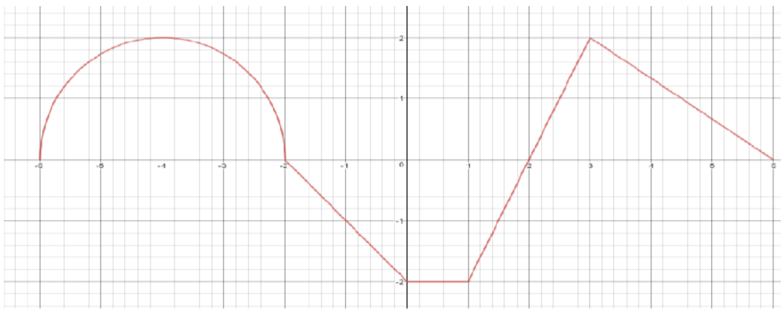
MATH 220 Exam 4 Review Exercises Part II Solutions

1. This is an exercise in finding area and signed area via the fundamental theorem of calculus and the area formulas of basic geometric shapes.



(a) $\int_{-6}^{-2} f(x)dx$, $\int_{-4}^{0} f(x)dx$, $\int_{0}^{3} |f(x)|dx$, $\int_{1}^{6} f(x)dx$, $\left|\int_{0}^{3} f(x)dx\right|$, $\int_{6}^{2} f(x)dx$, $\int_{3}^{3} f(x)dx$

(i) $\int_{-6}^{-2} f(x)dx$ represents the actual area and the signed area of a semicircle since the function lies above the x-axis. The region is centered on [–6, –2], so $r = |-6 - -2|/2 = 2$ (or just count the units on the axis). Hence, $\int_{-6}^{-2} f(x)dx = \frac{\pi r^2}{2} = \frac{\pi 2^2}{2} = 2\pi$.

(ii) $\int_{-4}^{0} f(x)dx$ is the signed area, $\int_{-4}^{-2} f(x)dx + \int_{-2}^{0} f(x)dx$. Notice the second definite integral will give a negative answer. What are these areas? The first is half the semicircle, or $\frac{2\pi}{2} = \pi$. The second is a triangle with $b = 2$ and $h = 2$. So $bh/2 = (2)(2)/2 = 2$. Hence,

$$\int_{-4}^{0} f(x)dx = \int_{-4}^{-2} f(x)dx - \int_{-2}^{0} f(x)dx = \pi - 2.$$

(iii) $\int_{0}^{3} |f(x)|dx$ indicates that the absolute value of the function be determined first. That is, we will break the region down as $\int_{0}^{2} |f(x)|dx + \int_{2}^{3} f(x)dx$. On [0, 2] there's a trapezoid. The formula $A_{\text{trapzoid}} = \frac{(b_1 + b_2)h}{2} = \frac{(2+1)2}{2} = 3$. But one can arrive at the same answer by breaking the region into a rectangle and a triangle, each with $b = 1$, $h = 2$. Hence,

$bh+\frac{bh}{2}=(1)(2)+\frac{(1)(2)}{2}=3$. In integral form, this is the same as negating the signed areas of the shapes under the *x*-axis. Hence, $\int_0^2 |f(x)|dx=3$. In detail, using properties of integrals,

$\int_0^2 |f(x)|dx=\int_0^1 |f(x)|dx+\int_1^2 |f(x)|dx=-\int_0^1 f(x)dx+-\int_1^2 f(x)dx=--2+--1=2+1=3.$

On [2, 3] the triangle sits, above the *x*-axis, *b* = 1, *h* = 2, so *A* = $\int_2^3 f(x)dx=\frac{bh}{2}=\frac{(1)(2)}{2}=1$.

