TECHNOLOGY CONNECTION

58–63. Use a calculator to check your answers to Exercises 17–22.

64. Business: growth of an investment. A company determines that the value of an investment is V, in millions of dollars, after time t, in years, where V is given by

$$V(t) = 5t^3 - 30t^2 + 45t + 5\sqrt{t}.$$

Note: Calculators often use only the variables y and x, so you may need to change the variables when entering this function.

a) Graph V over the interval $[0, 5]$.

b) Find the equation of the secant line passing through the points $(1, V(1))$ and $(5, V(5))$. Then graph this secant line using the same axes as in part (a).

c) Find the average rate of change of the investment between year 1 and year 5.

d) Repeat parts (b) and (c) for the following pairs of points: $(1, V(1))$ and $(4, V(4))$; $(1, V(1))$ and $(3, V(3))$; $(1, V(1))$ and $(1.5, V(1.5))$.

e) What *appears* to be the slope of the tangent line to the graph at the point $(1, V(1))$?

f) Approximate the rate at which the value of the investment is changing after 1 yr.

65. Use a calculator to determine where $f'(x)$ does not exist, if $f(x) = \sqrt[3]{x-5}$.

Answers to Quick Checks

1. $f'(x) = 3x^2 + 2x$; $f'(-2) = 8$, $f'(4) = 56$

2. $f'(x) = \dfrac{2}{x^2}$ **3.** At $x = -6$, the graph has a corner.

1.5 Differentiation Techniques: The Power and Sum–Difference Rules

OBJECTIVES

- Differentiate using the Power Rule or the Sum–Difference Rule.
- Differentiate a constant or a constant times a function.
- Determine points at which a tangent line has a specified slope.

Leibniz Notation

Let y be a function of x. A common way to express "the derivative of y with respect to x" is the notation

$$\frac{dy}{dx}.$$

This notation was invented by the German mathematician Leibniz. Using this notation, we can write the following sentence:

If $y = f(x)$, then the derivative of y with respect to x is $\dfrac{dy}{dx} = f'(x)$.

In practice, we often use *prime notation*, such as y' or $f'(x)$, to represent a derivative when there is no confusion as to which variables are involved. The dy/dx notation is a little more formal than the prime notation but has the same meaning. We will use both types of notation often.

When we wish to evaluate a derivative at a number, we write

$$\left.\frac{dy}{dx}\right|_{x=2} = f'(2).$$

The vertical line is interpreted as "evaluated at," so the above expression is read as "the derivative of y with respect to x evaluated at $x = 2$ is the value $f'(2)$."

We can also write

$$\frac{d}{dx}f(x).$$

This is identical in meaning to dy/dx and is another way to denote the derivative of the function. When placed next to a function, d/dx is treated as a command to find the function's derivative. Using functions from previous sections, we can write

$$\frac{d}{dx}x^2 = 2x, \quad \frac{d}{dx}x^3 = 3x^2, \quad \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}, \quad \text{and so on.}$$

Historical Note: The German mathematician and philosopher Gottfried Wilhelm von Leibniz (1646–1716) and the English mathematician, philosopher, and physicist Sir Isaac Newton (1642–1727) are both credited with the invention of calculus, though each performed his work independently. Newton used the dot notation $\dot{y}$ for dy/dt, where y is a function of time; this notation is still used, though it is not as common as Leibniz notation.

We will use all of these derivative forms often, and with practice, their use will become natural. In Chapter 2, we will discuss specific meanings of dy and dx.

The Power Rule

In Section 1.4, we calculated the derivative for some simple power functions. Look at the following table and see if you can identify a pattern:

Function	Derivative	
x^2	$2x^1$	Example 1 in Section 1.4
x^3	$3x^2$	Example 2 in Section 1.4
x^4	$4x^3$	Exercise 42 in Section 1.4
$\frac{1}{x} = x^{-1}$	$-1 \cdot x^{-2} = \frac{-1}{x^2}$	Example 4 in Section 1.4
$\frac{1}{x^2} = x^{-2}$	$-2 \cdot x^{-3} = \frac{-2}{x^3}$	Exercise 45 in Section 1.4
$\sqrt{x} = x^{1/2}$	$\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$	Exercise 46 in Section 1.4

The pattern can be described as follows: "To find the derivative of a power function, bring the exponent to the front of the variable as a coefficient and reduce the exponent by 1."

1. Write the exponent as the coefficient.

$$\frac{d}{dx}x^k = k \cdot x^{k-1}$$

2. Subtract 1 from the exponent.

This rule is summarized as the following theorem.

THEOREM 1 **The Power Rule**

For any real number k, if $y = x^k$, then

$$\frac{d}{dx}x^k = k \cdot x^{k-1}.$$

We proved this theorem for the cases where $k = 2$, 3, and -1 in Examples 1, 2, and 4 in Section 1.4 and for other cases as exercises at the end of that section. The proof of this theorem for the case where k is any positive integer is very elegant.

Proof. Let $f(x) = x^k$. We need to find the expanded form for $f(x + h) = (x + h)^k$ so that we can set up the difference quotient. When $(x + h)^k$ is multiplied (expanded), a pattern becomes evident, as the following shows:

$$\begin{aligned}
(x + h)^1 &= x + h, \\
(x + h)^2 &= x^2 + 2xh + h^2, \\
(x + h)^3 &= x^3 + 3x^2h + 3xh^2 + h^3, \\
(x + h)^4 &= x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4.
\end{aligned}$$

The first term is x^k, and the second term is $kx^{k-1}h$. The terms in the shaded triangle all contain h to the power of 2 or greater. Calling these "shaded terms," we can summarize the above expansion as follows:

$$(x + h)^k = x^k + kx^{k-1}h + (\textit{shaded terms}).$$

We now substitute for $(x + h)^k$ in the difference quotient:

$$\frac{f(x + h) - f(x)}{h} = \frac{(x + h)^k - x^k}{h} = \frac{x^k + kx^{k-1}h + (\textit{shaded terms}) - x^k}{h}.$$

The x^k terms in the numerator sum to 0, and h is factored out. The "shaded terms" now contain h to the power of 1 or greater, and we refer to them as the "reduced shaded terms." The h's in the numerator and the denominator cancel, and we have

$$\frac{h[kx^{k-1} + (\textit{reduced shaded terms})]}{h} = kx^{k-1} + (\textit{reduced shaded terms}).$$

When we take the limit as $h \to 0$, the "reduced shaded terms" become 0:

$$f'(x) = \lim_{h \to 0} kx^{k-1} + (\textit{reduced shaded terms}) = kx^{k-1}. \quad \blacksquare$$

Although we have proved the Power Rule only for the case where k is a positive integer, it is valid for all real numbers k. However, a complete proof of this fact is outside the scope of this book.

EXAMPLE 1 Differentiate each of the following:

a) $y = x^5$; **b)** $y = x$; **c)** $y = x^{-4}$.

Solution

a) $\frac{d}{dx}x^5 = 5 \cdot x^{5-1} = 5x^4$ Using the Power Rule

b) $\frac{d}{dx}x = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1$

c) $\frac{d}{dx}x^{-4} = -4 \cdot x^{-4-1} = -4x^{-5}$, or $-4 \cdot \frac{1}{x^5}$, or $-\frac{4}{x^5}$

❮ Quick Check 1

> **❯ Quick Check 1**
>
> **a)** Differentiate:
>
> (i) $y = x^{15}$; (ii) $y = x^{-7}$.
>
> **b)** Explain why $\frac{d}{dx}(\pi^2) = 0$, not 2π.

The Power Rule also allows us to differentiate expressions with rational exponents.

EXAMPLE 2 Differentiate:

a) $y = \sqrt[5]{x}$; **b)** $y = x^{0.7}$.

Solution

a) $\frac{d}{dx}\sqrt[5]{x} = \frac{d}{dx}x^{1/5} = \frac{1}{5} \cdot x^{(1/5)-1}$

$= \frac{1}{5}x^{-4/5}$, or $\frac{1}{5} \cdot \frac{1}{x^{4/5}}$, or $\frac{1}{5} \cdot \frac{1}{\sqrt[5]{x^4}}$, or $\frac{1}{5\sqrt[5]{x^4}}$

b) $\frac{d}{dx}x^{0.7} = 0.7x^{(0.7)-1} = 0.7x^{-0.3}$

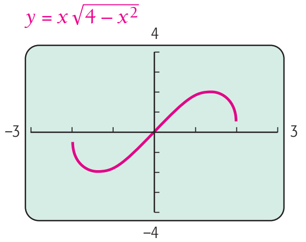
❮ Quick Check 2

> **❯ Quick Check 2**
>
> Differentiate:
>
> **a)** $y = \sqrt[4]{x}$;
>
> **b)** $y = x^{-1.25}$.

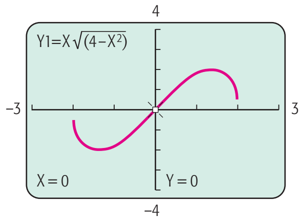
TECHNOLOGY CONNECTION

Numerical Differentiation and Tangent Lines

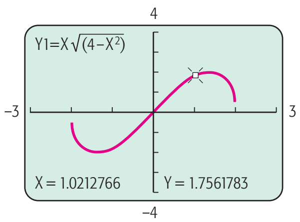
Consider $f(x) = x\sqrt{4 - x^2}$, graphed below.

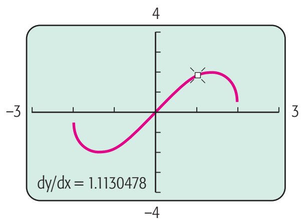


To find the value of dy/dx at a point, we select dy/dx from the CALC menu.

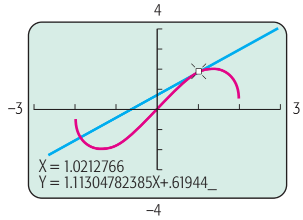


Next we key in the desired x-value or use the arrow keys to move the cursor to the desired point. We then press ENTER to obtain the value of the derivative at the given x-value.





We can also use the Tangent feature from the DRAW menu to draw the tangent line at the point where the derivative was found. Both the line and its equation will appear on the calculator screen.



EXERCISES

For each of the following functions, use dy/dx to find the derivative, and then draw the tangent line at the given point. When selecting the viewing window, be sure to include the specified x-values.

1. $f(x) = x(200 - x)$;
$x = 24, x = 138, x = 150, x = 190$

2. $f(x) = x^3 - 6x^2$;
$x = -2, x = 0, x = 2, x = 4, x = 6.3$

3. $f(x) = -4.32 + 1.44x + 3x^2 - x^3$;
$x = -0.5, x = 0.5, x = 2.1$

In Section 1.3, we found the simplified difference quotient for two common functions: $f(x) = 1/x$ and $f(x) = \sqrt{x}$. By taking the limit of each difference quotient as $h \to 0$, we find the derivative of the function as follows:

- For $f(x) = 1/x$, we have

$$f'(x) = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}. \qquad \text{(See Example 7 in Section 1.3.)}$$

- For $f(x) = \sqrt{x}$, we have

$$f'(x) = \lim_{h \to 0} \left(\frac{1}{\sqrt{x} + \sqrt{x+h}} \right) = \frac{1}{2\sqrt{x}}. \qquad \text{(See Example 8 in Section 1.3.)}$$

These two functions are very common in calculus, so it may be helpful to memorize their derivative forms. The Power Rule can be used to confirm these results.

- For $f(x) = 1/x$, we rewrite the function as a power: $f(x) = x^{-1}$. The Power Rule then gives

$$f'(x) = (-1)x^{(-1)-1} = (-1)x^{-2} = -\frac{1}{x^2}.$$

- For $f(x) = \sqrt{x}$, we rewrite the function as a power: $f(x) = x^{1/2}$. The Power Rule then gives

$$f'(x) = \frac{1}{2}x^{(1/2)-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}.$$

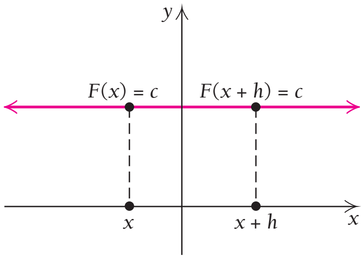
TECHNOLOGY CONNECTION

Exploratory

Graph the constant function $y = -3$. Then find the derivative of this function at $x = -6$, $x = 0$, and $x = 8$. What do you conclude about the derivative of a constant function?

The Derivative of a Constant Function

Consider the constant function given by $F(x) = c$. Note that the slope at each point on its graph is 0.



This suggests the following theorem.

THEOREM 2

The derivative of a constant function is 0. That is, $\frac{d}{dx}c = 0$.

Proof. Let F be the function given by $F(x) = c$. Then

$$\frac{F(x+h) - F(x)}{h} = \frac{c - c}{h}$$

$$= \frac{0}{h} = 0.$$

The difference quotient for this function is always 0. Thus, as h approaches 0, the limit of the difference quotient is 0, so $F'(x) = 0$. ■

The Derivative of a Constant Times a Function

Now let's consider differentiating functions such as

$$f(x) = 5x^2 \quad \text{and} \quad g(x) = -7x^4.$$

Note that we already know how to differentiate x^2 and x^4. Let's look for a pattern in the results of Section 1.4 and its exercise set.

Function	Derivative
$5x^2$	$10x$
$3x^{-1}$	$-3x^{-2}$
$\frac{3}{2}x^2$	$3x$
$1 \cdot x^3$	$3x^2$

Perhaps you have discovered the following theorem.

