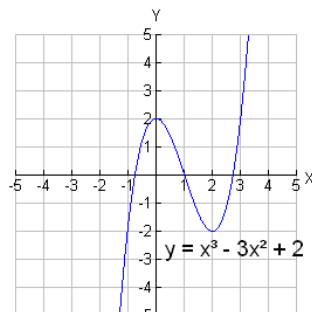


## Revised Summary of Sections 15-18

This unit shows how to use analytical methods of calculus to sketch graphs and interpret them.

Polynomial functions are *differentiable* at all values of their domain (the real numbers). Recall that **critical points** of a function are those points  $c$  where either  $f'(c) = 0$  or DNE.

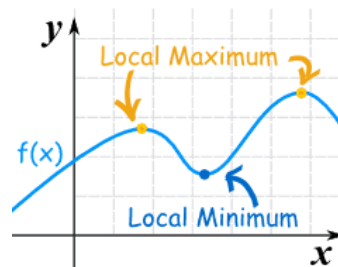
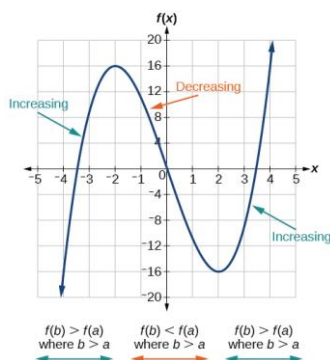


Polynomial functions are ideal for investigating intervals of increase and decrease, extremes, concavity and points of inflection.

### Definitions

A function  $f(x)$  is **increasing** on an interval  $I$  if for each  $a, b$  in  $I$ , when  $a < b$ ,  $f(a) < f(b)$ .

A function  $f(x)$  is **decreasing** on an interval  $I$  if for each  $a, b$  in  $I$ , when  $a < b$ ,  $f(a) > f(b)$ .



On an interval where  $f(x)$  is increasing,  $f'(x) > 0$ ; on an interval where  $f(x)$  is decreasing,  $f'(x) < 0$ .

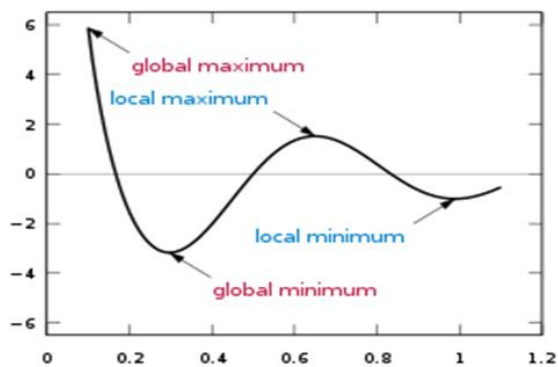
$f(x)$  has a **local maximum** at  $a$  if  $f(x) \leq f(a)$  for all  $x$  in an arbitrarily small neighborhood of  $a$ .

$f(x)$  has a **local minimum** at  $a$  if  $f(x) \geq f(a)$  for all  $x$  in an arbitrarily small neighborhood of  $a$ .

[Note: a function that is constant on an interval  $I$  has both a local a max and a local min on that interval.]

Local maximum and minimum values of a function are called **local extremes** of the function. When the restriction of the small neighborhood of  $a$  in interval  $I$  is removed, we inspect for the global extremes of the function.

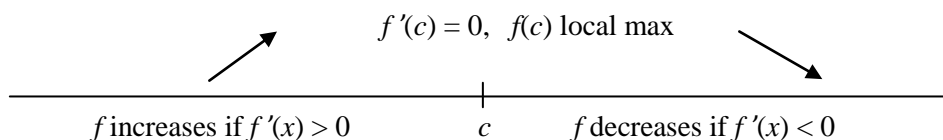
$M$  is a **global maximum** if for every  $x$  in  $I$ ,  $f(x) \leq M$ ;  $m$  is a **global minimum** if for every  $x$  in  $I$ ,  $f(x) \geq m$ .



