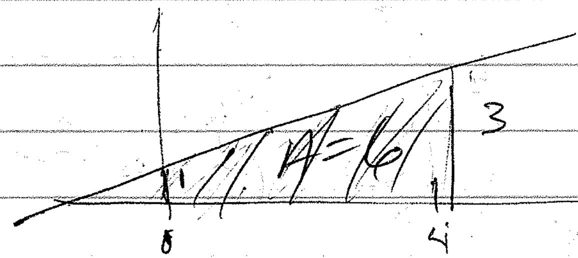
Sec 36 - Improper Integrals

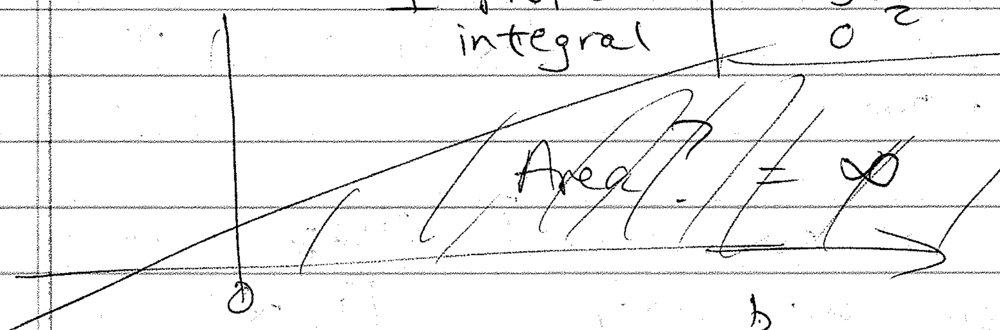


$$\int_0^4 \frac{x}{2} + 1 \, dx = 6$$

$$A_{TRAP} = \frac{b_1 + b_2}{2} \cdot \text{height}$$

Improper integral

$$\int_0^{\infty} \frac{x}{2} + 1 \, dx = ?$$



Rewrite as $\lim_{b \to \infty} \int_0^b \frac{x}{2} + 1 \, dx$

$$\int_0^b \frac{x}{2} + 1 \, dx = \frac{x^2}{4} + x \Big|_0^b = \frac{b^2}{4} + b - 0$$

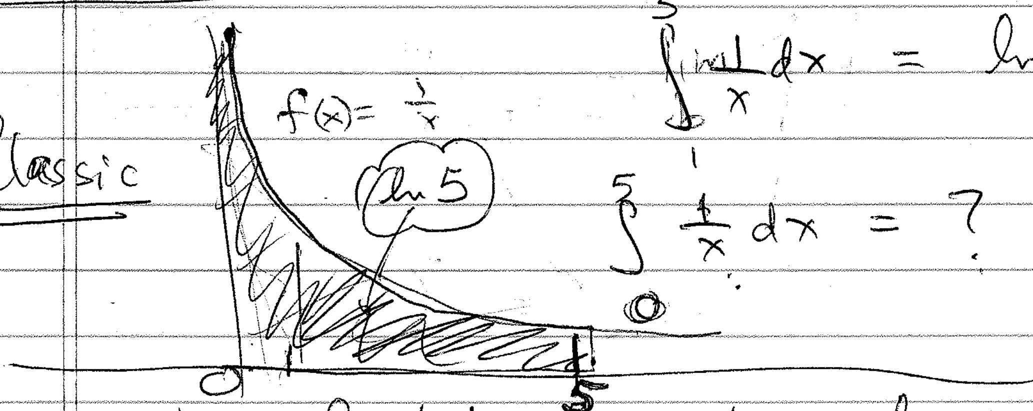
Take limit

$$\lim_{b \to \infty} \left(\frac{b^2}{4} + b \right) = \infty + \infty = \infty$$

(clearly)

Area under a line on $[0, \infty) = \infty$

Classic



$$\int_1^5 \frac{1}{x} dx = \ln|x| \Big|_1^5 = \ln 5$$

$$\int_0^5 \frac{1}{x} dx = ?$$

$$\lim_{a \to 0^-} \ln|5| - \ln|a| = \ln 5 - (-\infty) = \infty$$

More likely question (for the sake of series

Sum $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots = ?$

$$\int_1^{\infty} \frac{dx}{x} = \lim_{b \to \infty} \int_1^b \frac{dx}{x} = \cancel{\lim_{b\to\infty} (\ln b - \ln |x|)}$$