THEOREM 3

The derivative of a constant times a function is the constant times the derivative of the function. Using derivative notation, we can write this as

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx} f(x).$$

Proof. Let $F(x) = cf(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} && \text{Using the definition of a derivative} \\ &= \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} && \text{Substituting} \\ &= c \cdot \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h}\right] && \text{Factoring and using the Limit Properties} \\ &= c \cdot f'(x). && \text{Using the definition of a derivative} \quad \blacksquare \end{aligned}$$

Combining this rule with the Power Rule allows us to find many derivatives.

EXAMPLE 3 Find each of the following derivatives:

a) $\frac{d}{dx} 7x^4$; **b)** $\frac{d}{dx}(-9x)$; **c)** $\frac{d}{dx}\left(\frac{1}{5x^2}\right)$.

Solution

a) $\frac{d}{dx}7x^4 = 7\frac{d}{dx}x^4 = 7 \cdot 4 \cdot x^{4-1} = 28x^3$ With practice, this may be done in one step.

b) $\frac{d}{dx}(-9x) = -9\frac{d}{dx}x = -9 \cdot 1 = -9$

c)
$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{5x^2}\right) = \frac{d}{dx}\left(\frac{1}{5}x^{-2}\right) &= \frac{1}{5} \cdot \frac{d}{dx}x^{-2} \\ &= \frac{1}{5}(-2)x^{-2-1} \\ &= -\frac{2}{5}x^{-3}, \quad \text{or} \quad -\frac{2}{5x^3} \end{aligned}$$

❮ Quick Check 3

❯ **Quick Check 3**

Differentiate each of the following:

a) $y = 10x^9$;

b) $y = \pi x^3$;

c) $y = \frac{2}{3x^4}$.

A common mistake is to write an expression such as $1/(5x^2)$ as $(5x)^{-2}$, which is incorrect. The exponent 2 applies only to the x; the 5 is part of the coefficient $\frac{1}{5}$. Carefully examine Example 3c, noting how the final answer is simplified.

Recall from Section 1.4 that a function's derivative at x is also its instantaneous rate of change at x.

EXAMPLE 4 **Life Science: Volume of a Tumor.** The volume V of a spherical tumor can be approximated by

$$V(r) = \tfrac{4}{3}\pi r^3,$$

where r is the radius of the tumor, in centimeters.

a) Find the rate of change of the volume with respect to the radius.

b) Find the rate of change of the volume at $r = 1.2$ cm.

Solution

a) $\dfrac{dV}{dr} = V'(r) = 3 \cdot \dfrac{4}{3} \cdot \pi r^2 = 4\pi r^2$

(This expression turns out to be equal to the tumor's surface area.)

b) $V'(1.2) = 4\pi(1.2)^2 = 5.76\pi \approx 18\dfrac{\text{cm}^3}{\text{cm}} = 18\text{ cm}^2$

When the radius is 1.2 cm, the volume is changing at the rate of 18 cm^3 for every change of 1 cm in the radius.

The Derivative of a Sum or a Difference

In Exercise 11 of Exercise Set 1.4, you found that the derivative of

$$f(x) = x^2 + x$$

is
$$f'(x) = 2x + 1.$$

Note that the derivative of x^2 is $2x$, the derivative of x is 1, and the sum of these derivatives is $f'(x)$. This illustrates the following.

TECHNOLOGY CONNECTION

Exploratory

Let $y_1 = x(100 - x)$, and $y_2 = x\sqrt{100 - x^2}$. Using the Y-VARS option from the VARS menu, enter Y3 as Y1+Y2. Find the derivative of each of the three functions at $x = 8$ using the numerical differentiation feature.

Compare your answers. How do you think you can find the derivative of a sum?

THEOREM 4 **The Sum–Difference Rule**

Sum. The derivative of a sum is the sum of the derivatives:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Difference. The derivative of a difference is the difference of the derivatives:

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x).$$

Proof. The proof of the Sum Rule relies on the fact that the limit of a sum is the sum of the limits. Let $F(x) = f(x) + g(x)$. Then

$$\begin{aligned}\lim_{h\to 0}\frac{F(x+h) - F(x)}{h} &= \lim_{h\to 0}\frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\ &= \lim_{h\to 0}\left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right] = f'(x) + g'(x).\end{aligned}$$

To prove the Difference Rule, we note that

$$\begin{aligned}\frac{d}{dx}(f(x) - g(x)) &= \frac{d}{dx}(f(x) + (-1)g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}(-1)g(x) \\ &= f'(x) + (-1)\frac{d}{dx}g(x) = f'(x) - g'(x). \quad \blacksquare\end{aligned}$$

Any function that is a sum or difference of several terms can be differentiated term by term.

EXAMPLE 5 Find each of the following derivatives:

a) $\dfrac{d}{dx}(5x^3 - 7)$; **b)** $\dfrac{d}{dx}\left(24x - \sqrt{x} + \dfrac{5}{x}\right)$.

Solution

a)
$$\begin{aligned}\frac{d}{dx}(5x^3 - 7) &= \frac{d}{dx}(5x^3) - \frac{d}{dx}(7)\\ &= 5\frac{d}{dx}x^3 - 0\\ &= 5\cdot 3x^2\\ &= 15x^2\end{aligned}$$

b)
$$\begin{aligned}\frac{d}{dx}\left(24x - \sqrt{x} + \frac{5}{x}\right) &= \frac{d}{dx}(24x) - \frac{d}{dx}(\sqrt{x}) + \frac{d}{dx}\left(\frac{5}{x}\right)\\ &= 24\cdot\frac{d}{dx}x - \frac{d}{dx}x^{1/2} + 5\cdot\frac{d}{dx}x^{-1}\\ &= 24\cdot 1 - \frac{1}{2}x^{(1/2)-1} + 5(-1)x^{-1-1}\\ &= 24 - \frac{1}{2}x^{-1/2} - 5x^{-2}\\ &= 24 - \frac{1}{2\sqrt{x}} - \frac{5}{x^2}\end{aligned}$$

❮ Quick Check 4

❯ Quick Check 4

Differentiate:

$$y = 3x^5 + 2\sqrt[3]{x} + \frac{1}{3x^2} + \sqrt{5}.$$

A word of caution! The derivative of

$f(x) + c,$

a function plus a constant, is just the derivative of the function,

$f'(x).$

The derivative of

$c\cdot f(x),$

a function times a constant, is the constant times the derivative

$c\cdot f'(x).$

That is, the constant is retained for a product, but not for a sum.

Slopes of Tangent Lines

It is important to be able to determine points at which the tangent line to a curve has a certain slope, that is, points at which the derivative attains a certain value.

EXAMPLE 6 Find the points on the graph of $f(x) = -x^3 + 6x^2$ at which the tangent line is horizontal.

Solution The derivative is used to find the slope of a tangent line, and a horizontal tangent line has slope 0. Therefore, we are seeking all x for which $f'(x) = 0$:

$$f'(x) = 0 \qquad \text{Setting the derivative equal to 0}$$

$$\frac{d}{dx}(-x^3 + 6x^2) = 0$$

$$-3x^2 + 12x = 0. \qquad \text{Differentiating}$$

We factor and solve:

$$-3x(x - 4) = 0$$

$$-3x = 0 \quad or \quad x - 4 = 0$$

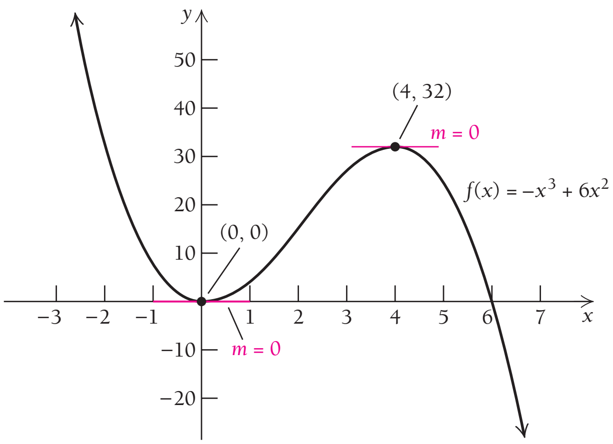
$$x = 0 \quad or \quad x = 4.$$

We are to find the points *on the graph*, so we must determine the second coordinates from the original equation, $f(x) = -x^3 + 6x^2$.

$$f(0) = -0^3 + 6 \cdot 0^2 = 0.$$

$$f(4) = -4^3 + 6 \cdot 4^2 = -64 + 96 = 32.$$

Thus, the points we are seeking are $(0, 0)$ and $(4, 32)$, as shown on the graph.



EXAMPLE 7 Find the points on the graph of $f(x) = -x^3 + 6x^2$ at which the tangent line has slope 9.

Solution We want to find values of x for which $f'(x) = 9$. That is, we want to find x such that

$$-3x^2 + 12x = 9. \qquad \text{As in Example 6, note that } \frac{d}{dx}(-x^3 + 6x^2) = -3x^2 + 12x.$$

To solve, we add -9 on both sides and get

$$-3x^2 + 12x - 9 = 0.$$

We then multiply both sides of the equation by $-\frac{1}{3}$, giving

$$x^2 - 4x + 3 = 0,$$

which is factored as follows:

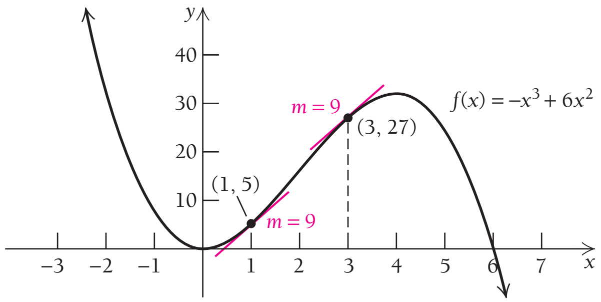
$$(x - 3)(x - 1) = 0.$$

TECHNOLOGY CONNECTION

EXERCISE

1. Graph $y = \frac{1}{3}x^3 - 2x^2 + 4x$, and draw tangent lines at various points. Estimate points at which the tangent line is horizontal. Then use calculus, as in Examples 6 and 7, to find the exact results.

We have two solutions: $x = 1$ or $x = 3$. We need the actual coordinates: when $x = 1$, we have $f(1) = -(1)^3 + 6(1)^2 = 5$. Therefore, at the point $(1, 5)$ on the graph of $f(x)$, the tangent line has a slope of 9. In a similar way, we can state that the tangent line at the point $(3, 27)$ has a slope of 9 as well. All of this is illustrated in the following graph.



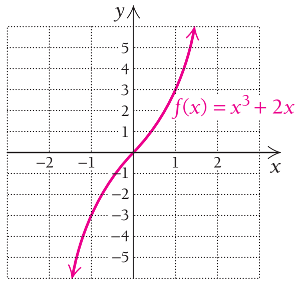
Quick Check 5

For the function in Example 7, find the x-values for which $f'(x) = -15$.

Quick Check 5

Analyzing a Function by Its Derivative

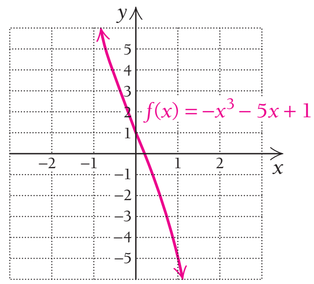
Some functions are always increasing or always decreasing. For example, the function $f(x) = x^3 + 2x$ is always increasing. That is, at no time does the graph of this function run "downhill" or lie flat. It is steadily increasing: all tangent lines have positive slopes. How can we use the function's derivative to demonstrate this fact?



The derivative of this function is $dy/dx = f'(x) = 3x^2 + 2$. For any x-value, x^2 will be nonnegative; the expression $3x^2 + 2$ is thus positive for all x. It is impossible to set this derivative equal to any negative quantity and solve for x (try it). The graph of the function shows the always increasing trend that the derivative proves has to be true.

EXAMPLE 8 Let $f(x) = -x^3 - 5x + 1$. Is this function always increasing or always decreasing? Use its derivative to support your conjecture.

Solution The graph of f is shown at the right.



Based on the graph alone, the function appears to be always decreasing, but how do we know we aren't missing something, since we are looking at only a small portion of the graph? The graph alone is not enough to "prove" our observation. We need to use the derivative:

$$f'(x) = -3x^2 - 5.$$

Since x^2 is always 0 or positive, $-3x^2$ is always negative or 0. Subtracting 5 from $-3x^2$ will always give a negative result. Therefore, the derivative is always negative for all real numbers x. This means all tangent lines to this graph have a negative ("downhill") slope. Thus, the graph is always decreasing.

Examples 6, 7, and 8 illustrate ways to use the derivative to analyze the behavior of a function much more accurately than can be done by observation alone. In fact, our eyes can deceive us! For example, the graph of $f(x) = x^3 - x^2$ appears to be always increasing if viewed on the standard window of the TI-83. However, it does have a small interval where it is decreasing, which will be shown in Exercise 133.

Section Summary

- Common forms of notation for the derivative of a function are y', $f'(x)$, $\frac{dy}{dx}$, and $\frac{d}{dx}f(x)$.
- The *Power Rule* for differentiation is $\frac{d}{dx}[x^k] = kx^{k-1}$, for all real numbers k.
- The derivative of a constant is zero: $\frac{d}{dx}c = 0$.
- The derivative of a constant times a function is the constant times the derivative of the function: $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}f(x)$.
- The derivative of a sum (or difference) is the sum (or difference) of the derivatives of the terms: $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$.

EXERCISE SET 1.5

Find $\frac{dy}{dx}$.

