Summary of Critical Values, Derivative Tests and Concavity

If $f\,'(c) = 0$, or if $f\,'(c)$ does not exist (DNE), then c is a critical number. The function could have a local extreme at c (including a nondifferentiable cusp or corner) or it could have an *inflection point.*

Before invoking the first derivative test to see if the derivative changes sign, we *could* go straight to the *second derivative test*, testing $f\,''(c)$ and drawing conclusions as follows:

- If $f\,''(c) > 0$, the function is concave up at c because the trend of the slope of the tangent is to increase. $f(c)$ is a local min.
- If $f\,''(c) < 0$, the function is concave down at c because the trend of the slope of the tangent is to decrease. $f(c)$ is a local max.
- If $f\,''(c) = 0$, we could have a local max or a local min, **or** we could have an inflection point. How do we decide what it is? There are a couple of things we can do:

EITHER

1. Resort to the f*irst derivative test*, checking values on either side of c to see if $f\,'(c)$ changes sign.
   If it does, we have a local extreme at $x = c$.
   If it doesn't, we have an inflection point at $x = c$.

OR

2. Stay with the *second derivative test*, testing for a value on either side of c to see if $f\,''(x)$ changes sign.
   If $f\,''(x) < 0$ to the left of c and > 0 to the right of c (or if it is > 0 to the left and < 0 to the right), then c is clearly an inflection point because concavity has changed.

   If $f\,''(x) > 0$ on *both sides* then $x = c$ is a local min

   If $f\,''(x) < 0$ on *both* sides, then $x = c$ is a local max.

Following are two examples that are illustrative because the functions are so simple you can already sketch them. Observe the derivative tests vis-a-vis the sketches to lock in your understanding.

Example 1: **Sketch the function $f(x) = x^4$**

$f\,'(x) = 4x^3 = 0$ at $x = 0$; $f\,''(x) = 12x^2 = 0$ at $x = 0$, also. What kind of critical point, then, is $c = 0$? There are two ways to find out:

First derivative test: Checking values into $f\,'(x)$ on either side of 0, $f\,'(-1) = 4(-1)^3 = -4$ and $f\,'(1) = 4(1)^3 = 4$. Because $f\,'$ changes sign, negative to positive, $c = 0$ is a local min.

Second derivative test: Checking values of $f\,''(x)$ on either side of 0, $f\,''(-1) = 12(-1)^2 = 12$ and $f\,''(1) = 12(1)^2 = 12$. No change in sign, $f\,''$ is positive on either side, so the function is concave up and $c = 0$ is a local min.

Example 2: **Sketch the function $f(x) = x^3$**

$f\,'(x) = 3x^2 = 0$ at $x = 0$; $f\,''(x) = 6x = 0$ at $x = 0$ also. So, what kind of critical point is $c = 0$? There are two ways to find out:

First derivative test: Checking values of $f'(x)$ on either side of 0, $f'(-1) = 3(-1)^2 = 3$, and $f'(1) = 3(1)^2 = 3$. Thus, the function is increasing on either side of $c = 0$, so c is an inflection point.

Second derivative test: Checking values of $f''(x)$ on either side of 0, $f''(-1) = 6(-1) = -6 < 0$. $f''(1) = 6(1) = 6 > 0$. The change in sign indicates the graph is concave down to the left of $x = 0$ and concave up to the right of it, at $x = 0$ is an inflection.

Example 3: Sketch the function $f'(x) = x^{1/3}$

$f'(x) = \dfrac{1}{3x^{2/3}}$. The function is differentiable on its domain, but its derivative cannot equal zero. However, $f'(0)$ does not exist (division by zero). In the DNE sense, $c = 0$ is a critical value of the function. (You should see from your graph that the tangent line to the function at $x = 0$ is a vertical line.)

First derivative test: The function is increasing everywhere, as is easily seen in the graph; algebraically, $f'(x)$ is positive on the domain, observable by the fact that $x^{2/3}$ is the square of a cube root. Thus, by the first derivative test, the function is everywhere increasing. There is no max or min. Is it an inflection point ?

$$f''(x) = -\frac{2}{9x^{5/3}}$$

Second derivative test: $f''(x)$ DNE at $x = 0$, for the same reason (division by zero). Checking values of $f''(x)$ on either side of 0, $f''(-1) = -\dfrac{2}{9(-1)^{5/3}} = \dfrac{2}{9} > 0$ (concave up). $f'(1) = -\dfrac{2}{9(1)^{5/3}} = -\dfrac{2}{9} < 0$ (concave down). $x = 0$ is a point of inflection.