

Wednesday Math 224 - Lecture

1. Brief quiz will be posted (chain rule, implicit diff)

2. Warm-ups - comment on difficulty level of

2.7 and 2.8, esp Q' , Q'' one.

2.6

\checkmark
 $\frac{d}{dt}$

3. $s(t)$, $v(t)$, $a(t) = \frac{v''}{v} = \frac{a''}{v}$ straight-line motion.

The signs of v and a relate as follows:

Where If $v > 0$, then object is moving forward.

Where If $v < 0$, then object is moving backward.

Where If $a > 0$ and $v > 0$ then object is speeding up.

Where If $a < 0$ and $v > 0$ then object is slowing down.

Where If $a < 0$ and $v < 0$ then object is speeding up.

Where If $a > 0$ and $v < 0$ then object is slowing down.

Where If $a > 0$ and $v < 0$ then object is speeding up.

Where If $a < 0$ and $v > 0$ then object is slowing down.

Why? Consider that direction of acceleration is also indicated by sign of derivative. So acceleration in the positive direction (a push forward) means $a > 0$, and acceleration in the negative direction (a push backward) means $a < 0$.

A push in the same direction as motion causes acceleration; a push in the opposite direction means deceleration.

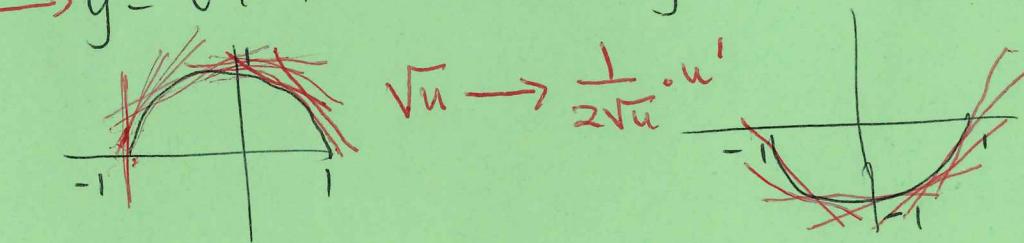
What do brakes on a car do? Essentially (or effectively) they push against the direction of motion, so the car slows down. The gas pedal is called the accelerator; maybe brakes are the "decelerator"? :)

4. Implicit differentiation - I should have discussed this on Monday. Here's the story.

Consider $\underbrace{f(x)}_y = x^2 - 2x + 1$. y is clearly a fn. of x . y is the output to input x

Now consider a circle: $x^2 + y^2 = 1$. This is the familiar unit circle, and while not a fn, it can be written as two fns, each

with y as output to input x : $y^2 = 1 - x^2$
 $\rightarrow y = \sqrt{1-x^2}$ and $y = -\sqrt{1-x^2}$ ← $y = \pm \sqrt{1-x^2}$
 2 fns.



The derivatives are:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}} \cdot -2x = \frac{-x}{\sqrt{1-x^2}}$$

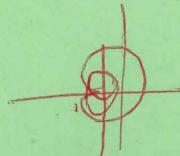
and

$$\frac{dy}{dx} = \frac{-1}{2\sqrt{1-x^2}} \cdot -2x = \frac{x}{\sqrt{1-x^2}}$$

But many curves (graphs of eqns) cannot be separated this way. They cannot be written so that $y = f(x)$. For these, we regard the y terms as "implicit fns" of x . For example:



$$8y^2 - x^2 y^3 = 6$$



There's no way to separate y from all x terms.

Try it! $y^2(8 - x^2 y) = 6 \rightarrow y^2 = \frac{6}{8 - x^2 y}$ (nope)

Any other ideas? We still want to see the behavior of the slope of the tangent line at any point where a derivative exists (i.e., where ~~a~~ a tangent line ~~is~~ may be drawn). Many curves are smooth throughout their domains; they just are not fns. Sec 2.6 contains many problems in science & other fields where explicit fns describe the IROC phenomena.

But there are of course many that do not.

In those cases, determining IROC requires "ID" "implicit differentiation"

Look at $8y^2 - x^2 y^3 = 6$, Desmos graphed it for me and I assure you it fails the vertical line test (but I can't do the picture of it justice)..