1. $y = x^7$

2. $y = x^8$

3. $y = -3x$

4. $y = -0.5x$

5. $y = 12$

6. $y = 7$

7. $y = 2x^{15}$

8. $y = 3x^{10}$

9. $y = x^{-6}$

10. $y = x^{-8}$

11. $y = 4x^{-2}$

12. $y = 3x^{-5}$

13. $y = x^3 + 3x^2$

14. $y = x^4 - 7x$

15. $y = 8\sqrt{x}$

16. $y = 4\sqrt{x}$

17. $y = x^{0.9}$

18. $y = x^{0.7}$

19. $y = \frac{1}{2}x^{4/5}$

20. $y = -4.8x^{1/3}$

21. $y = \frac{7}{x^3}$

22. $y = \frac{6}{x^4}$

23. $y = \frac{4x}{5}$

24. $y = \frac{3x}{4}$

Find each derivative.

25. $\frac{d}{dx}\left(\sqrt[4]{x} - \frac{3}{x}\right)$

26. $\frac{d}{dx}\left(\sqrt[5]{x} - \frac{2}{x}\right)$

27. $\frac{d}{dx}\left(\sqrt{x} - \frac{2}{\sqrt{x}}\right)$

28. $\frac{d}{dx}\left(\sqrt[3]{x} + \frac{4}{\sqrt{x}}\right)$

29. $\frac{d}{dx}\left(-2\sqrt[3]{x^5}\right)$

30. $\frac{d}{dx}\left(-\sqrt[4]{x^3}\right)$

31. $\frac{d}{dx}(5x^2 - 7x + 3)$

32. $\frac{d}{dx}(6x^2 - 5x + 9)$

Find $f'(x)$.

33. $f(x) = 0.6x^{1.5}$

34. $f(x) = 0.3x^{1.2}$

35. $f(x) = \frac{2x}{3}$

36. $f(x) = \frac{3x}{4}$

37. $f(x) = \frac{4}{7x^3}$

38. $f(x) = \frac{2}{5x^6}$

39. $f(x) = \frac{5}{x} - x^{2/3}$

40. $f(x) = \frac{4}{x} - x^{3/5}$

41. $f(x) = 4x - 7$

42. $f(x) = 7x - 14$

43. $f(x) = \frac{x^{4/3}}{4}$

44. $f(x) = \frac{x^{3/2}}{3}$

45. $f(x) = -0.01x^2 - 0.5x + 70$

46. $f(x) = -0.01x^2 + 0.4x + 50$

Find y'.

47. $y = 3x^{-2/3} + x^{3/4} + x^{6/5} + \frac{8}{x^3}$

48. $y = x^{-3/4} - 3x^{2/3} + x^{5/4} + \frac{2}{x^4}$

49. $y = \frac{2}{x} - \frac{x}{2}$

50. $y = \frac{x}{7} + \frac{7}{x}$

51. If $f(x) = x^2 + 4x - 5$, find $f'(10)$.

52. If $f(x) = \sqrt{x}$, find $f'(4)$.

53. If $y = \frac{4}{x^2}$, find $\left.\frac{dy}{dx}\right|_{x=-2}$

54. If $y = x + \frac{2}{x^3}$, find $\left.\frac{dy}{dx}\right|_{x=1}$

55. If $y = x^3 + 2x - 5$, find $\left.\frac{dy}{dx}\right|_{x=-2}$

56. If $y = \sqrt[3]{x} + \sqrt{x}$, find $\left.\frac{dy}{dx}\right|_{x=64}$

57. If $y = \frac{1}{3x^4}$, find $\left.\frac{dy}{dx}\right|_{x=-1}$

58. If $y = \frac{2}{5x^3}$, find $\left.\frac{dy}{dx}\right|_{x=4}$

59. Find an equation (in $y = mx + b$ form) of the tangent line to the graph of $f(x) = x^3 - 2x + 1$
a) at $(2, 5)$; **b)** at $(-1, 2)$; **c)** at $(0, 1)$.

60. Find an equation of the tangent line to the graph of $f(x) = x^2 - \sqrt{x}$
a) at $(1, 0)$; **b)** at $(4, 14)$; **c)** at $(9, 78)$.

61. Find an equation of the tangent line to the graph of $f(x) = \frac{1}{x^2}$
a) at $(1, 1)$; **b)** at $\left(3, \frac{1}{9}\right)$; **c)** at $\left(-2, \frac{1}{4}\right)$.

62. Find the equation of the tangent line to the graph of $g(x) = \sqrt[3]{x^2}$
a) at $(-1, 1)$; **b)** at $(1, 1)$; **c)** at $(8, 4)$.

For each function, find the points on the graph at which the tangent line is horizontal. If none exist, state that fact.

63. $y = x^2 - 3$

64. $y = -x^2 + 4$

65. $y = -x^3 + 1$

66. $y = x^3 - 2$

67. $y = 3x^2 - 5x + 4$

68. $y = 5x^2 - 3x + 8$

69. $y = -0.01x^2 - 0.5x + 70$

70. $y = -0.01x^2 + 0.4x + 50$

71. $y = 2x + 4$

72. $y = -2x + 5$

73. $y = 4$

74. $y = -3$

75. $y = -x^3 + x^2 + 5x - 1$

76. $y = -\frac{1}{3}x^3 + 6x^2 - 11x - 50$

77. $y = \frac{1}{3}x^3 - 3x + 2$

78. $y = x^3 - 6x + 1$

79. $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2$

80. $f(x) = \frac{1}{3}x^3 - 3x^2 + 9x - 9$

For each function, find the points on the graph at which the tangent line has slope 1.

81. $y = 20x - x^2$

82. $y = 6x - x^2$

83. $y = -0.025x^2 + 4x$

84. $y = -0.01x^2 + 2x$

85. $y = \frac{1}{3}x^3 + 2x^2 + 2x$

86. $y = \frac{1}{3}x^3 - x^2 - 4x + 1$

APPLICATIONS

Life Sciences

87. **Healing wound.** The circular area A, in square centimeters, of a healing wound is approximated by

$A(r) = 3.14r^2$,

where r is the wound's radius, in centimeters.

a) Find the rate of change of the area with respect to the radius.

b) Explain the meaning of your answer to part (a).

88. **Healing wound.** The circumference C, in centimeters, of a healing wound is approximated by

$C(r) = 6.28r$,

where r is the wound's radius, in centimeters.

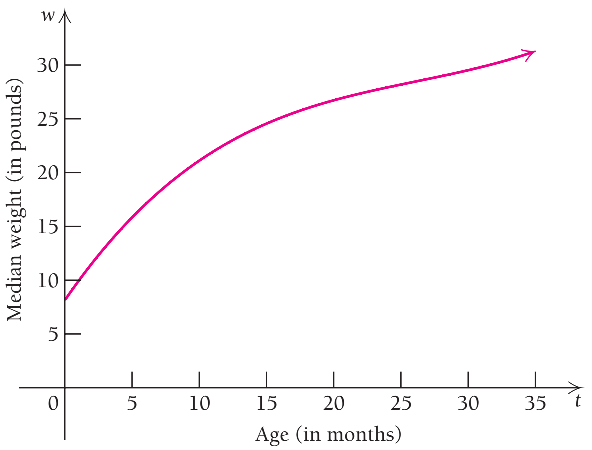
a) Find the rate of change of the circumference with respect to the radius.

b) Explain the meaning of your answer to part (a).

89. **Growth of a baby.** The median weight of a boy whose age is between 0 and 36 months can be approximated by the function

$w(t) = 8.15 + 1.82t - 0.0596t^2 + 0.000758t^3$,

where t is measured in months and w is measured in pounds.



(*Source*: Centers for Disease Control. Developed by the National Center for Health Statistics in collaboration with the National Center for Chronic Disease Prevention and Health Promotion, 2000.)

Use this approximation to find the following for a boy with median weight:

a) The rate of change of weight with respect to time.
b) The weight of the baby at age 10 months.
c) The rate of change of the baby's weight with respect to time at age 10 months.

90. Temperature during an illness. The temperature T of a person during an illness is given by

$$T(t) = -0.1t^2 + 1.2t + 98.6,$$

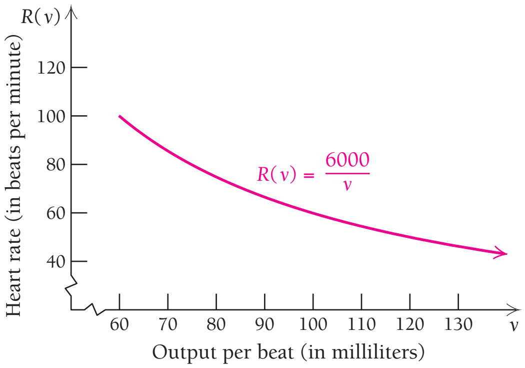
where T is the temperature, in degrees Fahrenheit, at time t, in days.

a) Find the rate of change of the temperature with respect to time.
b) Find the temperature at $t = 1.5$ days.
c) Find the rate of change at $t = 1.5$ days.

91. Heart rate. The equation

$$R(v) = \frac{6000}{v}$$

can be used to determine the heart rate, R, of a person whose heart pumps 6000 milliliters (mL) of blood per minute and v milliliters of blood per beat. (*Source*: *Mathematics Teacher*, Vol. 99, No. 4, November 2005.)

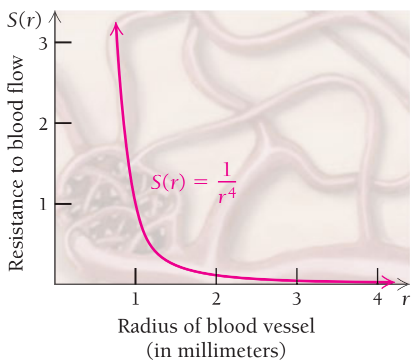


a) Find the rate of change of heart rate with respect to v, the output per beat.
b) Find the heart rate at $v = 80$ mL per beat.
c) Find the rate of change at $v = 80$ mL per beat.

92. Blood flow resistance. The equation

$$S(r) = \frac{1}{r^4}$$

can be used to determine the resistance to blood flow, S, of a blood vessel that has radius r, in millimeters (mm). (*Source*: *Mathematics Teacher*, Vol. 99, No. 4, November 2005.)



a) Find the rate of change of resistance with respect to r, the radius of the blood vessel.
b) Find the resistance at $r = 1.2$ mm.
c) Find the rate of change of S with respect to r when $r = 0.8$ mm.

Social Sciences

93. Population growth rate. The population of a city grows from an initial size of 100,000 to a size P given by

$$P(t) = 100{,}000 + 2000t^2,$$

where t is in years.

a) Find the **growth rate**, dP/dt.
b) Find the population after 10 yr.
c) Find the growth rate at $t = 10$.
d) Explain the meaning of your answer to part (c).

94. Median age of women at first marriage. The median age of women at first marriage can be approximated by the linear function

$$A(t) = 0.08t + 19.7,$$

where $A(t)$ is the median age of women marrying for the first time at t years after 1950.

a) Find the rate of change of the median age A with respect to time t.
b) Explain the meaning of your answer to part (a).

General Interest

95. View to the horizon. The view V, or distance in miles, that one can see to the horizon from a height h, in feet, is given by

$$V = 1.22\sqrt{h}.$$

a) Find the rate of change of V with respect to h.
b) How far can one see to the horizon from an airplane window at a height of 40,000 ft?
c) Find the rate of change at $h = 40{,}000$.
d) Explain the meaning of your answers to parts (a) and (c).

96. **Baseball ticket prices.** The average price, in dollars, of a ticket for a Major League baseball game x years after 1990 can be estimated by

$$p(x) = 9.41 - 0.19x + 0.09x^2.$$

a) Find the rate of change of the average ticket price with respect to the year, dp/dx.
b) What is the average ticket price in 2010?
c) What is the rate of change of the average ticket price in 2010?

SYNTHESIS

For each function, find the interval(s) for which $f'(x)$ is positive.

97. $f(x) = x^2 - 4x + 1$

98. $f(x) = x^2 + 7x + 2$

99. $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 5$

100. Find the points on the graph of
$$y = x^4 - \tfrac{4}{3}x^2 - 4$$
at which the tangent line is horizontal.

101. Find the points on the graph of
$$y = 2x^6 - x^4 - 2$$
at which the tangent line is horizontal.

Use the derivative to help show whether each function is always increasing, always decreasing, or neither.

102. $f(x) = x^5 + x^3$

103. $f(x) = x^3 + 2x$

104. $f(x) = \dfrac{1}{x}, \quad x \neq 0$

105. $f(x) = \sqrt{x}, \quad x \geq 0$

106. The function $f(x) = x^3 + ax$ is always increasing if $a > 0$, but not if $a < 0$. Use the derivative of f to explain why this observation is true.

Find dy/dx. Each function can be differentiated using the rules developed in this section, but some algebra may be required beforehand.

107. $y = (x + 3)(x - 2)$

108. $y = (x - 1)(x + 1)$

109. $y = \dfrac{x^5 - x^3}{x^2}$

110. $y = \dfrac{5x^2 - 8x + 3}{8}$

111. $y = \dfrac{x^5 + x}{x^2}$

112. $y = \dfrac{x^5 - 3x^4 + 2x + 4}{x^2}$

113. $y = (-4x)^3$

114. $y = \sqrt{7x}$

115. $y = \sqrt[3]{8x}$

116. $y = (x - 3)^2$

117. $y = \left(\sqrt{x} - \dfrac{1}{\sqrt{x}}\right)^2$

118. $y = (\sqrt{x} + \sqrt[3]{x})^2$

119. $y = (x + 1)^3$

120. Use Theorem 1 to prove that the derivative of 1 is 0.

121. When might Leibniz notation be more convenient than function notation?

122. Write a short biographical paper on Leibniz and/or Newton. Emphasize the contributions each man made to many areas of science and society.

