HOFFMAN, INTEGRATION BY PARTS

SECTION 6.1 Integration by Parts; Integral Tables

Integration by parts is a technique of integration based on the product rule for differentiation. In particular, if $u(x)$ and $v(x)$ are both differentiable functions of x, then

$$\frac{d}{dx}[u(x)v(x)] = u(x)\frac{dv}{dx} + v(x)\frac{du}{dx}$$

so that

$$u(x)\frac{dv}{dx} = \frac{d}{dx}[u(x)v(x)] - v(x)\frac{du}{dx}$$

Integrating both sides of this equation with respect to x, we obtain

$$\begin{aligned}\int\left[u(x)\frac{dv}{dx}\right]dx &= \int\frac{d}{dx}[u(x)v(x)]\,dx - \int\left[v(x)\frac{du}{dx}\right]dx\\ &= u(x)v(x) - \int\left[v(x)\frac{du}{dx}\right]dx\end{aligned}$$

since $u(x)v(x)$ is an antiderivative of $\frac{d}{dx}[u(x)v(x)]$. Moreover, we can write this integral formula in the more compact form

$$\int u\,dv = uv - \int v\,du$$

since

$$dv = \frac{dv}{dx}dx \qquad \text{and} \qquad du = \frac{du}{dx}dx$$

The equation $\int u\,dv = uv - \int v\,du$ is called the **integration by parts formula.**

The great value of this formula is that if we can find functions u and v so that a given integral $\int f(x)\,dx$ can be expressed in the form $\int f(x)\,dx = \int u\,dv$, then we have

$$\int f(x)\,dx = \int u\,dv = uv - \int v\,du$$

and the given integral is effectively exchanged for the integral $\int v\,du$. If the integral $\int v\,du$ is easier to compute than $\int u\,dv$, the exchange facilitates finding $\int f(x)\,dx$. Here is an example.

EXAMPLE 6.1.1

Find $\int x^2 \ln x \, dx$.

Solution

Our strategy is to express $\int x^2 \ln x \, dx$ as $\int u \, dv$ by choosing u and v so that $\int v \, du$ is easier to evaluate than $\int u \, dv$. This strategy suggests that we choose

$$u = \ln x \qquad \text{and} \qquad dv = x^2 \, dx$$

since

$$du = \frac{1}{x} dx$$

is a simpler expression than $\ln x$, while v can be obtained by the relatively easy integration

$$v = \int x^2 \, dx = \frac{1}{3}x^3$$

(For simplicity, we leave the "+ C" out of the calculation until the final step.) Substituting this choice for u and v into the integration by parts formula, we obtain

$$\begin{aligned}
\int x^2 \ln x \, dx &= \int \underbrace{(\ln x)}_{u}\underbrace{(x^2 \, dx)}_{dv} = \underbrace{(\ln x)}_{u}\underbrace{\left(\frac{1}{3}x^3\right)}_{v} - \int \underbrace{\left(\frac{1}{3}x^3\right)}_{v}\underbrace{\left(\frac{1}{x} \, dx\right)}_{du} \\
&= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\left(\frac{1}{3}x^3\right) + C \\
&= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C
\end{aligned}$$

Here is a summary of the procedure we have just illustrated.

Integration by Parts

To find an integral $\int f(x) \, dx$ using the integration by parts formula:

Step 1. Choose functions u and v so that $f(x) \, dx = u \, dv$. Try to pick u so that du is simpler than u and a dv that is easy to integrate.

Step 2. Organize the computation of du and v as

$$\begin{array}{cc} u & dv \\ du & v = \int dv \end{array}$$

Step 3. Complete the integration by finding $\int v \, du$. Then

$$\int f(x) \, dx = \int u \, dv = uv - \int v \, du$$

Add "+ C" only at the end of the computation.

Choosing a suitable u and dv for integration by parts requires insight and experience. For instance, in Example 6.1.1, things would not have gone so smoothly if we had chosen $u = x^2$ and $dv = \ln x\,dx$. Certainly $du = 2x\,dx$ is simpler than $u = x^2$, but what is $v = \int \ln x\,dx$? In fact, finding this integral is just as hard as finding the original integral $\int x^2 \ln x\,dx$ (see Example 6.1.4). Examples 6.1.2, 6.1.3, and 6.1.4 illustrate several ways of choosing u and dv in integrals that can be handled using integration by parts.