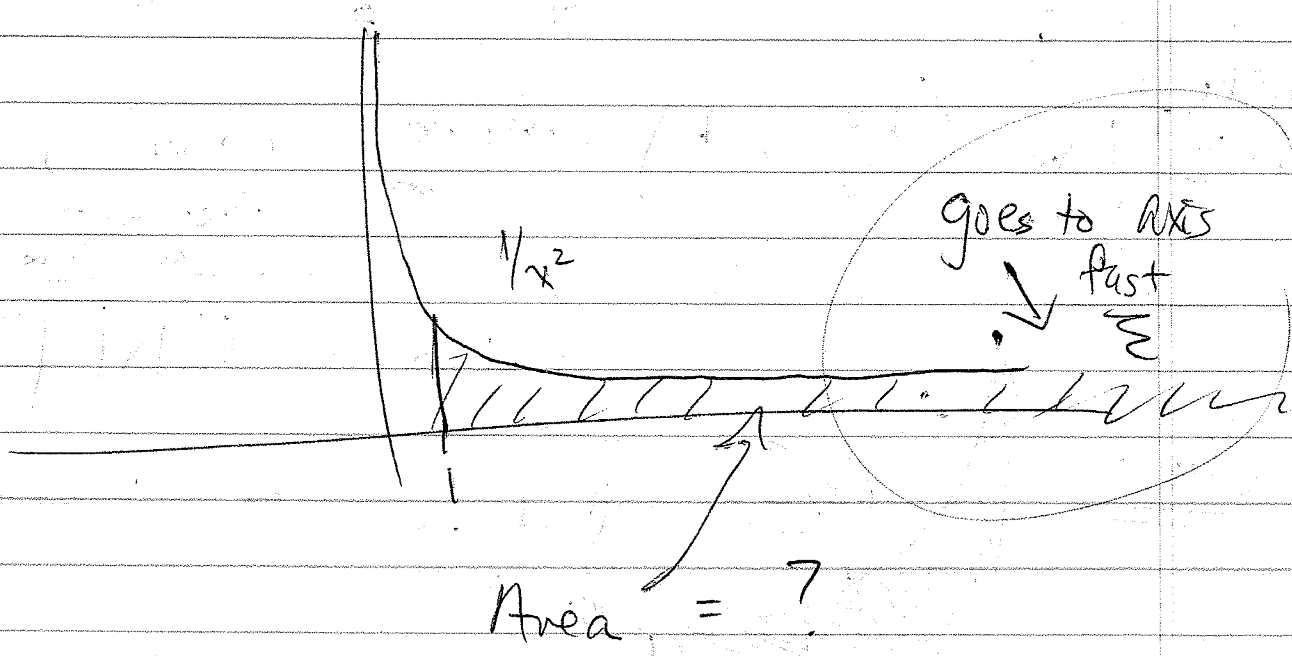
$$= \lim_{b \to \infty} (\ln b - \ln 1) = \ln \infty - 0 = \infty$$

Def

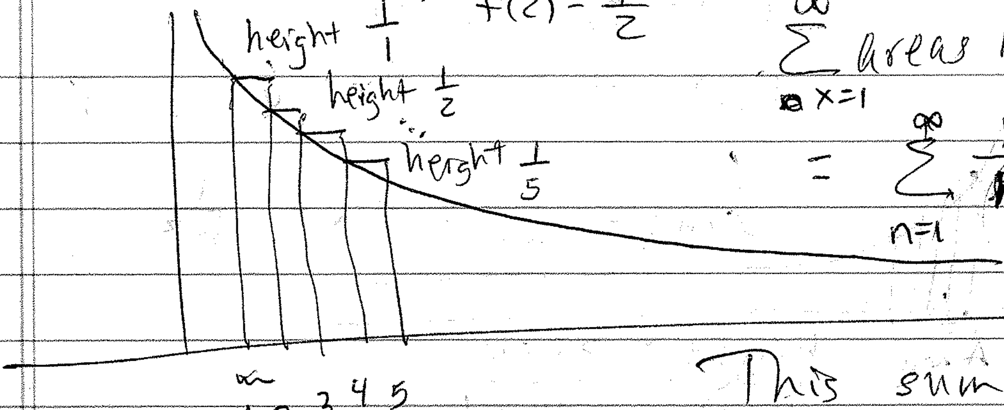
We say that an integral ~~value~~ that is infinite on an interval $[1, \infty)$ is __divergent__.