TECHNOLOGY CONNECTION

Graph each of the following. Then estimate the x-values at which tangent lines are horizontal.

123. $f(x) = x^4 - 3x^2 + 1$

124. $f(x) = 1.6x^3 - 2.3x - 3.7$

125. $f(x) = 10.2x^4 - 6.9x^3$

126. $f(x) = \dfrac{5x^2 + 8x - 3}{3x^2 + 2}$

For each of the following, graph f and f' and then determine $f'(1)$. For Exercises 131 and 132, use nDeriv *on the TI-83.*

127. $f(x) = 20x^3 - 3x^5$

128. $f(x) = x^4 - 3x^2 + 1$

129. $f(x) = x^3 - 2x - 2$

130. $f(x) = x^4 - x^3$

131. $f(x) = \dfrac{4x}{x^2 + 1}$

132. $f(x) = \dfrac{5x^2 + 8x - 3}{3x^2 + 2}$

133. The function $f(x) = x^3 - x^2$ (mentioned after Example 8) appears to be always increasing, or possibly flat, on the default viewing window of the TI-83.

a) Graph the function in the default window; then zoom in until you see a small interval in which f is decreasing.
b) Use the derivative to determine the point(s) at which the graph has horizontal tangent lines.
c) Use your result from part (b) to infer the interval for which f is decreasing. Does this agree with your calculator's image of the graph?
d) Is it possible there are other intervals for which f is decreasing? Explain why or why not.

Answers to Quick Checks

1. (a) (i) $y' = 15x^{14}$, (ii) $y' = -7x^{-8}$;
(b) because π^2 is a constant

2. (a) $y' = \frac{1}{4}x^{-3/4} = \dfrac{1}{4\sqrt[4]{x^3}}$; **(b)** $y' = -1.25x^{-2.25}$

3. (a) $y' = 90x^8$; **(b)** $y' = 3\pi x^2$; **(c)** $y' = -\dfrac{8}{3x^5}$

4. $y' = 15x^4 + \dfrac{2}{3\sqrt[3]{x^2}} - \dfrac{2}{3x^3}$

5. $x = -1$ and $x = 5$

1.6 Differentiation Techniques: The Product and Quotient Rules

OBJECTIVES

- Differentiate using the Product and Quotient Rules.
- Use the Quotient Rule to differentiate the average cost, revenue, and profit functions.

The Product Rule

A function can be written as the product of two other functions. For example, the function $F(x) = x^3 \cdot x^4$ can be viewed as the product of the two functions $f(x) = x^3$ and $g(x) = x^4$, yielding $F(x) = f(x) \cdot g(x)$. Is the derivative of $F(x)$ the product of the derivatives of its factors, $f(x)$ and $g(x)$? The answer is no. To see this, note that the product of x^3 and x^4 is x^7, and the derivative of this product is $7x^6$. However, the derivatives of the two functions are $3x^2$ and $4x^3$, and the product of these derivatives is $12x^5$. This example shows that, in general, *the derivative of a product is* ***not*** *the product of the derivatives*. The following is a rule for finding the derivative of a product.

THEOREM 5 **The Product Rule**

Let $F(x) = f(x) \cdot g(x)$. Then

$$F'(x) = \frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot \left[\frac{d}{dx} g(x)\right] + g(x) \cdot \left[\frac{d}{dx} f(x)\right].$$

The derivative of a product is the first factor times the derivative of the second factor, plus the second factor times the derivative of the first factor.

The proof of the Product Rule is outlined in Exercise 123 at the end of this section.

Let's check the Product Rule for $x^2 \cdot x^5$. There are five steps:

$$\frac{d}{dx}(x^2 \cdot x^5)$$
$$= \underbrace{x^2}_{1} \cdot \underbrace{5x^4}_{2} + \underbrace{x^5}_{3} \cdot \underbrace{2x}_{4}$$
$$= 5x^6 + 2x^6 \quad 5$$
$$= 7x^6$$

1. Write down the first factor.
2. Multiply it by the derivative of the second factor.
3. Write down the second factor.
4. Multiply it by the derivative of the first factor.
5. Add the result of steps (1) and (2) to the result of steps (3) and (4).

Usually we try to write the results in simplified form. In Examples 1 and 2, we do not simplify in order to better emphasize the steps being performed.

EXAMPLE 1 Find $\frac{d}{dx}[(x^4 - 2x^3 - 7)(3x^2 - 5x)]$. Do not simplify.

Solution We let $f(x) = x^4 - 2x^3 - 7$ and $g(x) = 3x^2 - 5x$. We differentiate each of these, obtaining $f'(x) = 4x^3 - 6x^2$ and $g'(x) = 6x - 5$. By the Product Rule, the derivative of the given function is then

$$\frac{d}{dx}[\underbrace{(x^4 - 2x^3 - 7)}_{f(x)} \cdot \underbrace{(3x^2 - 5x)}_{g(x)}] = \underbrace{(x^4 - 2x^3 - 7)}_{f(x)} \cdot \underbrace{(6x - 5)}_{g'(x)} + \underbrace{(3x^2 - 5x)}_{g(x)} \cdot \underbrace{(4x^3 - 6x^2)}_{f'(x)}$$

In this example, we could have first multiplied the polynomials and then differentiated. Both methods give the same solution after simplification.

It makes no difference which factor of the given function is called $f(x)$ and which is called $g(x)$. We usually let the first function listed be $f(x)$ and the second function be $g(x)$, but if we switch the names, the process still gives the same answer. Try it by repeating Example 1 with $g(x) = x^4 - 2x^3 - 7$ and $f(x) = 3x^2 - 5x$.

❯ Quick Check 1

Use the Product Rule to differentiate each of the following functions. Do not simplify.

a) $y = (2x^5 + x - 1)(3x - 2)$

b) $y = \left(\sqrt{x} + 1\right)\left(\sqrt[5]{x} - x\right)$

EXAMPLE 2 For $F(x) = (x^2 + 4x - 11)\left(7x^3 - \sqrt{x}\right)$, find $F'(x)$. Do not simplify.

Solution We rewrite this as

$$F(x) = (x^2 + 4x - 11)(7x^3 - x^{1/2}).$$

Then, using the Product Rule, we have

$$F'(x) = (x^2 + 4x - 11)\left(21x^2 - \tfrac{1}{2}x^{-1/2}\right) + (7x^3 - x^{1/2})(2x + 4).$$

❮ Quick Check 1

The Quotient Rule

The derivative of a quotient is *not* the quotient of the derivatives. To see why, consider x^5 and x^2. The quotient x^5/x^2 is x^3, and the derivative of this quotient is $3x^2$. The individual derivatives are $5x^4$ and $2x$, and the quotient of these derivatives, $5x^4/(2x)$, is $(5/2)x^3$, which is not $3x^2$.

The rule for differentiating quotients is as follows.

THEOREM 6 **The Quotient Rule**

If $Q(x) = \dfrac{N(x)}{D(x)}$, then $Q'(x) = \dfrac{D(x) \cdot N'(x) - N(x) \cdot D'(x)}{[D(x)]^2}$.

The derivative of a quotient is the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

(If we think of the function in the numerator as the first function and the function in the denominator as the second function, then we can reword the Quotient Rule as "the derivative of a quotient is the second function times the derivative of the first function minus the first function times the derivative of the second function, all divided by the square of the second function.")

A proof of this result is outlined in Exercise 101 of Section 1.7 (on p. 176).

The Quotient Rule is illustrated below.

$$\frac{d}{dx}\left[\frac{N(x)}{D(x)}\right] = \frac{D(x) \cdot N'(x) - N(x) \cdot D'(x)}{[D(x)]^2}$$

There are six steps:

1. Write the denominator.
2. Multiply the denominator by the derivative of the numerator.
3. Write a minus sign.
4. Write the numerator.
5. Multiply it by the derivative of the denominator.
6. Divide by the square of the denominator.

EXAMPLE 3 For $Q(x) = x^5/x^2$, find $Q'(x)$.

Solution We have already seen that $x^5/x^2 = x^3$ and $\frac{d}{dx}x^3 = 3x^2$, but we wish to practice using the Quotient Rule. We have $D(x) = x^2$ and $N(x) = x^5$:

$$Q'(x) = \frac{\overset{D(x)}{x^2} \cdot \overset{N'(x)}{5x^4} - \overset{N(x)}{x^5} \cdot \overset{D'(x)}{2x}}{\underset{[D(x)]^2}{(x^2)^2}}$$

$$= \frac{5x^6 - 2x^6}{x^4} = \frac{3x^6}{x^4} = 3x^2. \qquad \text{This checks with the result above.}$$

EXAMPLE 4 Differentiate: $f(x) = \dfrac{1 + x^2}{x^3}$.

Solution

$$f'(x) = \frac{x^3 \cdot 2x - (1 + x^2) \cdot 3x^2}{(x^3)^2} \qquad \text{Using the Quotient Rule}$$

$$= \frac{2x^4 - 3x^2 - 3x^4}{x^6} = \frac{-x^4 - 3x^2}{x^6}$$

$$= \frac{x^2(-x^2 - 3)}{x^2 \cdot x^4} \qquad \text{Factoring}$$

$$= \frac{-x^2 - 3}{x^4} \qquad \text{Removing a factor equal to 1: } \frac{x^2}{x^2} = 1$$

EXAMPLE 5 Differentiate: $f(x) = \dfrac{x^2 - 3x}{x - 1}$.

Solution We have

$$f'(x) = \frac{(x - 1)(2x - 3) - (x^2 - 3x) \cdot 1}{(x - 1)^2} \qquad \text{Using the Quotient Rule}$$

$$= \frac{2x^2 - 5x + 3 - x^2 + 3x}{(x - 1)^2} \qquad \text{Using the distributive law}$$

$$= \frac{x^2 - 2x + 3}{(x - 1)^2}. \qquad \text{Simplifying}$$

It is not necessary to multiply out $(x - 1)^2$.

❮ Quick Check 2

❯ Quick Check 2

a) Differentiate: $f(x) = \dfrac{1 - 3x}{x^2 + 2}$.

Simplify your result.

b) Show that

$$\frac{d}{dx}\left[\frac{ax + 1}{bx + 1}\right] = \frac{a - b}{(bx + 1)^2}.$$

TECHNOLOGY CONNECTION

Checking Derivatives Graphically

To check Example 5, we first enter the function:

$$y_1 = \frac{x^2 - 3x}{x - 1}.$$

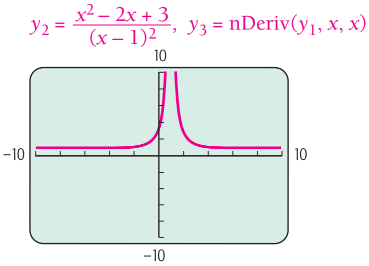
Then we enter the possible derivative:

$$y_2 = \frac{x^2 - 2x + 3}{(x - 1)^2}.$$

For the third function, we enter

$$y_3 = \text{nDeriv}(y_1, x, x).$$

Next, we deselect y_1 and graph y_2 and y_3. We use different graph styles and the Sequential mode to see each graph as it appears on the screen.



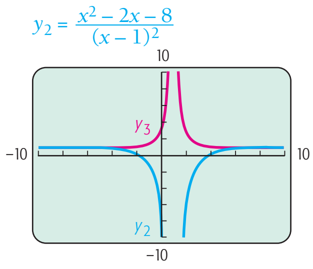
Since the graphs appear to coincide, it appears that $y_2 = y_3$ and we have a check. This is considered a partial check, however, because the graphs might not coincide at a point not in the viewing window.

We can also use a table to check that $y_2 = y_3$.

X	Y2	Y3
5.97	1.081	1.081
5.98	1.0806	1.0806
5.99	1.0803	1.0803
6	1.08	1.08
6.01	1.0797	1.0797
6.02	1.0794	1.0794
6.03	1.079	1.079

X = 5.97

You should verify that had we miscalculated the derivative as, say, $y_2 = (x^2 - 2x - 8)/(x - 1)^2$, neither the tables nor the graphs of y_2 and y_3 would agree.



EXERCISES

1. For the function

$$f(x) = \frac{x^2 - 4x}{x + 2},$$

use graphs and tables to determine which of the following seems to be the correct derivative.

a) $f'(x) = \dfrac{-x^2 - 4x - 8}{(x + 2)^2}$

b) $f'(x) = \dfrac{x^2 - 4x + 8}{(x + 2)^2}$

c) $f'(x) = \dfrac{x^2 + 4x - 8}{(x + 2)^2}$

2–5. Check the results of Examples 1–4 in this section.

Application of the Quotient Rule

The total cost, total revenue, and total profit functions, discussed in Section R.4, pertain to the accumulated cost, revenue, and profit when x items are produced. Because of economies of scale and other factors, it is common for the cost, revenue (price), and profit for, say, the 10th item to differ from those for the 1000th item. For this reason, a business is often interested in the *average* cost, revenue, and profit associated with the production and sale of x items.

DEFINITION

If $C(x)$ is the cost of producing x items, then the **average cost** of producing x items is $\dfrac{C(x)}{x}$;

if $R(x)$ is the revenue from the sale of x items, then the **average revenue** from selling x items is $\dfrac{R(x)}{x}$;

if $P(x)$ is the profit from the sale of x items, then the **average profit** from selling x items is $\dfrac{P(x)}{x}$.

