Inverse Trigonometric Functions

When we introduced the graph of the sine function, we remarked that it repeats every 2π (this corresponds to a full rotation around the circle). Because of this property, the function $y = \sin(x)$ is not one-to-one. However, if we restrict the function to the interval $[-\pi/2, \pi/2]$, then it *is* one-to-one. See the figure below.

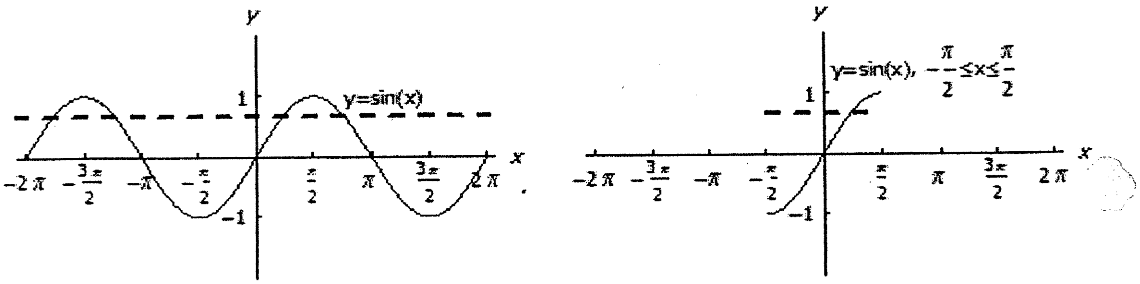

Figure 1: The unrestricted and restricted sine function

Two notations are commonly used to denote the inverse sine function:

$$y = \sin^{-1}(x) \quad \text{and} \quad y = \arcsin(x)$$

WARNING:

$y = \sin^{-1}(x)$ is **not** the same thing as $y = \frac{1}{\sin(x)}$.

For example, $\sin^{-1}(1) = \frac{\pi}{2} \approx 1.57 \neq 1.19 \approx \frac{1}{\sin(1)}$

The graph of $\sin^{-1}(x)$ can be found by reflecting the graph of the restricted sine function about the line $y = x$. Doing so, we have the following graph:

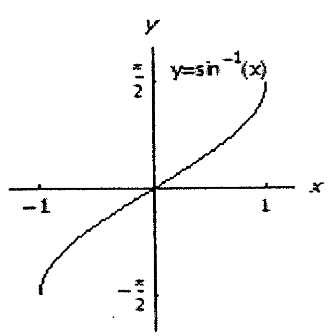

Figure 2: The graph of $y = \sin^{-1}(x)$

Example 1:

Evaluate (i) $\sin^{-1}\left(\frac{1}{2}\right)$ and (ii) $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$.

Solution:

(i) $\sin^{-1}\left(\frac{1}{2}\right)$ is the number in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ whose sine is $\frac{1}{2}$. Since $\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$, we conclude that $\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$.

(ii) $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$ is the number in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ whose sine is $-\frac{\sqrt{3}}{2}$. Since $\sin\left(-\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}$, we conclude that $\arcsin\left(-\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{3}$.

If $f(x)$ and $f^{-1}(x)$ are any pair of inverse functions, then by definition,

$$f[f^{-1}(x)] = x \quad \text{for every } x \text{ in the domain of } f^{-1}(x)$$

and

$$f^{-1}[f(x)] = x \quad \text{for every } x \text{ in the domain of } f(x)$$

Applying these facts to the restricted sine function and its inverse, we obtain the following two basic identities:

$$\sin(\sin^{-1}(x)) = x \quad \text{for every } x \text{ in the interval } [-1, 1]$$

$$\sin^{-1}(\sin(x)) = x \quad \text{for every } x \text{ in the interval } \left[-\frac{\pi}{2},\frac{\pi}{2}\right].$$

The following example indicates that the domain restrictions accompanying these two identities cannot be ignored.

Example 2:

Compute $\sin^{-1}(\sin(\pi))$.

Solution:

Notice that $\sin(\pi) = 0$, so $\sin^{-1}(\sin(\pi)) = \sin^{-1}(0)$, but $\sin^{-1}(0) = 0$. Thus, we have that $\sin^{-1}(\sin(\pi)) = 0$, not π. The reason why is because π is not in the domain of the restricted sine function.