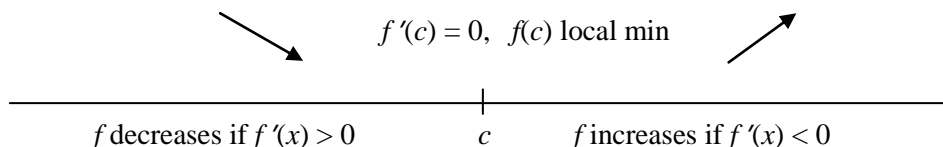
### First Derivative Test

Visually,  $f(c)$  is a local max if  $f$  is increasing when  $x < c$  and decreasing when  $x > c$ .  $f(c)$  is a local min if  $f$  is decreasing when  $x < c$  and increasing when  $x > c$ .

Analytically,  $f'(x)$  changes sign from positive to negative on either side of  $c$  when  $f(c)$  is a local max.



And  $f'(x)$  changes sign from negative to positive on either side of  $c$  when  $f(c)$  is a local min.

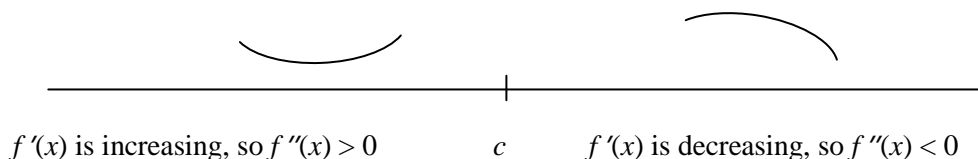


Thus, the FDT shows whether  $f(c)$  is a local max or min by the sign of  $f'(c)$  on either side of  $c$ .

### Second Derivative Test

Intervals of the graph where the shape is roughly similar to a cup **concave up**.

Intervals of the graph where the shape is roughly similar to a frown **concave down**.

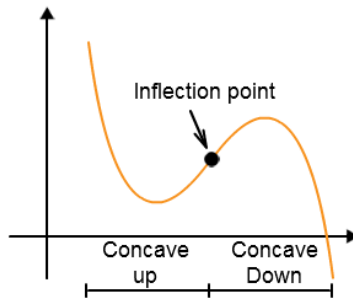


$f(x)$  is concave up on an interval where  $f'(x)$  is increasing because the progress of the slope of the tangent is to increase. Thus,  $f''(x) > 0$  on “concave up intervals.”

$f(x)$  is concave down on an interval where  $f'(x)$  is decreasing because the progress of the slope of the tangent is to decrease. Thus,  $f''(x) < 0$  on “concave down intervals.”

The SDT states that if  $f'(c) = 0$  and  $f''(c) > 0$ , then  $f(c)$  is a local min because the graph is concave up there. Likewise, if  $f'(c) = 0$  and  $f''(c) < 0$ , then  $f(c)$  is a local max because the graph is concave down there.

An **inflection point** is a point where the graph's concavity changes.



If  $f'(c) = 0$ ,  $f(c)$  could be an inflection point, but not necessarily. Note that  $f(x) = x^4$  has both a first and second derivative = 0 at  $x = 0$ , has an  $f(0)$  is a minimum of the function. (See examples below of basic functions and their critical and inflection points.)

To decide what the situation is in the case where  $f''(x) = 0$ , we do the following:

EITHER

1. Resort to the *first derivative test*, checking values on either side of  $c$  to see if  $f'(c)$  changes sign.

If it does, we have a local extreme at  $x = c$ .

If it doesn't, we have an inflection point at  $x = c$ .

OR

2. Stay with the *second derivative test*, testing values either side of  $c$  to see if  $f''(x)$  changes sign.

If  $f''(x)$  changes sign at  $c$ , then  $c$  is an *inflection point*, since concavity has changed.

If  $f''(x) > 0$  on *both sides* then  $x = c$  is a local min.

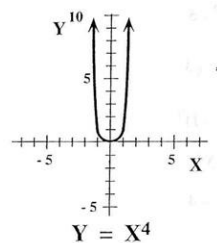
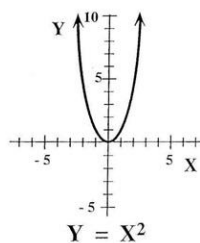
If  $f''(x) < 0$  on *both sides*, then  $x = c$  is a local max.

