Sec 5 Compounding of interest ("Compound interest")

Def Compound interest is the addition of interest to the principal sum of a loan or a deposit. in other words, it is interest on interest.

Interest reinvested is interest compounded.

Terms

P, principal (not profit here): initial investment or deposit or loan

r, annual rate of interest accrual: percent (as a decimal) paid on principal over time

n, number of compounding periods; times per year interest is compounded

t, time in years; length of time investment or deposit or loan (given time in months say, then $t = \frac{\text{months}}{12}$)

Suppose for some deposit P, a rate of 3% $(r = .03)$ is offered.
Suppose interest is paid only once a year $(n = 1)$.
After 1 yr, the interest earned is $P \cdot r = .03P$
The account has $P + .03P$ in it after 1 year. $(t = 1)$
Assuming no more principal is added, the account will have, after 2 years:

$$P + .03P + .03(\underbrace{P + .03P}_{\text{year 1}}) = \text{amt after year 2}$$

interest

Simplifying:

$$\boxed{P}(1 + .03) + .03\boxed{P}(1 + .03) = \boxed{P(1 + .03)}(1 + .03) = F$$

$$\text{Final value} = P(1 + .03)^2$$

Let F be "final value".

(2)

This is an example of simple interest, interest compounded once a year. The formula for simple interest is.

$$\boxed{F = P(1+r)^t}$$

r = annual rate, t = years

Repeating this analysis for $r = .03$, but compounding quarterly ($n = 4$ rather than 1), then interest is being added to the principal every quarter, so $n = 4$

After one quarter you will have:

$P + P\left(\frac{.03}{4}\right)$ since you've earned only $\frac{1}{4}$ of the 3%

by then. (That is, P + interest $= P + P\left(\frac{.03}{4}\right)$.

Simplifying this by factoring, we get $P\left(1+\frac{.03}{4}\right)$. After another quarter we will add this to more interest:

$$P\left(1+\frac{.03}{4}\right) + \left(P\left(1+\frac{.03}{4}\right)\right)\left(\frac{.03}{4}\right) = P\left(1+\frac{.03}{4}\right)\left(1+\frac{.03}{4}\right)$$

$$= P\left(1+\frac{.03}{4}\right)^{2} \leftarrow (\underset{n}{4}\cdot\underset{t}{\tfrac{1}{2}})$$

Notice the exponent 2 is half the number of compounding periods per year. Continuing in this way, after 3 quarters, we have in the account:

$P\left(1+\frac{.03}{4}\right)^{3} \leftarrow (4\cdot\frac{3}{4})$, where exponent 3 is $\frac{3}{4}$ of 4.

Finally, after 1 year ($t = 1$) we have a final value:

$$F = P\left(1+\frac{.03}{4}\right)^{4} \leftarrow (4\cdot 1).$$

The formula ~~for compound~~ interest, then, is found to be

$$\boxed{F = P\left(1+\frac{r}{n}\right)^{nt}}$$

Anticipating the more complex calculations entailing integration (calculus methods) we rename P and F as present value and future value of an investment or deposit or loan.

Continuous compounding and "e":

Per the video, we have an idea that the more periods each year we ~~of~~ compound the interest, the greater the interest earned. The "intermittent principals" are growing

Fact (for another section of text):

As n goes to infinity, $(1+\frac{1}{n})^n$ goes to e.

(Later we'll write $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.)

For now, we claim (correctly) that "compounding at each instant" of the year (i.e., $n \to \infty$) affects the ~~sum~~ quantity $(1+\frac{r}{n})^n$ by making it approach e^r. But back up a minute. On page 53 of the text, some trick algebra leads to the formula for $n \Rightarrow \infty$ periods

Continuous Compounding of Interest

$$\boxed{F = Pe^{rt}}$$

n, infinite

whereas, for n being finite,

$$\boxed{F = P(1+\frac{r}{n})^{nt}}$$

n, finite

④ Suppose you have a final value in mind and you know rate and time. To get the principal needed to invest, do the following:

- $F = P\left(1+\frac{r}{n}\right)^{nt}$ n being finite compounding

$$\frac{F}{\left(1+\frac{r}{n}\right)^{nt}} = P \rightarrow \boxed{P = F\left(1+\frac{r}{n}\right)^{-nt}}$$

- $F = Pe^{rt}$ n infinite or continuous compounding

$$\frac{F}{e^{rt}} = P \longrightarrow \boxed{P = Fe^{-rt}}$$

Finally, "principal" + "final" value are often renamed in other contexts (later in text) as "present" + "future" value of an account.

We can still use P and F, respectively.

Effective interest rate – This rate is a little greater than the annual interest rate as stated, according to number of compounding periods. Since we kept adding $P(1+\frac{r}{n})\cdot\frac{r}{n}$ to the new principal P we effectively built up the r to the value we call

$$\boxed{n \text{ finite}} \quad \boxed{\text{effective interest rate} = \left(1+\frac{r}{n}\right)^{n\cdot 1} - 1}$$

For example: $\left(1+\frac{.03}{4}\right)^4 - 1 = 1.030339191 - 1$

$\boxed{n=4}$

$$= \underbrace{.030339191}_{\text{eff interest}} > .03$$

Also, for continuous compounding:

$$\boxed{n \to \infty} \quad \boxed{\text{effective interest rate} = e^r - 1}$$

For example: $e^{.03} - 1 = 0.0304545\ldots > .03$

So, we see effective annual rate > annual rate as stated.

for short: $\boxed{\text{Eff rate} > \text{Rate}}$ by a bit