Go to (3)

Consider now $f(x) = \frac{1}{x^2}$



$f(x) = \frac{1}{x}$, $f(1) = 1$, $f(2) = \frac{1}{2}$



$\sum_{x=1}^{\infty}$ areas height x 1

$= \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$

This sum is infinite.

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \dots$

$1 + \quad 26/24 > 1 \qquad > 1$

$= 1 + 1 + + \dots = \infty$

42
35
30
107

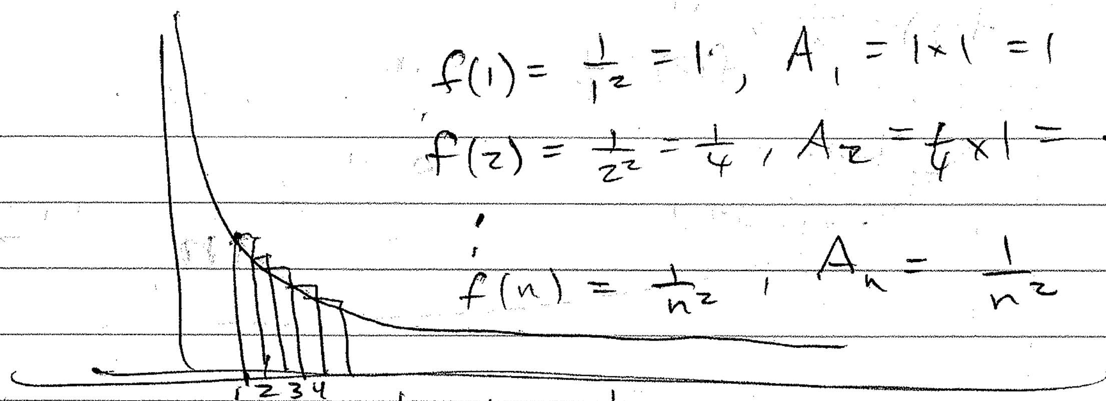
$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \iff \int_1^{\infty} \frac{dx}{x} = \infty$$

$f(x) = \frac{1}{x^2}$ → The denominator is growing a lot faster than it does for $\frac{1}{x}$.

Look at the sum of the rect. areas first:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots$$

Can we similarly group these fractions to give $1 + 1 + 1 + \dots$ until the sum appears to be infinite? NO



$f(1) = \frac{1}{1^2} = 1, \quad A_1 = 1 \times 1 = 1$

$f(2) = \frac{1}{2^2} = \frac{1}{4}, \quad A_2 = \frac{1}{4} \times 1 = \frac{1}{4}$

$\vdots$

$f(n) = \frac{1}{n^2}, \quad A_n = \frac{1}{n^2}$

Area = height × base
height: $f(x)$. base: all = 1

$$\text{Area} = \int_1^{\infty} \frac{dx}{x^2} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{x^2} = ?$$

~~$\lim_{b\to\infty} \dots = \lim_{b\to\infty}\left(-\frac{1}{x} - -\frac{1}{b}\right)$~~

$$= \lim_{b\to\infty} \left. \frac{-1}{x} \right|_1^b$$ ~~$= \lim_{b\to\infty} \dots$~~

$$= \lim_{b\to\infty} \left(\frac{-1}{b} - -\frac{1}{1}\right) = \lim_{b\to\infty} \left(1 - \frac{1}{b}\right)$$

$$= \lim_{b\to\infty} 1 - \lim_{b\to\infty} \frac{1}{b} = 1 - \frac{1}{\infty} \to 0$$

$$\boxed{= 1}$$

What does this represent?

Go to (2) for answer.

__Def__ An integral that is finite on an interval $[1, \infty]$ is said to __converge__.

When we have a two-sided limit, i.e.

$$\int_{-\infty}^{\infty} f(x)\,dx,$$ we look at

each half. $$\int_{-\infty}^{0} f(x)\,dx \quad + \quad \int_{0}^{\infty} f(x)\,dx$$

If both integrals converge then so does $\int_{-\infty}^{\infty} f(x)\,dx$.

But if only one converges + the other diverges, then $\int_{-\infty}^{\infty} f(x)\,dx$ diverges.