EXAMPLE 6 **Business.** Paulsen's Greenhouse finds that the cost, in dollars, of growing x hundred geraniums is modeled by

$$C(x) = 200 + 100\sqrt[4]{x}.$$

If the revenue from the sale of x hundred geraniums is modeled by

$$R(x) = 120 + 90\sqrt{x},$$

find each of the following.

a) The average cost, the average revenue, and the average profit when x hundred geraniums are grown and sold.

b) The rate at which average profit is changing when 300 geraniums are being grown and sold.

Solution

a) We let A_C, A_R, and A_P represent average cost, average revenue, and average profit, respectively. Then

$$A_C(x) = \frac{C(x)}{x} = \frac{200 + 100\sqrt[4]{x}}{x};$$

$$A_R(x) = \frac{R(x)}{x} = \frac{120 + 90\sqrt{x}}{x};$$

$$A_P(x) = \frac{P(x)}{x} = \frac{R(x) - C(x)}{x} = \frac{-80 + 90\sqrt{x} - 100\sqrt[4]{x}}{x}.$$

b) To find the rate at which average profit is changing when 300 geraniums are being grown, we calculate $A_P'(3)$ (remember that x is in hundreds):

$$\begin{aligned}
A_P'(x) &= \frac{d}{dx}\left[\frac{-80 + 90x^{1/2} - 100x^{1/4}}{x}\right] \\
&= \frac{x\left(\frac{1}{2}\cdot 90x^{1/2-1} - \frac{1}{4}\cdot 100x^{1/4-1}\right) - (-80 + 90x^{1/2} - 100x^{1/4})\cdot 1}{x^2} \\
&= \frac{45x^{1/2} - 25x^{1/4} + 80 - 90x^{1/2} + 100x^{1/4}}{x^2} = \frac{75x^{1/4} - 45x^{1/2} + 80}{x^2}; \\
A_P'(3) &= \frac{75\sqrt[4]{3} - 45\sqrt{3} + 80}{3^2} \approx 11.20.
\end{aligned}$$

When 300 geraniums are being grown, the average profit is increasing by \$11.20 per hundred plants, or about 11.2 cents per plant.

TECHNOLOGY CONNECTION

Using Y-VARS

One way to save keystrokes on most calculators is to use the Y-VARS option on the VARS menu.

To check Example 6, we let $y_1 = 200 + 100x^{0.25}$ and $y_2 = 120 + 90x^{0.5}$. To express the profit function as y_3, we press Y= and move the cursor to enter y_3. Next we press VARS and select Y-VARS and then FUNCTION. From the FUNCTION menu we select Y2, which then appears on the Y= screen. After pressing −, we repeat the procedure to get Y1 on the Y= screen.

```
Plot 1  Plot 2  Plot 3
\Y1 = 200 + 100X^0.25
\Y2 = 120 + 90X^0.5
\Y3 = Y2 – Y1
\Y4 =
\Y5 =
\Y6 =
```

EXERCISES

1. Use the Y-VARS option to enter $y_4 = y_1/x$, $y_5 = y_2/x$, and $y_6 = y_3/x$, and explain what each of the functions represents.

2. Use nDeriv from the MATH menu or dy/dx from the CALC menu to check part (b)of Example 6.

Section Summary

- The *Product Rule* is

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot \frac{d}{dx}[g(x)] + g(x) \cdot \frac{d}{dx}[f(x)].$$

- The *Quotient Rule* is

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot \frac{d}{dx}[f(x)] - f(x) \cdot \frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

- Be careful to note the order in which you write out the factors when using the Quotient Rule. Because the Quotient Rule involves subtraction and division, the order in which you perform the operations is important.

EXERCISE SET 1.6

Differentiate two ways: first, by using the Product Rule; then, by multiplying the expressions before differentiating. Compare your results as a check.

1. $y = x^5 \cdot x^6$

2. $y = x^9 \cdot x^4$

3. $f(x) = (2x + 5)(3x - 4)$

4. $g(x) = (3x - 2)(4x + 1)$

5. $G(x) = 4x^2(x^3 + 5x)$

6. $F(x) = 3x^4(x^2 - 4x)$

7. $y = (3\sqrt{x} + 2)x^2$

8. $y = (4\sqrt{x} + 3)x^3$

9. $g(x) = (4x - 3)(2x^2 + 3x + 5)$

10. $f(x) = (2x + 5)(3x^2 - 4x + 1)$

11. $F(t) = (\sqrt{t} + 2)(3t - 4\sqrt{t} + 7)$

12. $G(t) = (2t + 3\sqrt{t} + 5)(\sqrt{t} + 4)$

Differentiate two ways: first, by using the Quotient Rule; then, by dividing the expressions before differentiating. Compare your results as a check.

13. $y = \dfrac{x^7}{x^3}$

14. $y = \dfrac{x^6}{x^4}$

15. $f(x) = \dfrac{2x^5 + x^2}{x}$

16. $g(x) = \dfrac{3x^7 - x^3}{x}$

17. $G(x) = \dfrac{8x^3 - 1}{2x - 1}$

18. $F(x) = \dfrac{x^3 + 27}{x + 3}$

19. $y = \dfrac{t^2 - 16}{t + 4}$

20. $y = \dfrac{t^2 - 25}{t - 5}$

Differentiate each function.

21. $f(x) = (3x^2 - 2x + 5)(4x^2 + 3x - 1)$

22. $g(x) = (5x^2 + 4x - 3)(2x^2 - 3x + 1)$

23. $y = \dfrac{5x^2 - 1}{2x^3 + 3}$

24. $y = \dfrac{3x^4 + 2x}{x^3 - 1}$

25. $G(x) = (8x + \sqrt{x})(5x^2 + 3)$

26. $F(x) = (-3x^2 + 4x)(7\sqrt{x} + 1)$

27. $g(t) = \dfrac{t}{3 - t} + 5t^3$

28. $f(t) = \dfrac{t}{5 + 2t} - 2t^4$

29. $F(x) = (x + 3)^2$
[*Hint:* $(x + 3)^2 = (x + 3)(x + 3)$.]

30. $G(x) = (5x - 4)^2$

31. $y = (x^3 - 4x)^2$

32. $y = (3x^2 - 4x + 5)^2$

33. $g(x) = 5x^{-3}(x^4 - 5x^3 + 10x - 2)$

34. $f(x) = 6x^{-4}(6x^3 + 10x^2 - 8x + 3)$

35. $F(t) = \left(t + \dfrac{2}{t}\right)(t^2 - 3)$

36. $G(t) = (3t^5 - t^2)\left(t - \dfrac{5}{t}\right)$

37. $y = \dfrac{x^2 + 1}{x^3 - 1} - 5x^2$

38. $y = \dfrac{x^3 - 1}{x^2 + 1} + 4x^3$

39. $y = \dfrac{\sqrt[3]{x} - 7}{\sqrt{x} + 3}$

40. $y = \dfrac{\sqrt{x} + 4}{\sqrt[3]{x} - 5}$

41. $f(x) = \dfrac{x}{x^{-1} + 1}$

42. $f(x) = \dfrac{x^{-1}}{x + x^{-1}}$

43. $F(t) = \dfrac{1}{t - 4}$

44. $G(t) = \dfrac{1}{t + 2}$

45. $f(x) = \dfrac{3x^2 + 2x}{x^2 + 1}$

46. $f(x) = \dfrac{3x^2 - 5x}{x^2 - 1}$

47. $g(t) = \dfrac{-t^2 + 3t + 5}{t^2 - 2t + 4}$

48. $f(t) = \dfrac{3t^2 + 2t - 1}{-t^2 + 4t + 1}$

49–96. Use a graphing calculator to check the results of Exercises 1–48.

97. Find an equation of the tangent line to the graph of $y = 8/(x^2 + 4)$ at **(a)** $(0, 2)$; **(b)** $(-2, 1)$.

98. Find an equation of the tangent line to the graph of $y = \sqrt{x}/(x + 1)$ at **(a)** $x = 1$; **(b)** $x = \frac{1}{4}$.

99. Find an equation of the tangent line to the graph of $y = x^2 + 3/(x - 1)$ at **(a)** $x = 2$; **(b)** $x = 3$.

100. Find an equation of the tangent line to the graph of $y = 4x/(1 + x^2)$ at **(a)** $(0, 0)$; **(b)** $(-1, -2)$.

APPLICATIONS

Business and Economics

101. Average cost. Summertime Fabrics finds that the cost, in dollars, of producing x jackets is given by $C(x) = 950 + 15\sqrt{x}$. Find the rate at which the average cost is changing when 400 jackets have been produced.

102. Average cost. Tongue-Tied Sauces, Inc., finds that the cost, in dollars, of producing x bottles of barbecue sauce is given by $C(x) = 375 + 0.75x^{3/4}$. Find the rate at which the average cost is changing when 81 bottles of barbecue sauce have been produced.

103. Average revenue. Summertime Fabrics finds that the revenue, in dollars, from the sale of x jackets is given by $R(x) = 85\sqrt{x}$. Find the rate at which average revenue is changing when 400 jackets have been produced.

104. Average revenue. Tongue-Tied Sauces, Inc., finds that the revenue, in dollars, from the sale of x bottles of barbecue sauce is given by $R(x) = 7.5x^{0.7}$. Find the rate at which average revenue is changing when 81 bottles of barbecue sauce have been produced.

105. Average profit. Use the information in Exercises 101 and 103 to determine the rate at which Summertime Fabrics' average profit per jacket is changing when 400 jackets have been produced and sold.

106. Average profit. Use the information in Exercises 102 and 104 to determine the rate at which Tongue-Tied Sauces' average profit per bottle of barbecue sauce is changing when 81 bottles have been produced and sold.

107. Average profit. Sparkle Pottery has determined that the cost, in dollars, of producing x vases is given by

$$C(x) = 4300 + 2.1x^{0.6}.$$

If the revenue from the sale of x vases is given by $R(x) = 65x^{0.9}$, find the rate at which the average profit per vase is changing when 50 vases have been made and sold.

108. Average profit. Cruzin' Boards has found that the cost, in dollars, of producing x skateboards is given by

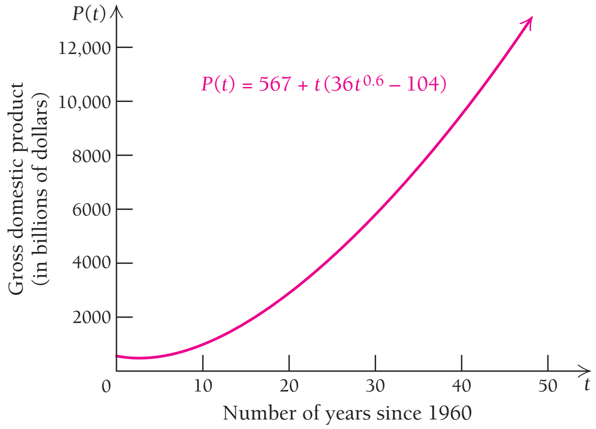
$$C(x) = 900 + 18x^{0.7}.$$

If the revenue from the sale of x skateboards is given by $R(x) = 75x^{0.8}$, find the rate at which the average profit per skateboard is changing when 20 skateboards have been built and sold.

109. Gross domestic product. The U.S. gross domestic product (in billions of dollars) can be approximated using the function

$$P(t) = 567 + t(36t^{0.6} - 104),$$

where t is the number of the years since 1960.



(*Source*: U.S. Bureau of Economic Analysis.)

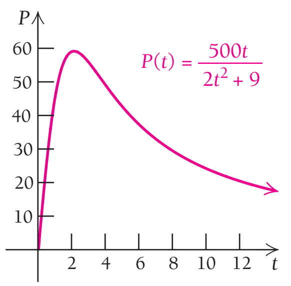
a) Find $P'(t)$.
b) Find $P'(45)$.
c) In words, explain what $P'(45)$ means.

Social Sciences

110. Population growth. The population P, in thousands, of a small city is given by

$$P(t) = \frac{500t}{2t^2 + 9},$$

where t is the time, in years.



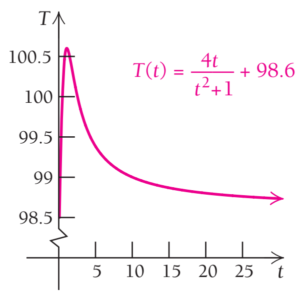
a) Find the growth rate.
b) Find the population after 12 yr.
c) Find the growth rate at $t = 12$ yr.

Life and Physical Sciences

111. Temperature during an illness. The temperature T of a person during an illness is given by

$$T(t) = \frac{4t}{t^2 + 1} + 98.6,$$

where T is the temperature, in degrees Fahrenheit, at time t, in hours.



a) Find the rate of change of the temperature with respect to time.
b) Find the temperature at $t = 2$ hr.
c) Find the rate of change of the temperature at $t = 2$ hr.

SYNTHESIS

Differentiate each function.

112. $f(x) = \dfrac{7 - \dfrac{3}{2x}}{\dfrac{4}{x^2} + 5}$ (*Hint:* Simplify before differentiating.)

113. $y(t) = 5t(t - 1)(2t + 3)$

114. $f(x) = x(3x^3 + 6x - 2)(3x^4 + 7)$

115. $g(x) = (x^3 - 8) \cdot \dfrac{x^2 + 1}{x^2 - 1}$

116. $f(t) = (t^5 + 3) \cdot \dfrac{t^3 - 1}{t^3 + 1}$

117. $f(x) = \dfrac{(x - 1)(x^2 + x + 1)}{x^4 - 3x^3 - 5}$

118. Let $f(x) = \dfrac{x}{x + 1}$ and $g(x) = \dfrac{-1}{x + 1}$.

a) Compute $f'(x)$.
b) Compute $g'(x)$.
c) What can you conclude about f and g on the basis of your results from parts (a) and (b)?