We can do the same thing for the cosine function. The graph of cosine repeats every 2π (this corresponds to a full rotation around the circle). Because of this property, the function $y = \cos(x)$ is also not one-to-one. However, if we restrict the function to the interval $[0, \pi]$, then it *is* one-to-one. See the figure below.

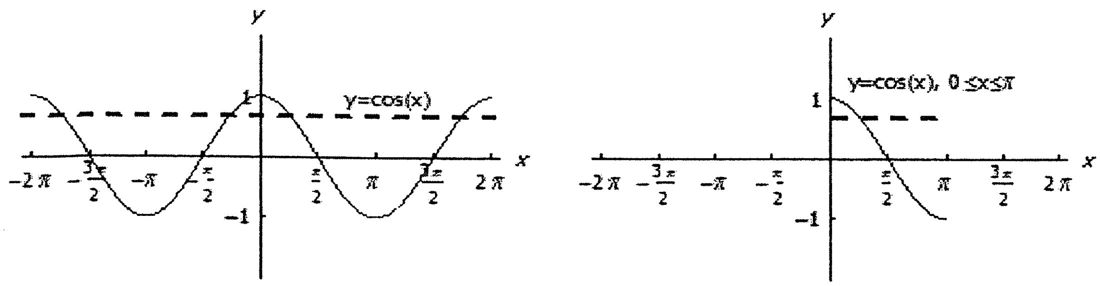

Figure 3: The unrestricted and restricted cosine function

Two notations are commonly used to denote the inverse cosine function:

$$y = \cos^{-1}(x) \quad \text{and} \quad y = \arccos(x)$$

WARNING:

$y = \cos^{-1}(x)$ is **not** the same thing as $y = \frac{1}{\cos(x)}$.

For example, $\cos^{-1}(-1) = \pi \approx 3.14 \neq 1.85 \approx \frac{1}{\cos(-1)}$

The graph of $\cos^{-1}(x)$ can be found by reflecting the graph of the restricted cosine function about the line $y = x$. Doing so, we have the following graph:

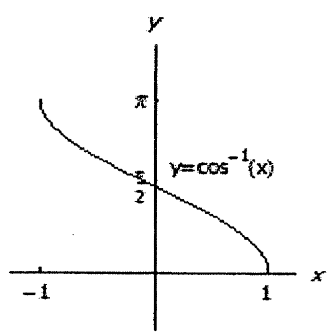

Figure 4: The graph of $y = \cos^{-1}(x)$

Again, we have two basic identities relating the function $\cos(x)$ and $\cos^{-1}(x)$.

$\cos(\cos^{-1}(x)) = x$ for every x in the interval [-1, 1]

$\cos^{-1}(\cos(x)) = x$ for every x in the interval $[0, \pi]$.

Example 3:

Evaluate (i) $\cos^{-1}(0)$ and (ii) $\arccos\left(\frac{\sqrt{2}}{2}\right)$.

Solution:

(i) $\cos^{-1}(0)$ is the number in the interval $[0, \pi]$ whose cosine is 0. Since $\cos\left(\frac{\pi}{2}\right) = 0$, we conclude that $\cos^{-1}(0) = \frac{\pi}{2}$.

(ii) $\arccos\left(\frac{\sqrt{2}}{2}\right)$ is the number in the interval $[0, \pi]$ whose cosine is $\frac{\sqrt{2}}{2}$. Since $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, we conclude that $\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

Just as there is a basic identity concerning $\sin(x)$ and $\cos(x)$, namely $\sin^2(x) + \cos^2(x) = 1$, there is also an identity concerning $\sin^{-1}(x)$ and $\cos^{-1}(x)$.

$$\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2} \quad \text{for every } x \text{ in the interval } [-1, 1]$$

Finally, we introduce the restricted tangent function and inverse tangent function. The graph of cosine repeats every π. Because of this property, the function $y = \tan(x)$ is also not one-to-one. However, if we restrict the function to the interval [-π/2, π/2], then it *is* one-to-one. See the figure below.

