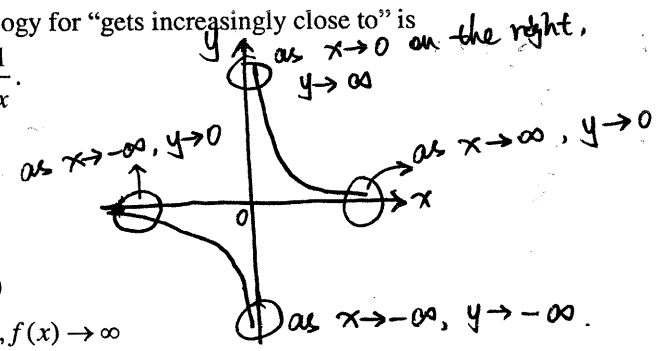
Ch 6 Summary of Rational Function Graphing

Rational functions have the form $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials (and $Q(x)$ is not the zero polynomial).

The domain of a rational function is the set of all x such that $Q(x) \neq 0$.

Graphs of rational functions are distinguished by their asymptotes, those lines which the graph of $f(x)$ gets increasingly close to, but never touches. The terminology for "gets increasingly close to" is "approaches," and is symbolized by "$\rightarrow$". Recall $f(x) = \dfrac{1}{x}$.



In its graph:

as $x \rightarrow \pm\infty, f(x) \rightarrow 0$

as $x \rightarrow 0$ on the right, $f(x) \rightarrow \infty$

as $x \rightarrow 0$ on the left, $f(x) \rightarrow -\infty$

The x-axis (that is, the line y = 0) is the horizontal asymptote (HA).
The y-axis (that is, the line $x = 0$) is the vertical asymptote (VA).

The location of the HA and the VA depends on a couple of features of $f(x)$:

The VA is *usually* determined from those values of the denominator that are omitted from the domain. (Sometimes, these omitted values lead to a hole in the graph rather than a VA, as we'll see.)

So for the parent function, $f(x) = \dfrac{1}{x}$, whose domain is all $x \neq 0$, the equation of the ~~HA~~ VA is $x = 0$.

The HA is *generally* determined by comparing the degrees of $P(x)$ and $Q(x)$:

- If deg of $P(x)$ < deg $Q(x)$, the HA is the line $y = 0$ (that is, the x-axis).
- If deg of $P(x)$ = deg $Q(x)$, the HA is the line $y = \dfrac{\text{leading coeff of } P(x)}{\text{leading coeff of } Q(x)}$ (that is, the x-axis).
- If deg $P(x)$ > deg $Q(x)$, there is no HA, *but* …
- If deg $P(x)$ exceeds deg $Q(x)$ by 1, there is a slant asymptote (SA).

Examples:

$f(x) = \dfrac{1}{x}$ deg $P(x)$ < deg $Q(x)$ VA: $x = 0$ HA: $y = 0$

$f(x) = \dfrac{-2x}{x+4}$ deg $P(x)$ = deg $Q(x)$ VA: $x = -4$ HA: $y = -2/1 = -2$

$f(x) = \dfrac{x^3+1}{x-2}$ deg $P(x)$ > deg $Q(x)$ VA: $x = 2$ HA: none

SA: none (deg $P(x)$ exceeds deg $Q(x)$ by more than 1)

$f(x) = \dfrac{x^3+1}{x^2+1}$ deg $P(x)$ > deg $Q(x)$ VA: none HA: none

SA: deg $P(x)$ exceeds deg $Q(x)$ by more than 1

The last example is a little unusual. It is a rational function without a VA. The denominator cannot equal zero. It has no HA because the deg of the top is greater than the bottom, but it has an SA because the top's degree exceeds the bottom by just 1. To find the SA, we proceed by long division of polynomials:

$$\begin{array}{r} x \\ x^2+1 \overline{\big) x^3+0+1} \\ x^3+x+0 \\ \hline -x+1 \end{array}$$

$y = x$.

$$f(x) = x + \frac{1-x}{x^2+1}$$

We don't need to look at the remainder of the division. The equation of the SA is simply the polynomial part of the quotient.

Finally, we consider the possibility of *holes* in the graph. These occur only when the top and bottom factor in such a way as one of the binomial factors cancel.

For example: $f(x) = \dfrac{x^2+4x-5}{x^2-1}$. Factor the top and bottom (as always).

$$= \frac{(x+5)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} \qquad (x \neq \pm 1)$$

We found there was a cancellation of the factor $x - 1$. That means there is a hole at $f(1)$. Find the coordinates by evaluating the *reduced function*:

$f(x) = \dfrac{x+5}{x+1}$, $f(1) = \dfrac{6}{2} = 3$ Hole is at $(1, 3)$

Practice problems

Find all the features asked for. Asymptotes must be in equation form. Holes are ordered pairs.
Also, find y-intercept and any roots:

$f(x)=\dfrac{3}{x+1}$

VA: $x=-1$ HA: $y=0$. SA: \
Hole: \ . $y-$intercept: $(0,3)$ roots: \

$f(x)=\dfrac{2x-5}{x^2-1}$

$$\begin{array}{r} 9x-9 \\ x+1 \overline{\big) 9x^2+0-1} \\ 9x^2+9x+0 \\ \hline -9x-1 \\ -9x-9 \\ \hline -10 \end{array}$$

VA: $x=1$ and $x=-1$ HA: $y=0$. SA: \
Hole: \ $y-$intercept: $(0,5)$ roots: $x=\frac{5}{2}$

$f(x)=\dfrac{9x^2-1}{x+1}$

VA: $x=-1$ HA: none SA: $y=9x-9$
Hole: \ . $y-$intercept: $(0,-1)$ roots: $x=\frac{1}{3}$ or $-\frac{1}{3}$

$f(x)=\dfrac{x^2+8x-9}{x^2-7x+6} = \dfrac{(x+9)\cancel{(x-1)}}{(x-6)\cancel{(x-1)}} = \dfrac{x+9}{x-6}$

VA: $x=6$ and $x=1$ HA: $y=1$ SA: \
Hole: $(1,-2)$ $y-$intercept: $(0,-\frac{3}{2})$ roots: $x=-9$.

$f(x)=\dfrac{x^3+1}{3x^2+2x-1} = \dfrac{x^3+1}{(3x+1)(x-1)}$

$$\begin{array}{r} \frac{1}{3}x-\frac{2}{9} \\ 3x^2+2x-1 \overline{\big) x^3+0x^2+0x+1} \\ x^3+\frac{2}{3}x^2-\frac{1}{3}x \\ \hline -\frac{2}{3}x^2+\frac{1}{3}x+1 \\ -\frac{2}{3}x^2-\frac{4}{9}x-\frac{2}{9} \\ \hline \frac{7}{9}x+\frac{7}{9} \end{array}$$

VA: $x=-\frac{1}{3}$ and $x=1$ HA: none SA: $y=\frac{1}{3}x-\frac{2}{9}$
Hole: \ $y-$intercept: $(0,-1)$ roots: $x=-1$