Yet we might like to know $\frac{dy}{dx}$, even though it won't likely be a fn either. It will describe the slope of the tangent line to the curve, which is all we seek. | Here's the process:

$$\frac{d}{dx} [8y^2 - x^2 y^3] = \frac{d}{dx}[6]$$

$$\frac{d}{dx}[8y^2] - \frac{d}{dx}[x^2 y^3] = 0$$

Since y is regarded as an implicit fn of x , we treat the differentiation like we would a $u(x)$ in the chain rule

"Imagine the 'buried' y as an implicit (implied) fn of x "

This is true whether it's standing alone or in a product or quotient (or later in an exponential or log term).

$$\frac{d}{dx}[8y^2] = 8 \cdot 2y \cdot \frac{dy}{dx}; \quad \frac{d}{dx}[x^2 y^3] = 2x y^3 + 3x^2 y^2 \frac{dy}{dx}$$

Thus, $\frac{d}{dx} [\text{each side}]$ gives:

$$16y \frac{dy}{dx} - 2x y^3 - 3x^2 y^2 \frac{dy}{dx} = 0 \quad | \text{ so far}$$

Keeping all $\frac{dy}{dx}$ terms on one side gives :

$$16y \frac{dy}{dx} + 3x^2 y^2 \frac{dy}{dx} = 2xy^3$$

Factoring out $\frac{dy}{dx}$:

$$\frac{dy}{dx} [16y + 3x^2 y^2] = 2xy^3$$

Finally, solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{2xy^3}{16y + 3x^2 y^2}$$

Notice the derivative is not an $f'(x)$.

It has both x and y in the expression.

A more reasonable example, and one where the graph is accessible is an ellipse, but like the circle, it can be written as two explicit functions of x .

$$\frac{x^2}{4} + \frac{y^2}{1} = 1 \rightarrow y = \pm \sqrt{1 - \frac{x^2}{4}}$$

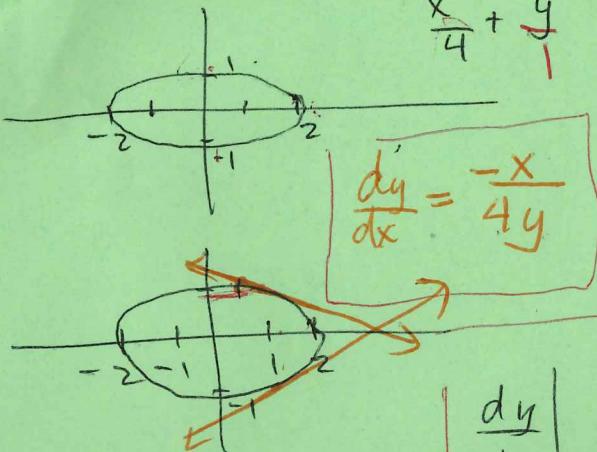
Yet we can express both its derivatives in one expression by using ID:

$$\left(\frac{d}{dx} \right) \left[\frac{x^2}{4} + y^2 \right] = \left(\frac{d}{dx} \right) [1]$$

$$\frac{2x}{24} + 2y \frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = -\frac{x/2}{2y} = -\frac{x}{4y}$$

$$\frac{m + an}{1} = \frac{-x}{4y} = \frac{dy}{dx}$$

Solv



$$\frac{x^2}{4} + \frac{y^2}{1} = 1 \rightarrow y = \pm \sqrt{1 - \frac{x^2}{4}}$$

Find the slopes of the two tangents we can draw to the curve at $x = 1$

$$\left. \frac{dy}{dx} \right|_{\substack{x=1 \\ y=?}} = ?$$

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First we'll need the two y-values when $x = 1$:

$$y = ? \quad \frac{1^2}{4} + y^2 = 1 \quad \rightarrow \quad y^2 = \frac{3}{4} \quad \rightarrow \quad y = \pm \sqrt{\frac{3}{4}} = \pm \frac{\sqrt{3}}{2}$$

$$\left. \frac{dy}{dx} \right|_{(1, \frac{\sqrt{3}}{2})} = \frac{-1}{4 \cdot \frac{\sqrt{3}}{2}} = -\frac{1}{2\sqrt{3}} \quad \text{m top} \quad \left. \frac{dy}{dx} \right|_{(1, -\frac{\sqrt{3}}{2})} = \frac{1}{2\sqrt{3}} \quad \text{m bottom}$$