EXAMPLE 6.1.2

Find $\int xe^{2x}\,dx$.

Solution

Although both factors x and e^{2x} are easy to integrate, only x becomes simpler when differentiated. Therefore, we choose $u = x$ and $dv = e^{2x}\,dx$ and find

$$
\begin{aligned}
u &= x & dv &= e^{2x}\,dx \\
du &= dx & v &= \frac{1}{2}e^{2x}
\end{aligned}
$$

Substituting into the integration by parts formula, we obtain

$$
\begin{aligned}
\int \underbrace{x}_{u}\underbrace{(e^{2x}\,dx)}_{dv} &= \underbrace{x}_{u}\underbrace{\left(\frac{1}{2}e^{2x}\right)}_{v} - \int \underbrace{\left(\frac{1}{2}e^{2x}\right)}_{v}\underbrace{dx}_{du} \\
&= \frac{1}{2}xe^{2x} - \frac{1}{2}\left(\frac{1}{2}e^{2x}\right) + C \\
&= \frac{1}{2}\left(x - \frac{1}{2}\right)e^{2x} + C
\end{aligned}
$$

EXAMPLE 6.1.3

Find $\int x\sqrt{x+5}\,dx$.

Solution

Again, both factors x and $\sqrt{x+5}$ are easy to differentiate and to integrate, but x is simplified by differentiation, while the derivative of $\sqrt{x+5}$ is even more complicated than $\sqrt{x+5}$ itself. This observation suggests that you choose

$$u = x \qquad dv = \sqrt{x+5}\,dx = (x+5)^{1/2}\,dx$$

so that

$$du = dx \qquad v = \frac{2}{3}(x+5)^{3/2}$$

Substituting into the integration by parts formula, you obtain

$$\int \underbrace{x}_{u}\underbrace{(\sqrt{x+5}\,dx)}_{dv} = \underbrace{x}_{u}\underbrace{\left[\frac{2}{3}(x+5)^{3/2}\right]}_{v} - \int \underbrace{\left[\frac{2}{3}(x+5)^{3/2}\right]}_{v}\underbrace{dx}_{du}$$

$$= \frac{2}{3}x(x+5)^{3/2} - \frac{2}{3}\left[\frac{2}{5}(x+5)^{5/2}\right] + C$$

$$= \frac{2}{3}x(x+5)^{3/2} - \frac{4}{15}(x+5)^{5/2} + C$$

NOTE Some integrals can be evaluated by either substitution or integration by parts. For instance, the integral in Example 6.1.3 can be found by substituting as follows:

Let $u = x + 5$. Then $du = dx$ and $x = u - 5$, and

$$\int x\sqrt{x+5}\,dx = \int (u-5)\sqrt{u}\,du = \int (u^{3/2} - 5u^{1/2})\,du$$

$$= \frac{u^{5/2}}{5/2} - \frac{5u^{3/2}}{3/2} + C$$

$$= \frac{2}{5}(x+5)^{5/2} - \frac{10}{3}(x+5)^{3/2} + C$$

This form of the integral is not the same as that found in Example 6.1.3. To show that the two forms are equivalent, note that the antiderivative in Example 6.1.3 can be expressed as

$$\frac{2x}{3}(x+5)^{3/2} - \frac{4}{15}(x+5)^{5/2} = (x+5)^{3/2}\left[\frac{2x}{3} - \frac{4}{15}(x+5)\right]$$

$$= (x+5)^{3/2}\left(\frac{2x}{5} - \frac{4}{3}\right) = (x+5)^{3/2}\left[\frac{2}{5}(x+5) - \frac{10}{3}\right]$$

$$= \frac{2}{5}(x+5)^{5/2} - \frac{10}{3}(x+5)^{3/2}$$

which is the form of the antiderivative obtained by substitution. This example shows that it is quite possible for you to do everything right and still not get the answer given at the back of the book. ■