119. Let $f(x) = \dfrac{x^2}{x^2 - 1}$ and $g(x) = \dfrac{1}{x^2 - 1}$.

a) Compute $f'(x)$.
b) Compute $g'(x)$.
c) What can you conclude about the graphs of f and g on the basis of your results from parts (a) and (b)?

120. Write a rule for finding the derivative of $f(x) \cdot g(x) \cdot h(x)$. Describe the rule in words.

121. Is the derivative of the reciprocal of $f(x)$ the reciprocal of the derivative of $f'(x)$? Why or why not?

122. Sensitivity. The reaction R of the body to a dose Q of medication is often represented by the general function

$$R(Q) = Q^2\left(\frac{k}{2} - \frac{Q}{3}\right),$$

where k is a constant and R is in millimeters of mercury (mmHg) if the reaction is a change in blood pressure or in degrees Fahrenheit (°F) if the reaction is a change in temperature. The rate of change dR/dQ is defined to be the body's *sensitivity* to the medication.

a) Find a formula for the sensitivity.
b) Explain the meaning of your answer to part (a).

123. A proof of the Product Rule appears below. Provide a justification for each step.

a) $$\frac{d}{dx}[f(x)\cdot g(x)] = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

b) $$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

c) $$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x)}{h}$$

d) $$= \lim_{h\to 0}\left[f(x+h)\cdot\frac{g(x+h) - g(x)}{h}\right] + \lim_{h\to 0}\left[g(x)\cdot\frac{f(x+h) - f(x)}{h}\right]$$

e) $$= f(x)\cdot\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} + g(x)\cdot\lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

f) $$= f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

g) $$= f(x)\cdot\left[\frac{d}{dx}g(x)\right] + g(x)\cdot\left[\frac{d}{dx}f(x)\right]$$

TECHNOLOGY CONNECTION

124. Business. Refer to Exercises 102, 104, and 106. At what rate is Tongue-Tied Sauces' profit changing at the break-even point? At what rate is the average profit per bottle of barbecue sauce changing at that point?

125. Business. Refer to Exercises 101, 103, and 105. At what rate is Summertime Fabrics' profit changing at the break-even point? At what rate is the average profit per jacket changing at that point?

For the function in each of Exercises 126–131, graph f and f′. Then estimate points at which the tangent line to f is horizontal. If no such point exists, state that fact.

126. $f(x) = x^2(x-2)(x+2)$

127. $f(x) = \left(x + \frac{2}{x}\right)(x^2 - 3)$

128. $f(x) = \dfrac{x^3 - 1}{x^2 + 1}$

129. $f(x) = \dfrac{0.3x}{0.04 + x^2}$

130. $f(x) = \dfrac{0.01x^2}{x^4 + 0.0256}$

131. $f(x) = \dfrac{4x}{x^2 + 1}$

132. Use a graph to decide which of the following seems to be the correct derivative of the function in Exercise 131.

$$y_1 = \frac{2}{x}$$

$$y_2 = \frac{4 - 4x}{x^2 + 1}$$

$$y_3 = \frac{4 - 4x^2}{(x^2 + 1)^2}$$

$$y_4 = \frac{4x^2 - 4}{(x^2 + 1)^2}$$

Answers to Quick Checks

1. (a) $y' = (2x^5 + x - 1)(3) + (3x - 2)(10x^4 + 1)$

(b) $y' = \left(\sqrt{x} + 1\right)\left(\dfrac{1}{5\sqrt[5]{x^4}} - 1\right) + \left(\sqrt[5]{x} - x\right)\left(\dfrac{1}{2\sqrt{x}}\right)$

2. (a) $y' = \dfrac{3x^2 - 2x - 6}{(x^2 + 2)^2}$

(b) $$\frac{(bx + 1)(a) - (ax + 1)(b)}{(bx + 1)^2} = \frac{abx + a - abx - b}{(bx + 1)^2} = \frac{a - b}{(bx + 1)^2}$$

1.7 The Chain Rule

OBJECTIVES

- Find the composition of two functions.
- Differentiate using the Extended Power Rule or the Chain Rule.

The Extended Power Rule

Some functions are considered simple. *Simple* is a subjective concept, and it may be best to illustrate what it means with examples. The following functions are considered simple and are similar to those we have seen in Sections 1.5 and 1.6:

$$f(x) = 2x, \quad g(x) = 3x^2 - 5x, \quad h(x) = 2\sqrt{x}, \quad j(x) = \frac{2x - 1}{x^2 - 3}.$$

On the other hand, these functions are not considered simple:

$$f(x) = (x^3 + 2x)^5, \quad g(x) = \sqrt{2x + 5}, \quad h(x) = \left(\frac{3x - 7}{4x^2 + 1}\right)^3.$$

In these cases, we see that the variable x is part of one or more expressions that are raised to some power. How can we use the concepts of differentiation from Sections 1.5 and 1.6 to differentiate functions of this form? In this section, we will introduce and discuss the *Chain Rule*, but we begin our discussion with a special case of the Chain Rule called the *Extended Power Rule*.

The Extended Power Rule

The function $y = 1 + x^2$ is considered simple, and its derivative can be found directly from the Power Rule. However, if we nest this function in some manner, for example, $y = (1 + x^2)^3$, we now have a more complicated form. How do we determine the derivative? We might guess the following:

$$\frac{d}{dx}[(1 + x^2)^3] \stackrel{?}{=} 3(1 + x^2)^2. \qquad \text{Remember, this is a guess.}$$

To check this, we expand the function $y = (1 + x^2)^3$:

$$\begin{aligned} y &= (1 + x^2)^3 \\ &= (1 + x^2) \cdot (1 + x^2) \cdot (1 + x^2) \\ &= (1 + 2x^2 + x^4) \cdot (1 + x^2) && \text{Multiplying the first two factors} \\ &= 1 + 3x^2 + 3x^4 + x^6. && \text{Multiplying by the third factor} \end{aligned}$$

Taking the derivative of this function, we have

$$y' = 6x + 12x^3 + 6x^5.$$

Now we can factor out $6x$:

$$y' = 6x(1 + 2x^2 + x^4).$$

We rewrite $6x$ as $3 \cdot 2x$ and factor the expression within the parentheses:

$$y' = 3(1 + x^2)^2 \cdot 2x$$

Thus, it seems our original *guess* was close: it lacked only the extra factor, $2x$, which is the derivative of the expression inside the parentheses. The correct derivative of $y = (1 + x^2)^3$ is $y' = 3(1 + x^2)^2 \cdot 2x$, which suggests a general pattern for differentiating functions of this form, in which an expression is raised to a power k.

THEOREM 7 **The Extended Power Rule**

Suppose that $g(x)$ is a differentiable function of x. Then, for any real number k,

$$\frac{d}{dx}[g(x)]^k = k[g(x)]^{k-1} \cdot \frac{d}{dx} g(x).$$

The Extended Power Rule allows us to differentiate functions such as $y = (1 + x^2)^{89}$ without having to expand the expression $1 + x^2$ to the 89th power (very time-consuming) and functions such as $y = (1 + x^2)^{1/3}$, for which "expanding" to the $\frac{1}{3}$ power is impractical.

Let's differentiate $(1 + x^3)^5$. There are three steps to carry out.

1. Mentally block out the "inside" function, $1 + x^3$. $\quad (1 + x^3)^5$
2. Differentiate the "outside" function, $(1 + x^3)^5$. $\quad 5(1 + x^3)^4$
3. Multiply by the derivative of the "inside" function. $\quad 5(1 + x^3)^4 \cdot 3x^2$
   $= 15x^2(1 + x^3)^4 \qquad$ Simplified

Step (3) is quite commonly overlooked. Do not forget it!

TECHNOLOGY CONNECTION

Exploratory

One way to check your differentiation of y_1 is to enter your derivative as y_2 and see if the graph of y_2 coincides with the graph of $y_3 = \text{nDeriv}(y_1, x, x)$. Use this approach to check Example 1. Be sure to use the Sequential mode and different graph styles for the two curves.

The Extended Power Rule is best illustrated by examples. Carefully examine each of the following examples, noting the three-step process for applying the rule.

EXAMPLE 1 Differentiate: $f(x) = (1 + x^3)^{1/2}$.

Solution

$$\begin{aligned}\frac{d}{dx}(1 + x^3)^{1/2} &= \frac{1}{2}(1 + x^3)^{1/2-1} \cdot 3x^2\\ &= \frac{3x^2}{2}(1 + x^3)^{-1/2}\\ &= \frac{3x^2}{2\sqrt{1 + x^3}}\end{aligned}$$

EXAMPLE 2 Differentiate: $y = (1 - x^2)^3 + (5 + 4x)^2$.

Solution Here we combine the Sum–Difference Rule and the Extended Power Rule:

$$\frac{dy}{dx} = 3(1 - x^2)^2(-2x) + 2(5 + 4x)^1 \cdot 4.$$

We differentiate each term using the Extended Power Rule.

Since $dy/dx = 3(1 - x^2)^2(-2x) + 2(5 + 4x) \cdot 4$, it follows that

$$\begin{aligned}\frac{dy}{dx} &= -6x(1 - x^2)^2 + 8(5 + 4x)\\ &= -6x(1 - 2x^2 + x^4) + 40 + 32x\\ &= -6x + 12x^3 - 6x^5 + 40 + 32x\\ &= 40 + 26x + 12x^3 - 6x^5.\end{aligned}$$

❯ Quick Check 1

a) Use the Extended Power Rule to differentiate $y = (x^4 + 2x^2 + 1)^3$.

b) Explain why $\frac{d}{dx}[(x^2 + 4x + 1)^4] = 4(x^2 + 4x + 1)^3 \cdot 2x + 4$ is incorrect.

❮ Quick Check 1

EXAMPLE 3 Differentiate: $f(x) = (3x - 5)^4(7 - x)^{10}$.

Solution Here we combine the Product Rule and the Extended Power Rule:

$$\begin{aligned}f'(x) &= (3x - 5)^4 \cdot 10(7 - x)^9(-1) + (7 - x)^{10}\, 4(3x - 5)^3(3)\\ &= -10(3x - 5)^4(7 - x)^9 + (7 - x)^{10}\, 12(3x - 5)^3\\ &= 2(3x - 5)^3(7 - x)^9[-5(3x - 5) + 6(7 - x)]\\ &= 2(3x - 5)^3(7 - x)^9(-15x + 25 + 42 - 6x)\\ &= 2(3x - 5)^3(7 - x)^9(67 - 21x).\end{aligned}$$

We factor out $2(3x - 5)^3(7 - x)^9$.

EXAMPLE 4 Differentiate: $f(x) = \sqrt[4]{\dfrac{x + 3}{x - 2}}$.

Solution Here we use the Quotient Rule to differentiate the inside function:

$$\begin{aligned}\frac{d}{dx}\sqrt[4]{\frac{x + 3}{x - 2}} = \frac{d}{dx}\left(\frac{x + 3}{x - 2}\right)^{1/4} &= \frac{1}{4}\left(\frac{x + 3}{x - 2}\right)^{1/4-1}\left[\frac{(x - 2)1 - 1(x + 3)}{(x - 2)^2}\right]\\ &= \frac{1}{4}\left(\frac{x + 3}{x - 2}\right)^{-3/4}\left[\frac{x - 2 - x - 3}{(x - 2)^2}\right]\\ &= \frac{1}{4}\left(\frac{x + 3}{x - 2}\right)^{-3/4}\left[\frac{-5}{(x - 2)^2}\right], \quad \text{or} \quad \frac{-5}{4(x + 3)^{3/4}(x - 2)^{5/4}}.\end{aligned}$$

❯ Quick Check 2

Differentiate:

$$f(x) = \frac{(2x^2 - 1)}{(3x^4 + 2)^2}.$$

❮ Quick Check 2

Author Marv Bittinger and his size-$11\frac{1}{2}$ running shoes

Composition of Functions and the Chain Rule

Before discussing the Chain Rule, let's consider *composition of functions*.

One author of this text exercises three times a week at a local YMCA. When he recently bought a pair of running shoes, he found a label on which the numbers at the bottom indicate equivalent shoe sizes in five countries.