Hence, $\int_0^3 |f(x)|dx=\int_0^2 |f(x)|dx+\int_2^3 f(x)dx=3+1=4.$

(iv) $\int_1^6 f(x)dx$ is the signed area of the two regions, both triangles. The integral is simply the sum of the integrals of the two regions. The first is below the *x*-axis, so its definite integral is negative. The second is above the axis, so it is positive. Thus, $\int_1^6 f(x)dx=-\int_1^2 f(x)dx+\int_2^6 f(x)dx.$ The first integral was found to be 1; the second is *bh*/2 = (4)(2)/2 = 4. Hence,

$$\int_1^6 f(x)dx=-\int_1^2 f(x)dx+\int_2^6 f(x)dx=-1+4=3.$$

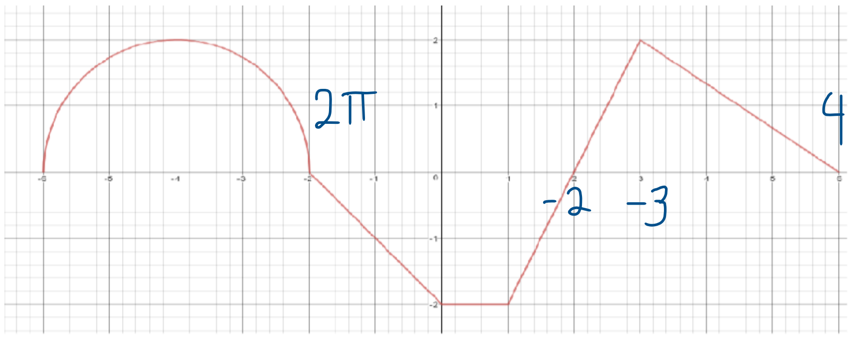
(v) $\left|\int_0^3 f(x)dx\right|=\left|\int_0^2 f(x)dx+\int_2^3 f(x)dx\right|=|-3+1|=|-2|=2$, which is the absolute value of the signed areas, already found.

(vi) $\int_6^2 f(x)dx=-\int_2^6 f(x)dx=-4$ from the property of definite integrals, $\int_a^b f(x)dx=-\int_b^a f(x)dx$

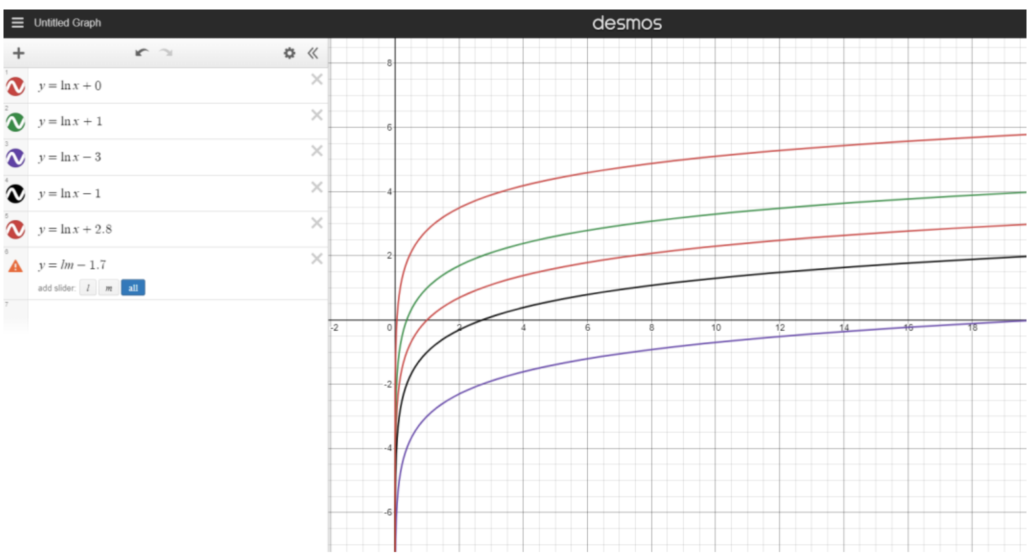
(vii) $\int_3^3 f(x)dx=0,$ since the area under a point has base equal to zero. From the property $\int_a^a f(x)dx$ $=F(a)-F(a)=0$

(b) On $[a,b]$, $f_{avg}=\frac{1}{b-a}\int_a^b f(x)dx$. Add up all the *signed areas* because f_{avg} seeks the average of all values of *f*, positive and negative, then divide by the length of the interval, [–6, 6]. The values of the signed areas are filled in to illustrate this easily. The average is, thus,

$$\frac{1}{6--6}(2\pi - 2 - 3 + 4) = \frac{1}{12}(2\pi - 1).$$



2. (a) $\int \frac{dx}{x} = \ln|x| + C = G(x)$



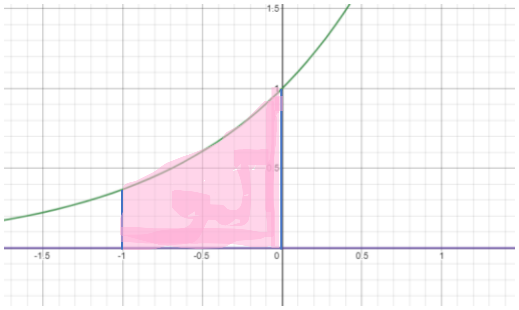
(b) Find the antiderivative that goes through the point (e, 0).