Look at the sketch — this is believable ☺

Find the tangent lines' eqns there:

$$y - \left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{2\sqrt{3}}(x-1) \quad \text{Quadrant IV}$$

$$y - \frac{\sqrt{3}}{2} = -\frac{1}{2\sqrt{3}}(x-1) \quad \text{Quadrant I}$$

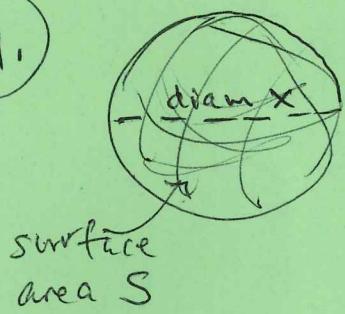
Warm-up See 2.8

"RR"

Related rates " $\frac{df}{dt}$ " where

each variable is itself a fn. of time t .

1.



sphere surface area $S = 4\pi r^2$

changes with time according to

$$\frac{ds}{dt} = -7 \frac{\text{cm}^2}{\text{min}}. \quad \text{Find rate at which}$$

the diameter x decreases at the instant that its 8 cm.

So, we freeze the shrinking ball at the instant when $x = 8$, knowing that $\frac{ds}{dt}$ will affect $\frac{dx}{dt}$ (and vice-versa \therefore) How do we solve this RR?

1. Draw picture (done)

2. Write all knowns + unknowns.

$$\frac{ds}{dt} = -7 \frac{\text{cm}^2}{\text{min}}, \quad x = 8 \text{ cm}, \quad \frac{dx}{dt} = ?$$

3. Write formula that relates the variables involved: $S = 4\pi r^2$ (wasn't given \therefore)

*4. Differentiate left and right with respect to time. Remember $S =: S(t)$, $x =: x(t)$

$$\text{So } \frac{d}{dt}(S) = \frac{dS}{dr} \cdot \frac{dr}{dt}$$

(We have to pass through r to get to t)

But wait, there's no r in the given. That's

ok, since diam $x = 2r$, so $\boxed{r = \frac{x}{2}}$

Rewrite $S = 4\pi r^2$ as $S = 4\pi \left(\frac{x}{2}\right)^2 = \pi x^2$

Note: ~~cancel r~~

Caution !! Don't substitute any knowns till you differentiate the formula.

$$S = \pi x^2, \quad \frac{dS}{dt} = \frac{d}{dt} (\pi x^2) = \frac{dS}{dx} \cdot \frac{dx}{dt}$$

$$\boxed{\frac{dS}{dt} = 2\pi x \cdot \frac{dx}{dt}}$$

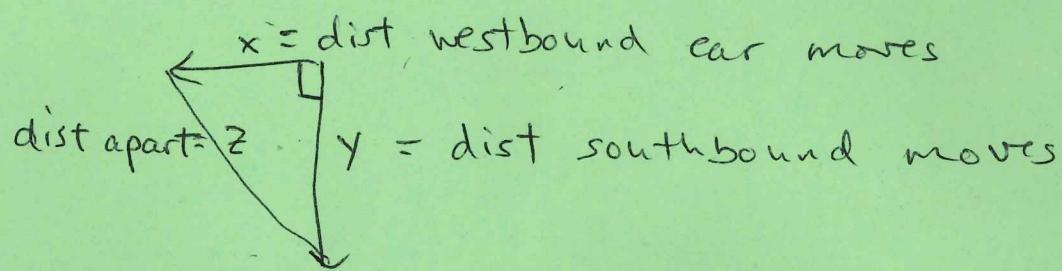
5. Plug in values; solve for unknown:

$$-\frac{7 \text{ cm}^2}{\text{min}} = 2\pi \cdot 8 \text{ cm} \cdot \frac{dx}{dt}$$

$$\boxed{\frac{dx}{dt} = -\frac{7}{16\pi} \frac{\text{cm}}{\text{min}}}$$

Rate of shrinking diameter.