This label suggests that there are functions that convert one country's shoe sizes to those used in another country. There is, indeed, a function g that gives a correspondence between shoe sizes in the United States and those in France:

$$g(x) = \frac{4x + 92}{3},$$

where x is the U.S. size and $g(x)$ is the French size. Thus, a U.S. size $11\frac{1}{2}$ corresponds to a French size

$$g(11\tfrac{1}{2}) = \frac{4 \cdot 11\frac{1}{2} + 92}{3}, \quad \text{or } 46.$$

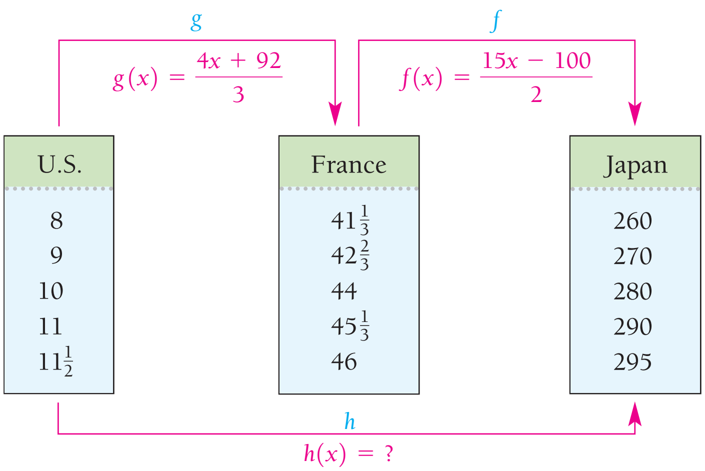
There is also a function f that gives a correspondence between shoe sizes in France and those in Japan. The function is given by

$$f(x) = \frac{15x - 100}{2},$$

where x is the French size and $f(x)$ is the corresponding Japanese size. Thus, a French size 46 corresponds to a Japanese size

$$f(46) = \frac{15 \cdot 46 - 100}{2}, \quad \text{or } 295.$$

It seems reasonable to conclude that a shoe size of $11\frac{1}{2}$ in the United States corresponds to a size of 295 in Japan and that some function h describes this correspondence. Can we find a formula for h?



A shoe size x in the United States corresponds to a shoe size $g(x)$ in France, where

$$g(x) = \frac{4x + 92}{3}.$$

Thus, $(4x + 92)/3$ represents a shoe size in France. If we replace x in $f(x)$ with $(4x + 92)/3$, we can find the corresponding shoe size in Japan:

$$\begin{aligned} f(g(x)) &= \frac{15\left(\frac{4x + 92}{3}\right) - 100}{2} \\ &= \frac{5(4x + 92) - 100}{2} = \frac{20x + 460 - 100}{2} \\ &= \frac{20x + 360}{2} = 10x + 180. \end{aligned}$$

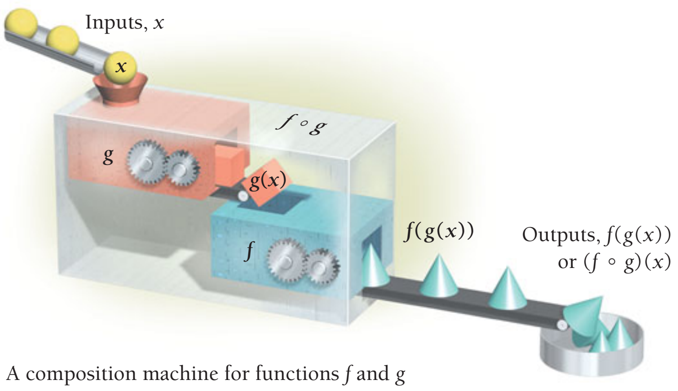
This gives a formula for h: $h(x) = 10x + 180$. As a check, a shoe size of $11\frac{1}{2}$ in the United States corresponds to a shoe size of $h\left(11\frac{1}{2}\right) = 10\left(11\frac{1}{2}\right) + 180 = 295$ in Japan. The function h is the *composition* of f and g, symbolized by $f \circ g$ and read as "f composed with g," or simply "f circle g."

DEFINITION

The **composed** function $f \circ g$, the **composition** of f and g, is defined as

$$(f \circ g)(x) = f(g(x)).$$

We can visualize the composition of functions as shown below.



A composition machine for functions f and g

To find $(f \circ g)(x)$, we substitute $g(x)$ for x in $f(x)$. The function $g(x)$ is nested within $f(x)$.

EXAMPLE 5 For $f(x) = x^3$ and $g(x) = 1 + x^2$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution Consider each function separately:

$$f(x) = x^3 \qquad \text{This function cubes each input.}$$

and

$$g(x) = 1 + x^2. \qquad \text{This function adds 1 to the square of each input.}$$

a) The function $f \circ g$ first does what g does (adds 1 to the square) and then does what f does (cubes). We find $f(g(x))$ by substituting $g(x)$ for x:

$$\begin{aligned}(f \circ g)(x) = f(g(x)) &= f(1 + x^2) && \text{Using } g(x) \text{ as an input}\\ &= (1 + x^2)^3 \\ &= 1 + 3x^2 + 3x^4 + x^6.\end{aligned}$$

b) The function $g \circ f$ first does what f does (cubes) and then does what g does (adds 1 to the square). We find $g(f(x))$ by substituting $f(x)$ for x:

$$\begin{aligned}(g \circ f)(x) = g(f(x)) &= g(x^3) && \text{Using } f(x) \text{ as an input}\\ &= 1 + (x^3)^2 = 1 + x^6.\end{aligned}$$

EXAMPLE 6 For $f(x) = \sqrt{x}$ and $g(x) = x - 1$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution

$$(f \circ g)(x) = f(g(x)) = f(x - 1) = \sqrt{x - 1}$$
$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{x} - 1$$

❮ Quick Check 3

❯ Quick Check 3

For the functions in Example 6, find:

a) $(f \circ f)(x)$;

b) $(g \circ g)(x)$.

Keep in mind that, in general, $(f \circ g)(x) \neq (g \circ f)(x)$. We see this fact demonstrated in Examples 5 and 6.

How do we differentiate a composition of functions? The following theorem tells us.

THEOREM 8 The Chain Rule

The derivative of the composition $f \circ g$ is given by

$$\frac{d}{dx}[(f \circ g)(x)] = \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x).$$

As we noted earlier, the Extended Power Rule is a special case of the Chain Rule. Consider $f(x) = x^k$. For any other function $g(x)$, we have $(f \circ g)(x) = [g(x)]^k$, and the derivative of the composition is

$$\frac{d}{dx}[g(x)]^k = k[g(x)]^{k-1} \cdot g'(x).$$

The Chain Rule often appears in another form. Suppose that $y = f(u)$ and $u = g(x)$. Then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

To better understand the Chain Rule, suppose that a video game manufacturer wished to determine its rate of profit, in *dollars* per *minute*. One way to find this rate would be to multiply the rate of profit, in *dollars* per *item*, by the production rate, in *items* per *minute*. That is,

$$\begin{bmatrix}\text{Change in profits}\\ \text{with respect to time}\end{bmatrix} = \begin{bmatrix}\text{Change in profits with respect}\\ \text{to number of games produced}\end{bmatrix} \cdot \begin{bmatrix}\text{Change in number of games}\\ \text{produced with respect to time.}\end{bmatrix}$$

EXAMPLE 7 For $y = 2 + \sqrt{u}$ and $u = x^3 + 1$, find dy/du, du/dx, and dy/dx.

Solution First we find dy/du and du/dx:

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} \quad \text{and} \quad \frac{du}{dx} = 3x^2.$$

Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= \frac{1}{2\sqrt{u}} \cdot 3x^2\\ &= \frac{3x^2}{2\sqrt{x^3+1}}. \qquad \text{Substituting } x^3+1 \text{ for } u\end{aligned}$$

❯ **Quick Check 4**

If $y = u^2 + u$ and $u = x^2 + x$, find $\frac{dy}{dx}$.

❮ Quick Check 4

EXAMPLE 8 **Business.** A new product is placed on the market and becomes very popular. Its quantity sold N is given as a function of time t, where t is measured in weeks:

$$N(t) = \frac{250{,}000t^2}{(2t+1)^2}, \qquad t > 0.$$

Differentiate this function. Then use the derivative to evaluate $N'(52)$ and $N'(208)$, and interpret these results.

Solution To determine $N'(t)$, we use the Quotient Rule along with the Extended Power Rule:

$$\begin{aligned}N'(t) = \frac{d}{dt}\left[\frac{250{,}000t^2}{(2t+1)^2}\right] &= \frac{(2t+1)^2 \cdot \frac{d}{dt}[250{,}000t^2] - 250{,}000t^2 \cdot \frac{d}{dt}[(2t+1)^2]}{[(2t+1)^2]^2}\\ &= \frac{(2t+1)^2 \cdot (500{,}000t) - 250{,}000t^2 \cdot 2(2t+1)^1 \cdot 2}{(2t+1)^4} \qquad \text{The Extended Power Rule is used here.}\\ &= \frac{(2t+1)^2(500{,}000t) - 1{,}000{,}000t^2(2t+1)}{(2t+1)^4}\\ &= \frac{500{,}000t(2t+1)[(2t+1) - 2t]}{(2t+1)^4}. \qquad \text{The expression } 500{,}000t(2t+1) \text{ is factored out in the numerator; the } 2t \text{ terms inside the square brackets sum to 0.}\end{aligned}$$

Therefore,

$$N'(t) = \frac{500{,}000t}{(2t+1)^3}. \qquad \text{The factor } (2t+1) \text{ in the numerator cancels } (2t+1) \text{ in the denominator.}$$

We evaluate $N'(t)$ at $t = 52$:

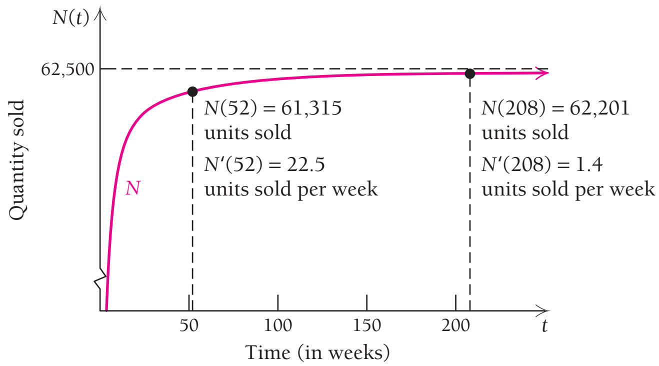
$$N'(52) = \frac{500{,}000(52)}{(2(52)+1)^3} \approx 22.5.$$

Thus, after 52 weeks (1 yr), the quantity sold is increasing by about 22.5 units per week.

For $t = 208$ weeks (4 yr), we get

$$N'(208) = \frac{500{,}000(208)}{(2(208)+1)^3} \approx 1.4.$$

After 4 yr, the quantity sold is increasing at about 1.4 units per week. What is happening here? Consider the graph of $N(t)$:



We see that the slopes of the tangent lines, representing the change in numbers of units sold per week, are leveling off as t increases. Perhaps the market is becoming saturated with this product: while sales continue to increase, the rate of the sales increase per week is leveling off.

Section Summary

- The *Extended Power Rule* tells us that if $y = [f(x)]^k$, then

$$y' = \frac{d}{dx}[f(x)]^k = k[f(x)]^{k-1} \cdot f'(x).$$

- The *composition* of $f(x)$ with $g(x)$ is written $(f \circ g)(x)$ and is defined as $(f \circ g)(x) = f(g(x))$.
- In general, $(f \circ g)(x) \neq (g \circ f)(x)$.
- The *Chain Rule* is used to differentiate a composition of functions. If

$$F(x) = (f \circ g)(x) = f(g(x)),$$

then

$$F'(x) = \frac{d}{dx}[(f \circ g)(x)] = f'(g(x)) \cdot g'(x).$$

EXERCISE SET 1.7

Differentiate each function.

1. $y = (2x + 1)^2$
2. $y = (3 - 2x)^2$

$\Big\}$ Check by expanding and then differentiating.

3. $y = (7 - x)^{55}$

4. $y = (8 - x)^{100}$

5. $y = \sqrt{1 + 8x}$

6. $y = \sqrt{1 - x}$

7. $y = \sqrt{3x^2 - 4}$

8. $y = \sqrt{4x^2 + 1}$

9. $y = (8x^2 - 6)^{-40}$

10. $y = (4x^2 + 1)^{-50}$

11. $y = (x - 4)^8(2x + 3)^6$

12. $y = (x + 5)^7(4x - 1)^{10}$

13. $y = \dfrac{1}{(3x + 8)^2}$

14. $y = \dfrac{1}{(4x + 5)^2}$

15. $y = \dfrac{4x^2}{(7 - 5x)^3}$

16. $y = \dfrac{7x^3}{(4 - 9x)^5}$

17. $f(x) = (1 + x^3)^3 - (2 + x^8)^4$

18. $f(x) = (3 + x^3)^5 - (1 + x^7)^4$

19. $f(x) = x^2 + (200 - x)^2$

20. $f(x) = x^2 + (100 - x)^2$

21. $g(x) = \sqrt{x} + (x - 3)^3$

22. $G(x) = \sqrt[3]{2x - 1} + (4 - x)^2$

23. $f(x) = -5x(2x - 3)^4$

24. $f(x) = -3x(5x + 4)^6$

25. $g(x) = (3x - 1)^7(2x + 1)^5$

26. $F(x) = (5x + 2)^4(2x - 3)^8$

27. $f(x) = x^2\sqrt{4x - 1}$

28. $f(x) = x^3\sqrt{5x + 2}$

29. $G(x) = \sqrt[3]{x^5 + 6x}$

30. $F(x) = \sqrt[4]{x^2 - 5x + 2}$

31. $f(x) = \left(\dfrac{3x - 1}{5x + 2}\right)^4$

32. $f(x) = \left(\dfrac{2x}{x^2 + 1}\right)^3$

33. $g(x) = \sqrt{\dfrac{4 - x}{3 + x}}$

34. $g(x) = \sqrt{\dfrac{3 + 2x}{5 - x}}$

35. $f(x) = (2x^3 - 3x^2 + 4x + 1)^{100}$

36. $f(x) = (7x^4 + 6x^3 - x)^{204}$

37. $g(x) = \left(\dfrac{2x + 3}{5x - 1}\right)^{-4}$

38. $h(x) = \left(\dfrac{1 - 3x}{2 - 7x}\right)^{-5}$

39. $f(x) = \sqrt{\dfrac{x^2 + x}{x^2 - x}}$

40. $f(x) = \sqrt[3]{\dfrac{4 - x^3}{x - x^2}}$

41. $f(x) = \dfrac{(2x + 3)^4}{(3x - 2)^5}$

42. $f(x) = \dfrac{(5x - 4)^7}{(6x + 1)^3}$

43. $f(x) = 12(2x + 1)^{2/3}(3x - 4)^{5/4}$

44. $y = 6\sqrt[3]{x^2 + x}(x^4 - 6x)^3$

Find $\dfrac{dy}{du}$, $\dfrac{du}{dx}$, *and* $\dfrac{dy}{dx}$.

