

The following problem, #9 in Sec 9,  
requires Sec 7 + 8 material.

The solution given here includes  
notes on <sup>the</sup> difference quotient and  
how it relates to the derivative  
of the fcn. at  $x = a$  (in this  
case,  $a = 0$ )

The text develops this very problem,  
but I think it's clearer here.

You will recognize what we did  
on the board with the Digression  
on p. 2 of this file.

\* This is shown in the worked out Ex. 7.4 p. 71

9. Show  $f(x) = |x|$  is continuous at  $x=0$  but that the derivative  $f'(x)$  DNE.

This problem requires Section 8 definition of "derivative of a function".

Def The derivative of  $f(x)$  is given by the limit:

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

If this limit exists for all  $x$  in the domain of  $f$ , we say the fn. is

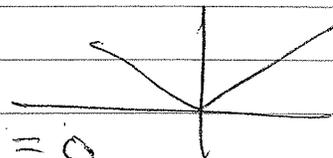
"differentiable for all  $x$ ". We could also look at a value of  $x$  and determine the fn.'s "differentiability" at that value.

We'll see soon that to be differentiable, a fn. must be continuous and have no sharp corners. Right away, we know  $f(x) = |x|$  is known to have a sharp corner at  $x=0$ . It is in fact differentiable for all  $x$  EXCEPT at  $x=0$ .

First, is  $f(x) = |x|$  cts. at  $x=0$ ?

Yes, since we do not take our pen off the paper when graph it.

Notice  $\lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^+} |x| = 0$

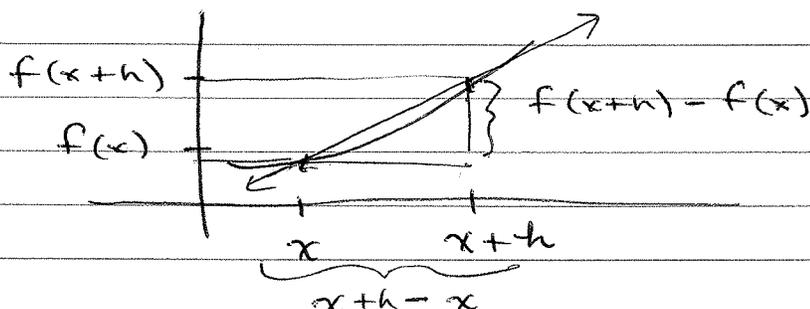


Now let's define the so-called "difference quotient" of the function. It is this:

$$\frac{f(x+h) - f(x)}{(x+h) - x} \quad \text{or} \quad \frac{f(x+h) - f(x)}{h}$$

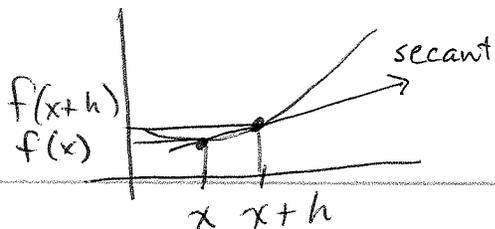
Digression

where  $h$  is some distance from  $x$ . For ex:

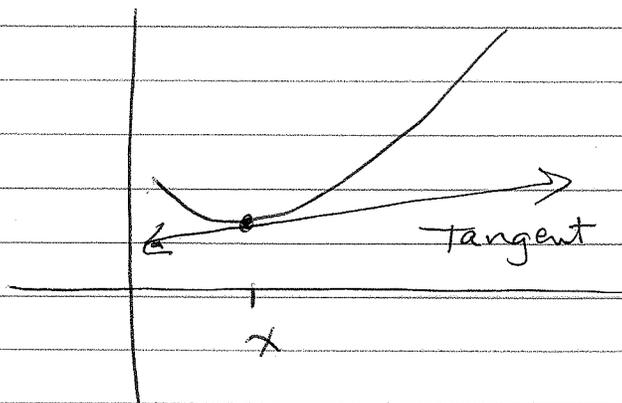


The quotient shown should remind you of the ratio called the slope of a line. This is not a line, however. It's a curve that is nonlinear. The rate of change between the two points shown is  $\frac{f(x+h) - f(x)}{x+h-x}$

It is also the slope of the <sup>red</sup> secant line. As these points move closer together, the distance  $h$  decreases (or vice-versa, as  $h$  decreases, the points move closer).



The slope of the secant line is changing as  $h$  gets smaller. In fact, it is approaching the slope of the line tangent to the curve at  $x$ .



The limit of the slope of the secant line as  $h \rightarrow 0$  is the slope of the tangent to the curve at the point  $(x, f(x))$ . This limit is called the derivative  $f'(x)$  of the function, and it can be generalized that for any ~~value~~ <sup>value</sup> of  $x$ , the derivative exists if this limit exists.

Def of Derivative  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

In the problem, to show  $f(x) = |x|$  is continuous at  $x=0$  but that  $f'(x)$  DNE at  $x=0$ , we have to take the difference quotient limit of each piece of the function  $f(x) = |x|$

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

limit  $h \rightarrow 0^+$  and  $h \rightarrow 0^-$

Back to #9

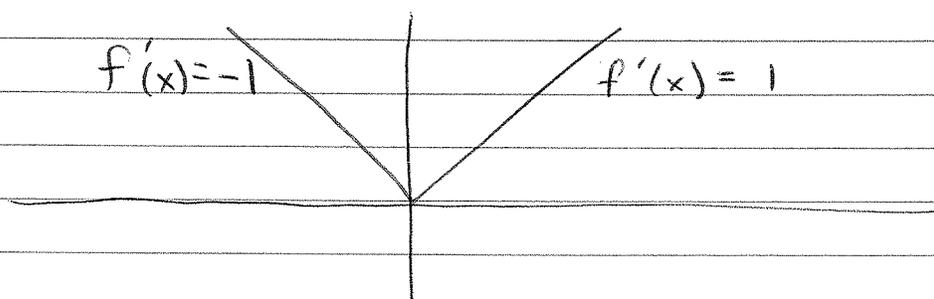
$$\lim_{h \rightarrow 0} \frac{|x+h| - |x|}{h} = \begin{cases} \lim_{h \rightarrow 0} \frac{x+h-x}{h}, & x \geq 0 \\ = 1 \\ \lim_{h \rightarrow 0} \frac{-x-h-(-x)}{h}, & x < 0 \\ = -\frac{h}{h} \end{cases}$$

Simplifying:

$$f'(x) = \lim_{h \rightarrow 0} \frac{|x+h| - |x|}{h} = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

On either side of  $x=0$ , the limit of the difference quotient ~~exists~~ differs, that is, the LHL  $\neq$  RHL.

Therefore,  $f'(x)$  does not exist at  $x=0$ .



The picture shows a constant slope on each of the two halves of the domain, but these have different values, so  $f(x)$  is not differentiable at  $x=0$ .

NOTE: This does not mean  $f'(x) \nexists$  for all  $x$ ; it does for any  $x$  other than zero.  $f'(x) = 1, x > 0, f'(x) = -1, x < 0$ .