2. Cars moving apart from common starting pt.



Known: $\frac{dx}{dt} = 48 \frac{\text{mi}}{\text{h}}$, $\frac{dy}{dt} = 20 \frac{\text{mi}}{\text{h}}$, $t = 4 \text{ hrs}$

Unknowns: $\frac{dz}{dt}$ (it's the question to answer)

x, y, z

Formula:
$$x^2 + y^2 = z^2$$
 where x, y, z are all funcs

Differentiate w.r.t. t

$$\left[2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt} \right] \text{ of time } t \quad (\text{implicit diff w.r.t. } t)$$

Fill in what is known: $2x \cdot 48 + 2y \cdot 20 = 2z \frac{dz}{dt}$

(Reduce by 2: $48x + 20y = z \frac{dz}{dt}$)

This one requires a little more thought.

Since $t = 4 \text{ hr}$, then $\frac{dx}{dt} \cdot 4 = 48 \cdot 4 = 192 \frac{\text{mi}}{\text{hr}}$
 $(\frac{dt}{dt} = 4)$

and $\frac{dy}{dt} \cdot 4 = 20 \cdot 4 = 80 \frac{\text{mi}}{\text{hr}}$

$$\text{Finally, } z = \sqrt{192^2 + 80^2} = 208 \text{ mi}$$

Cars
cond

$$\text{Hence, } \frac{dz}{dt} = \frac{(218 \cdot 192 + 20 \cdot 80) \text{ mi}^2/\text{hr}}{208 \text{ mi}}$$
$$= 52 \text{ mi/hr}$$

Ex One more: This time, let's use y' notation:

From Stewart #11 p. 169 Sec 2.6

$$\frac{d}{dx}(\sin x) + \left(\frac{d}{dx} \cos y \right) = \frac{d}{dx}(2x - \frac{d^3 y}{dx})$$

$$\cos x + (-\sin y)(y') = 2 - 3y'$$

derivative of outer fn derivative of inner fn.

"Pass thru
y to get to x
 $\frac{d}{dx}$ "

$$\rightarrow \cos x - \sin y \cdot y' = 2 - 3y'$$

$$\rightarrow (\cos x) - 2 = \sin y \cdot y' - 3y'$$

$$\cos x - 2 = y'(\sin y - 3)$$

$$y' = \frac{\cos x - 2}{\sin y - 3}$$

isolate
 y' to
get soln.

Ex Find eqn. of line tangent to at $(\pi/2, \pi/4)$

$$y \sin 2x = x \cos 2y$$

Soln A good review of trig is to check that the point is on the curve! $\frac{\pi}{4} \cdot \sin \frac{\pi}{2} = 0$ left ✓

$$\frac{\pi}{2} \cdot \cos \frac{\pi}{4} = 0$$
 right ✓

$$\frac{d}{dx} (y \cdot \sin 2x) = \frac{d}{dx} (x \cdot \cos 2y)$$

product rule
needed

$$y' \cdot \sin 2x + \cancel{\cos 2x \cdot 2} \cdot y = 1 \cdot \cos 2y + x \cdot (-\sin y) 2y'$$

implicit
step
chain rule

implicit step with chain rule!

$$y' \sin 2x - 2x \sin 2y \cdot y' = \cos 2y - 2y \cos 2x$$

$$y' (\sin 2x - 2x \sin 2y) = \cos 2y - 2y \cos 2x$$

$$y' = \frac{\cos 2y - 2y \cos 2x}{\sin 2x - 2x \sin 2y}$$

horrible.

Find m at

$$\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$$

$$y - \frac{\pi}{4} = \frac{\cos \frac{\pi}{2} - \frac{\pi}{2} \cos \frac{\pi}{4}}{\sin \frac{\pi}{2} - \frac{\pi}{2} \sin \frac{\pi}{4}} \left(x - \frac{\pi}{2}\right)$$

m

$$y - \frac{\pi}{4} = \frac{0 - \left(\frac{\pi}{2}\right)(-1)}{0 - \pi(1)} \left(x - \frac{\pi}{2}\right)$$

$$y - \frac{\pi}{4} = \frac{\pi/2}{-\pi} \left(x - \frac{\pi}{2}\right)$$

$$y - \frac{\pi}{4} = -\frac{1}{2} \left(x - \frac{\pi}{2}\right)$$

Warm-up Sec 2.6

Graph $Q'(t)$ is considered to be $y = f(t)$. Then

$y' = f'(t)$ is the fn

to analyze for maximum / minimum.