$$G(x) = \ln|x| + C,\ G(e) = \ln e + C = 0$$

$$C = 0 - \ln e = 0 - 1 = 1$$

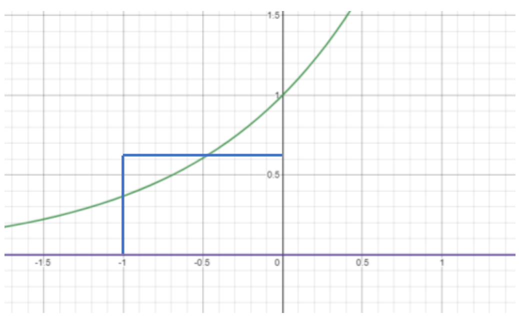
$$G(x) = \ln|x| - 1$$

3. (a) Sketch the region and find the area enclosed by $y = e^x$ and $y = 0$ on the interval [–1, 0].



$$\text{area} = \int_{-1}^{0} e^x dx = e^0 - e^{-1} = 1 - \tfrac{1}{e}$$

(b) The rectangle whose area is equal to the shaded region will have height = f_{avg}. This is the mean value theorem for integrals. I estimated it visually from comparing the areas between the function and the rectangle. However, since the interval length = 1 then f_{avg} is in fac the area calculated! That is, $f_{\text{avg}} = \frac{1}{0 - -1} \int_{-1}^{0} e^x dx = 1\left(1 - \tfrac{1}{e}\right) = 1 - \tfrac{1}{e}$, which is about 0.64. My rectanlge height is close to this value.



4. (a) You are finding F when P = \$1000, $r = 0.03$, $n = 12$, $t = 40/12$ years:

$$F = P(1 + r/n)^{nt} = 1000(1 + 0.03/12)^{40/12} \text{ dollars}$$

(b) Interest = F – 1000 dollars, where F is given above.

5. (a) $P = Fe^{-rt} = 250{,}000e^{-(0.01)(100)} = 250{,}000/e$

(b) $PV = \int_{0}^{100} (t + 200)e^{-0.01t} dt$

6. (a) $F = Pe^{rt}$, where $F = 2P$. Thus, $6{,}000 = 3{,}000e^{0.05t}$.

Solve for t by reducing first, then taking the natural log of both sides:

$$6{,}000 = 3{,}000e^{0.05t}$$

$$2 = e^{0.05t} \qquad \textit{(divide by coeff before taking the log)}$$

$$\ln 2 = \ln e^{0.05t} = 0.05t \ln e$$

$$\ln 2 = 0.05t$$

$$t = (\ln 2)/0.05 \text{ years}$$

The formula for time to double money under continuous growth is $t = \ln 2 / r$. The size of P doesn't matter.

(b) $f(t)$ = \$3,000, $r = 0.05$, FV = \$5,000, T is unknown. The income stream formula to use is:

$$6,000 = e^{(0.05)(T)}\int_0^T 3,000e^{-0.05t}\,dt$$

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$$2 = e^{0.05T}\int_0^T e^{-0.05t}\,dt$$

integrate $\qquad 2 = \dfrac{e^{0.05T}}{-0.05}\left[e^{-0.05T} - 1\right]$

distribute $e^{0.05T}$ $\qquad 2 = \dfrac{-1}{0.05}\left[1 - e^{0.05T}\right]$

cross multiply $\qquad -2(0.05) = 1 - e^{0.05T}$

simplify $\qquad -0.1 = 1 - e^{0.05T}$

rearrange $\qquad e^{0.05T} = 1 + 0.1$

ln each side $\qquad \ln e^{0.05T} = \ln 1.1$

simplify $\qquad 0.05T = \ln 1.1$

solve $\qquad T = \dfrac{\ln 1.1}{0.05}$