45. $y = \sqrt{u}$ and $u = x^2 - 1$

46. $y = \dfrac{15}{u^3}$ and $u = 2x + 1$

47. $y = u^{50}$ and $u = 4x^3 - 2x^2$

48. $y = \dfrac{u + 1}{u - 1}$ and $u = 1 + \sqrt{x}$

49. $y = u(u + 1)$ and $u = x^3 - 2x$

50. $y = (u + 1)(u - 1)$ and $u = x^3 + 1$

Find $\dfrac{dy}{dx}$ *for each pair of functions.*

51. $y = 5u^2 + 3u$ and $u = x^3 + 1$

52. $y = u^3 - 7u^2$ and $u = x^2 + 3$

53. $y = \sqrt[3]{2u + 5}$ and $u = x^2 - x$

54. $y = \sqrt{7 - 3u}$ and $u = x^2 - 9$

55. Find $\dfrac{dy}{dt}$ if $y = \dfrac{1}{u^2 + u}$ and $u = 5 + 3t$.

56. Find $\dfrac{dy}{dt}$ if $y = \dfrac{1}{3u^5 - 7}$ and $u = 7t^2 + 1$.

57. Find an equation for the tangent line to the graph of $y = \sqrt{x^2 + 3x}$ at the point $(1, 2)$.

58. Find an equation for the tangent line to the graph of $y = (x^3 - 4x)^{10}$ at the point $(2, 0)$.

59. Find an equation for the tangent line to the graph of $y = x\sqrt{2x + 3}$ at the point $(3, 9)$.

60. Find an equation for the tangent line to the graph of $y = \left(\dfrac{2x + 3}{x - 1}\right)^3$ at the point $(2, 343)$.

61. Consider

$$f(x) = \frac{x^2}{(1 + x)^5}.$$

a) Find $f'(x)$ using the Quotient Rule and the Extended Power Rule.
b) Note that $f(x) = x^2(1 + x)^{-5}$. Find $f'(x)$ using the Product Rule and the Extended Power Rule.
c) Compare your answers to parts (a) and (b).

62. Consider

$$g(x) = \left(\frac{6x + 1}{2x - 5}\right)^2.$$

a) Find $g'(x)$ using the Extended Power Rule.
b) Note that

$$g(x) = \frac{36x^2 + 12x + 1}{4x^2 - 20x + 25}.$$

Find $g'(x)$ using the Quotient Rule.
c) Compare your answers to parts (a) and (b). Which approach was easier, and why?

In Exercises 63–66, find $f(x)$ and $g(x)$ such that $h(x) = (f \circ g)(x)$. Answers may vary.

63. $h(x) = (3x^2 - 7)^5$

64. $h(x) = \dfrac{1}{\sqrt{7x + 2}}$

65. $h(x) = \dfrac{x^3 + 1}{x^3 - 1}$

66. $h(x) = \left(\sqrt{x} + 5\right)^4$

Do Exercises 67–70 in two ways. First, use the Chain Rule to find the answer. Next, check your answer by finding $f(g(x))$, taking the derivative, and substituting.

67. $f(u) = u^3$, $g(x) = u = 2x^4 + 1$
Find $(f \circ g)'(-1)$.

68. $f(u) = \dfrac{u + 1}{u - 1}$, $g(x) = u = \sqrt{x}$
Find $(f \circ g)'(4)$.

69. $f(u) = \sqrt[3]{u}$, $g(x) = u = 1 + 3x^2$
Find $(f \circ g)'(2)$.

70. $f(u) = 2u^5$, $g(x) = u = \dfrac{3 - x}{4 + x}$
Find $(f \circ g)'(-10)$.

For Exercises 71–74, use the Chain Rule to differentiate each function. You may need to apply the rule more than once.

71. $f(x) = (2x^3 + (4x - 5)^2)^6$

72. $f(x) = \left(-x^5 + 4x + \sqrt{2x + 1}\right)^3$

73. $f(x) = \sqrt{x^2 + \sqrt{1 - 3x}}$

74. $f(x) = \sqrt[3]{2x + (x^2 + x)^4}$

APPLICATIONS

Business and Economics

75. Total revenue. A total-revenue function is given by

$$R(x) = 1000\sqrt{x^2 - 0.1x},$$

where $R(x)$ is the total revenue, in thousands of dollars, from the sale of x items. Find the rate at which total revenue is changing when 20 items have been sold.

76. Total cost. A total-cost function is given by

$$C(x) = 2000(x^2 + 2)^{1/3} + 700,$$

where $C(x)$ is the total cost, in thousands of dollars, of producing x items. Find the rate at which total cost is changing when 20 items have been produced.

77. Total profit. Use the total-cost and total-revenue functions in Exercises 75 and 76 to find the rate at which total profit is changing when x items have been produced and sold.

78. Total cost. A company determines that its total cost, in thousands of dollars, for producing x items is

$$C(x) = \sqrt{5x^2 + 60},$$

and it plans to boost production t months from now according to the function

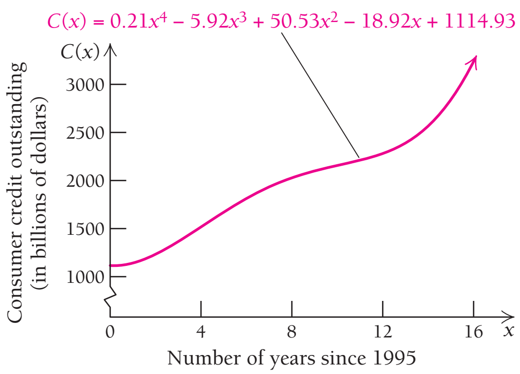
$$x(t) = 20t + 40.$$

How fast will costs be rising 4 months from now?

79. Consumer credit. The total outstanding consumer credit of the United States (in billions of dollars) can be modeled by the function

$$C(x) = 0.21x^4 - 5.92x^3 + 50.53x^2 - 18.92x + 1114.93,$$

where x is the number of years since 1995.



(*Source*: Federal Reserve Board.)

a) Find dC/dx.
b) Interpret the meaning of dC/dx.
c) Using this model, estimate how quickly outstanding consumer credit was rising in 2010.

80. Utility. Utility is a type of function that occurs in economics. When a consumer receives x units of a product, a certain amount of pleasure, or utility, U, is derived. Suppose that the utility related to the number of tickets x for a ride at a county fair is

$$U(x) = 80\sqrt{\frac{2x + 1}{3x + 4}}.$$

Find the rate at which the utility changes with respect to the number of tickets bought.

81. Compound interest. If \$1000 is invested at interest rate i, compounded annually, in 3 yr it will grow to an amount A given by (see Section R.1)

$$A = \$1000(1 + i)^3.$$

a) Find the rate of change, dA/di.
b) Interpret the meaning of dA/di.

82. Compound interest. If \$1000 is invested at interest rate i, compounded quarterly, in 5 yr it will grow to an amount, A, given by

$$A = \$1000\left(1 + \frac{i}{4}\right)^{20}.$$

a) Find the rate of change, dA/di.
b) Interpret the meaning of dA/di.

83. Consumer demand. Suppose that the demand function for a product is given by

$$D(p) = \frac{80{,}000}{p},$$

and that price p is a function of time given by $p = 1.6t + 9$, where t is in days.

a) Find the demand as a function of time t.
b) Find the rate of change of the quantity demanded when $t = 100$ days.

84. Business profit. A company is selling laptop computers. It determines that its total profit, in dollars, is given by

$$P(x) = 0.08x^2 + 80x,$$

where x is the number of units produced and sold. Suppose that x is a function of time, in months, where $x = 5t + 1$.

a) Find the total profit as a function of time t.
b) Find the rate of change of total profit when $t = 48$ months.

Life and Physical Sciences

85. Chemotherapy. The dosage for Carboplatin chemotherapy drugs depends on several parameters of the particular drug as well as the age, weight, and sex of the patient. For female patients, the formulas giving the dosage for such drugs are

$$D = 0.85A(c + 25) \quad \text{and} \quad c = (140 - y)\frac{w}{72x},$$

where A and x depend on which drug is used, D is the dosage in milligrams (mg), c is called the creatine clearance, y is the patient's age in years, and w is the patient's weight in kilograms (kg). (*Source*: *U.S. Oncology*.)

a) Suppose that a patient is a 45-year-old woman and the drug has parameters $A = 5$ and $x = 0.6$. Use this information to write formulas for D and c that give D as a function of c and c as a function of w.
b) Use your formulas from part (a) to compute dD/dc.
c) Use your formulas from part (a) to compute dc/dw.
d) Compute dD/dw.
e) Interpret the meaning of the derivative dD/dw.

SYNTHESIS

If $f(x)$ is a function, then $(f \circ f)(x) = f(f(x))$ is the composition of f with itself. This is called an iterated function, *and the composition can be repeated many times. For example, $(f \circ f \circ f)(x) = f(f(f(x)))$. Iterated functions are very useful in many areas, including finance (compound interest is a simple case) and the sciences (in weather forecasting, for example). For the each function, use the Chain Rule to find the derivative.*

86. If $f(x) = x^2 + 1$, find $\frac{d}{dx}[(f \circ f)(x)]$.

87. If $f(x) = x + \sqrt{x}$, find $\frac{d}{dx}[(f \circ f)(x)]$.

88. If $f(x) = x^2 + 1$, find $\frac{d}{dx}[(f \circ f \circ f)(x)]$

89. If $f(x) = \sqrt[3]{x}$, find $\frac{d}{dx}[(f \circ f \circ f)(x)]$.

Do you see a shortcut?

Differentiate.

90. $y = \sqrt{(2x - 3)^2 + 1}$

91. $y = \sqrt[3]{x^3 + 6x + 1} \cdot x^5$

92. $s = \sqrt[4]{t^4 + 3t^2 + 8} \cdot 3t$

93. $y = \left(\dfrac{x}{\sqrt{x - 1}}\right)^3$

94. $y = \left(x\sqrt{1 + x^2}\right)^3$

95. $y = \dfrac{\sqrt{1 - x^2}}{1 - x}$

96. $w = \dfrac{u}{\sqrt{1 + u^2}}$

97. $y = \left(\dfrac{x^2 - x - 1}{x^2 + 1}\right)^3$

98. $g(x) = \sqrt{\dfrac{x^2 - 4x}{2x + 1}}$

99. $f(t) = \sqrt{3t + \sqrt{t}}$

100. $F(x) = [6x(3 - x)^5 + 2]^4$

101. The following is the beginning of an alternative proof of the Quotient Rule that uses the Product Rule and the Power Rule. Complete the proof, giving reasons for each step.

Proof. Let

$$Q(x) = \frac{N(x)}{D(x)}.$$

Then

$$Q(x) = N(x) \cdot [D(x)]^{-1}.$$

Therefore, ...

102. The Extended Power Rule (for positive integer powers) can be verified using the Product Rule. For example, if $y = [f(x)]^2$, then the Product Rule is applied by recognizing that $[f(x)]^2 = [f(x)] \cdot [f(x)]$. Therefore,

$$\begin{aligned}\frac{d}{dx}([f(x)] \cdot [f(x)]) &= f(x) \cdot f'(x) + f'(x) \cdot f(x) \\ &= 2f(x) \cdot f'(x).\end{aligned}$$

a) Use the Product Rule to show that $\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 \cdot f'(x)$. [*Hint:* $[f(x)]^3 = [f(x)]^2 \cdot f(x)$.]

b) Use the Product Rule to show that $\frac{d}{dx}[f(x)]^4 = 4[f(x)]^3 \cdot f'(x)$.

TECHNOLOGY CONNECTION

For the function in each of Exercises 103 and 104, graph f and f' over the given interval. Then estimate points at which the tangent line is horizontal.

103. $f(x) = 1.68x\sqrt{9.2 - x^2}$; $[-3, 3]$

104. $f(x) = \sqrt{6x^3 - 3x^2 - 48x + 45}$; $[-5, 5]$

Find the derivative of each of the following functions analytically. Then use a calculator to check the results.

105. $f(x) = x\sqrt{4 - x^2}$

106. $g(x) = \dfrac{4x}{\sqrt{x - 10}}$

107. $f(x) = \left(\sqrt{2x - 1} + x^3\right)^5$

Answers to Quick Checks

1. (a) $y' = 3(x^4 + 2x^2 + 1)^2(4x^3 + 4x)$
(b) The result lacks parentheses around $2x + 4$. It should be written: $y' = 4(x^2 + 4x + 1)^3(2x + 4)$.

2. $y' = \dfrac{-36x^5 + 24x^3 + 8x}{(3x^4 + 2)^3}$

3. (a) $f(f(x)) = \sqrt{\sqrt{x}} = \sqrt[4]{x}$;
(b) $g(g(x)) = (x - 1) - 1 = x - 2$

4. $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = (2u + 1)(2x + 1) =$
$(2(x^2 + x) + 1)(2x + 1) = (2x^2 + 2x + 1)(2x + 1)$