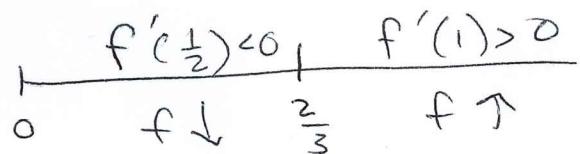
$$f = Q'(t) = 3t^2 - 4t + 7$$



$$f' = Q''(t) = 6t - 4 = 0$$

at $t = \frac{2}{3}$ sec or $\frac{2}{3}$ sec

Where is $f' < 0$? > 0 ?



Sec 2.7 Slides

$$f(t) = t^3 - 15t^2 + 72t$$

Basic shape

Straight-line motion

(s) gives position of an object at time t seconds.

But understand that the cubic fn shape describes the displacement of object from $t = 0$ till $t = \text{end}$.

$f(0) = 0$ so we're assuming it begins at origin, not several meters left or right of it.

Where is motion forward? Backward? Stopped?

We need $v(t)$ (that is, $f'(t)$) to determine the intervals where $v = s'$, so $v = f'(t)$.

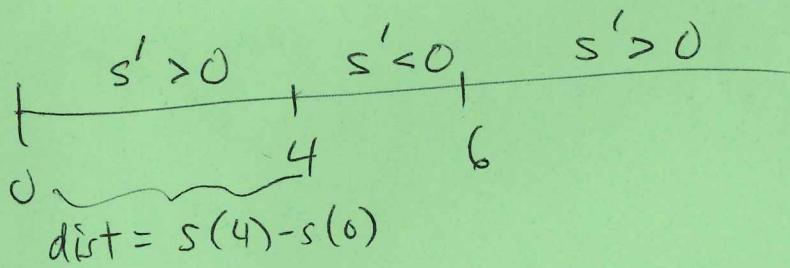
$$s = f(t) = t^3 - 15t^2 + 72t$$

$$v = f'(t) = 3t^2 - 30t + 72 = 3(t-6)(t-4) = 0$$

$$a = f''(t) = 6t - 30$$

particle stopped at $t = 4, 6$ sec.

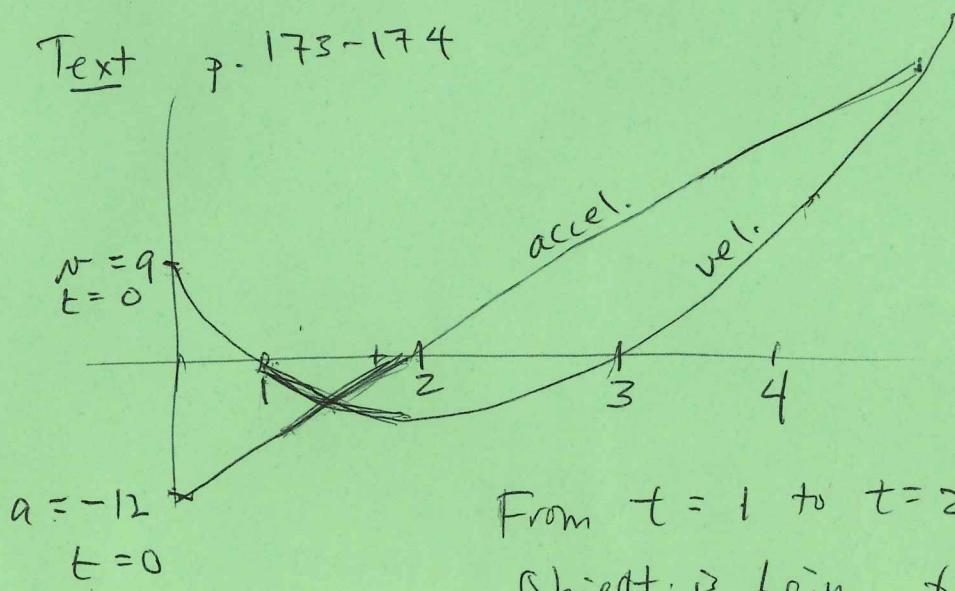
what's the difference between $s(5)$ and distance traveled the first 5 sec?



$$\text{dist} = \sqrt{s(6) - s(4)}$$

Def Fact Particle speeds up when ~~both~~
 $s' > s'' > 0$ or $s', s'' < 0$. Why?

Text p. 173-174



From $t = 1$ to $t = 2$, $\text{accel}, \text{vel} < 0$

Object is being pushed in same direction it moves.