit, and state the maximum profit. (*Hint:* Since the graph of the profit function is a plane, you will not need to check the interior for possible critical points.)

b) Suppose the farmer's profit function is instead $P(x, y) = -x^2 - y^2 + 600y - 75{,}000$. Assuming the same constraints, how many acres of celery and lettuce should he plant to maximize profit, and what is that maximum profit?

SYNTHESIS

Find the indicated maximum or minimum values of f subject to the given constraint.

32. Minimum: $f(x, y) = xy$; $x^2 + y^2 = 9$

33. Minimum: $f(x, y) = 2x^2 + y^2 + 2xy + 3x + 2y$; $y^2 = x + 1$

34. Maximum: $f(x, y, z) = x + y + z$; $x^2 + y^2 + z^2 = 1$

35. Maximum: $f(x, y, z) = x^2y^2z^2$; $x^2 + y^2 + z^2 = 2$

36. Maximum: $f(x, y, z) = x + 2y - 2z$; $x^2 + y^2 + z^2 = 4$

37. Maximum: $f(x, y, z, t) = x + y + z + t$; $x^2 + y^2 + z^2 + t^2 = 1$

38. Minimum: $f(x, y, z) = x^2 + y^2 + z^2$; $x - 2y + 5z = 1$

39. Economics: the Law of Equimarginal Productivity. Suppose that $p(x, y)$ represents the production of a two-product firm. The company produces x units of the first product at a cost of c_1 each and y units of the second product at a cost of c_2 each. The budget constraint, B, is a constant given by

$$B = c_1x + c_2y.$$

Use the method of Lagrange multipliers to find the value of λ in terms of p_x, p_y, c_1, and c_2. The resulting equation holds for any production function p and is called the *Law of Equimarginal Productivity*.

40. Business: maximizing production. A computer company has the following Cobb–Douglas production function for a certain product:

$$p(x, y) = 800x^{3/4}y^{1/4},$$

where x is the labor, measured in dollars, and y is the capital, measured in dollars. Suppose that the company can make a total investment in labor and capital of \$1,000,000. How should it allocate the investment between labor and capital in order to maximize production?

41. Discuss the difference between solving a maximum–minimum problem using the method of Lagrange multipliers and the method of Section 6.3.

42. Write a brief report on the life and work of the mathematician Joseph Louis Lagrange (1736–1813).

TECHNOLOGY CONNECTION

43–50. Use a 3D graphics program to graph both equations in each of Exercises 1–8. Then visually check the results that you found analytically.

Answers to Quick Checks

1. $A = \dfrac{225}{2}$ at $x = 15, y = \dfrac{15}{2}$

2. $g = \dfrac{1}{10}$ at $x = \dfrac{3}{10}, y = -\dfrac{1}{10}$

3. $r \approx 4.3$ cm, $h \approx 8.6$ cm, $s \approx 348.73$ cm^2

4. $x = 5, y = 8$, maximum revenue = \$122