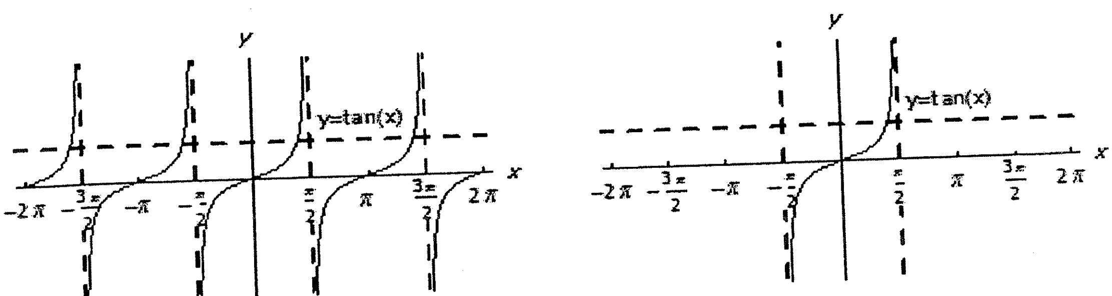

Figure 5: The unrestricted and restricted tangent function

Two notations are commonly used to denote the inverse tangent function:

$$y = \tan^{-1}(x) \quad \text{and} \quad y = \arctan(x)$$

WARNING:

$y = \tan^{-1}(x)$ is **not** the same thing as $y = \frac{1}{\tan(x)}$.

For example, $\tan^{-1}(1) = \frac{\pi}{4} \approx 0.785 \neq 0.642 \approx \frac{1}{\tan(1)}$

The graph of $\tan^{-1}(x)$ can be found by reflecting the graph of the restricted tangent function about the line $y = x$. Doing so, we have the following graph:

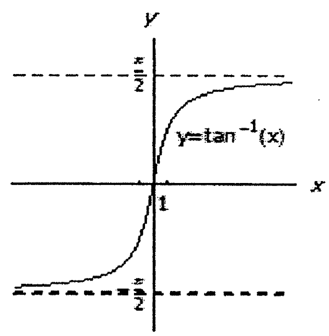

Figure 6: The graph of $y = \tan^{-1}(x)$

Again, we have two basic identities relating the function $\tan(x)$ and $\tan^{-1}(x)$.

$\tan(\tan^{-1}(x)) = x$ for every real number x

$\tan^{-1}(\tan(x)) = x$ for every x in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Example 4:

Evaluate (i) $\tan^{-1}(-1)$ and (ii) $\arctan(\sqrt{3})$.

Solution:

(i) $\tan^{-1}(-1)$ is the number in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose tangent is -1. Since

$\tan\left(-\frac{\pi}{4}\right) = -1$, we conclude that $\tan^{-1}(-1) = \frac{\pi}{4}$.

(ii) $\arctan(\sqrt{3})$ is the number in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose tangent is $\sqrt{3}$. Since

$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$, we conclude that $\arctan(\sqrt{3}) = \frac{\pi}{3}$.

Example 5:

Simplify the quantity $\csc(\tan^{-1}(x))$, where $x > 0$.

Solution:

We let $\theta = \tan^{-1}(x)$. That is, we have that $\tan(\theta) = x = x/1$. Using this information, we can sketch a right triangle with an angle θ whose tangent is x. See Figure 7.

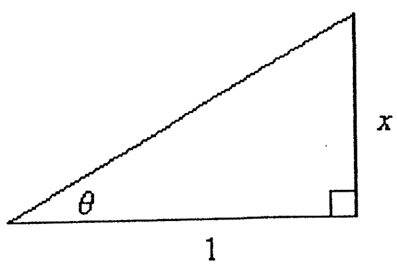

Figure 7: Graph of $\theta = \tan^{-1}(x)$

The Pythagorean Theorem tells us that the length of the hypotenuse in this triangle is equal to $\sqrt{1+x^2}$. Consequently, we have:

$$\csc(\tan^{-1}(\theta)) = \csc(\theta) = \frac{\text{hypotenuse}}{\text{side opposite}} = \frac{\sqrt{1+x^2}}{x}.$$