The following examples are illustrative because the functions are so simple to inspect through a sketch.

Example 1

$$f(x) = x^2 \text{ (parabola)}$$

$$f(x) = x^4 \text{ (quartic)}$$



Analyzing the parabola is easy:

*FDT*:  $f'(x) = 2x = 0$  at  $x = 0$ .  $f'(-1) = -2 < 0$ ,  $f$  is decreasing left of zero;  $f'(1) = 2 > 0$ ,  $f$  is increasing right of zero.  $f(0)$  is a min.

*SDT*:  $f''(x) = 2 > 0$  for all  $x$ , so  $f$  is concave up everywhere. Thus,  $f(0)$  is a min, as the graph shows.

Now the quartic:

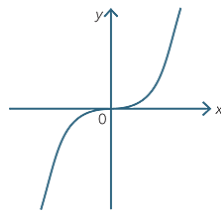
$f'(x) = 4x^3 = 0$  at  $x = 0$ ;  $f''(x) = 12x^2 = 0$  at  $x = 0$ , also. What kind of critical point, then, is  $c = 0$ ?

There are two ways to find out:

*FDT*: Checking values into  $f'(x)$  on either side of 0,  $f'(-1) = 4(-1)^3 = -4$  and  $f'(1) = 4(1)^3 = 4$ . Because  $f'$  changes sign, negative to positive,  $c = 0$  is a local min.

*SDT*: Checking values of  $f''(x)$  on either side of 0,  $f''(-1) = 12(-1)^2 = 12$  and  $f''(1) = 12(1)^2 = 12$ . No change in sign,  $f''$  is positive on either side, so the function is concave up and  $c = 0$  is a local min, as the graph shows.

*Example 2*  $f(x) = x^3$  (cubic)



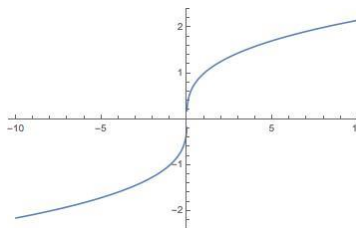
$f'(x) = 3x^2 = 0$  at  $x = 0$ ;  $f''(x) = 6x = 0$  at  $x = 0$  also. So, what kind of critical point is  $c = 0$ ? There are two ways to find out:

*FDT*: Checking values of  $f'(x)$  on either side of 0,  $f'(-1) = 3(-1)^2 = 3$ , and  $f'(1) = 3(1)^2 = 3$ . Thus, the function is increasing on either side of  $c = 0$ , so  $c$  is an inflection point, as the graph shows.

*SDT*: Checking values of  $f''(x)$  on either side of 0,  $f''(-1) = 6(-1) = -6 < 0$ .  $f''(1) = 6(1) = 6 > 0$ . The change in sign indicates the graph is concave down to the left of  $x = 0$  and concave up to the right of it, at  $x = 0$  is an inflection.

*Example 3*

$f(x) = x^{1/3}$  (cube root)



$$f'(x) = \frac{1}{3x^{2/3}}$$

We see that  $f'(0)$  does not exist (division by zero). In the DNE sense,  $c = 0$  is a critical value of the function. (You see from the graph that the tangent line to the function at  $x = 0$  is a vertical line.)

*FDT*: The function is increasing everywhere, as is easily seen in the graph; algebraically,  $f'(x)$  is positive everywhere it is defined, as  $x^{2/3}$  is the square of a cube root. Check  $f'(-1)$  and  $f'(1)$ .

Thus, by the first derivative test, the function is everywhere increasing. There is no max or min.

Is it an inflection point?  $f''(x) = -\frac{2}{9x^{5/3}}$

*SDT*:  $f''(x)$  DNE at  $x = 0$ , for the same reason (division by zero). Checking values of  $f''(x)$  on either side of 0:

$$f''(-1) = -\frac{2}{9(-1)^{5/3}} = \frac{2}{9} > 0 \text{ (concave up).}$$

$$f''(1) = -\frac{2}{9(1)^{5/3}} = -\frac{2}{9} < 0 \text{ (concave down). } x = 0 \text{ is a point of inflection, as the graph shows.}$$