Friday Mar 20 notes

This week's summary:

We're setting derivatives equal to zero, finding *x* where $f'(x)$ and $f''(x) = 0$, and gathering information from these data on the shape of more complex curves, primarily but not limited to polynomials. Herewith:

If $f'(c) = 0$, then c is a critical number. The function could have a local extreme at c or it could have an *inflection point.* Another type of critical number is that for which the derivative is undefined. Always be sure if you get this result that the function itself is defined there. Otherwise you would discard that value as a critical number.

We learned that if $f'(c) = 0$ and the derivative changes sign on either side of *c*¸then there is an extreme at *c*. But, we before invoking the first derivative test to see if the derivative changes sign, we could straight to the *second derivative test*, testing $f''(c)$ and drawing conclusions as follows:

- If $f''(c) > 0$, the function is concave up at c because the trend of the slope of the tangent is to increase. Thus, $f(c)$ is a local min.

- If $f''(c) < 0$, the function is concave down at c because the trend of the slope of the tangent is to decrease. Thus, $f(c)$ is a local max.

- If $f''(c) = 0$, we *might* have a local max or a local min, or we might have an inflection point (a place where concavity changes from up to down (bowl to inverted bowl shape). How do we interpret or conclude what the behavior is at $x = c$? We could do a couple of things:

  Resort to the first derivative test, checking values on either side of *c* to see if $f'(x)$ changes sign. If it does, we have a local extreme at $x = c$. If it doesn't, we have an inflection point at $x = c$.

  OR, we could stay with the second derivative, testing for a value on either side of c to see if $f''(x)$ changes sign. If $f'(x) < 0$ to the left of *c* and > 0 to the right of *c* (or if it is > 0 to the left and < 0 to the right) then *c* is clearly where there is an inflection point because concavity has changed.

  If $f'(x)$ does not change signs (it is either > 0 on both sides or < 0 on both sides), then $x = c$ is a local min or local max, resp.

Example 1: $f(x) = x^4$; $f'(x) = 4x^3 = 0$ at $x = 0$; $f''(x) = 12x^2 = 0$ at $x = 0$, also. What kind of critical point, then, is $c = 0$? Extreme or inflection? Two ways to find out:

First derivative test: Checking values into $f'(x)$ on either side of 0, $f'(-1) = 4(-1)^3 = -4$, and $f'(1) = 4(1)^3 = 4$. Thus, f' changes sign, negative to positive, so $c = 0$ is a local min.

Second derivative test: Checking values into $f''(x)$ on either side of 0, $f''(-1) = 12(-1)^2 = 12$ and $f''(1) = 12(1)^2 = 12$. No change in sign, f'' is positive on either side, so the function is concave up (why? The shape of concavity tells us about the rate of change of the derivative) and $c = 0$ is a local min.

Example 2: $f(x) = x^3$; $f'(x) = 3x^2 = 0$ at $x = 0$; $f''(x) = 6x = 0$ at $x = 0$, also. What kind of critical point, then, is $c = 0$? Two ways to find out:

First derivative test: Checking values into $f'(x)$ on either side of 0, $f'(-1) = 3(-1)^2 = 3$, and $f'(1) = 3(1)^2 = 3$. Thus, the function is increasing on either side of $c = 0$, so c is an inflection point. Does it change from concave up to down or the other way? Go to the second derivative test.

Second derivative test: Checking values into $f''(x)$ on either side of 0, $f''(-1) = 6(-1) = -6 < 0$. $f''(1) = 6(1) = 6 > 0$. Changes sign, the graph is concave down to the left of $c = 0$ and concave up to the right of it, at $c = 0$ is an inflection.

Important note: The function exists at c, so this investigation is valid. There are many graphs with asymptotes, and the derivative DNE exist there because the function does not. But you could still do a second derivative test on values on either side of c to discover concavity. The most accessible example of this is the hyperbola, $y = 1/x$. We'll investigate this in class. Try it yourself with second derivative test left and right of $x = 0$. Compare your answer to the graph.