~~Wkst #6, Sec 3.2 Sketches~~

Ch. 3 Section 3.4 HW Solns

#1) a) Are $f(x) = \sqrt[3]{x-1}$ and $g(x) = x^3 + 1$ inverses?

Dom $f = \mathbb{R}$ Dom $g = \mathbb{R}$

Rang $f = \mathbb{R}$ Range $g = \mathbb{R}$

For an inverse fcn, g of f, the requirement is met $\rightarrow$ $\left\{ \begin{array}{l} \text{dom } g = \text{range } f \\ \text{range } g = \text{dom } f \end{array} \right.$

Check algebraically:

$f(x) = y = \sqrt[3]{x-1} \longrightarrow x = \sqrt[3]{y-1} \longrightarrow x^3 = y - 1$

$\longrightarrow y = x^3 + 1 = f^{-1}(x)$

So f & g are inverses

$(f^{-1} = g$ & $g^{-1} = f)$

Check that $f^{-1}(f(x)) = x$ & $g^{-1}(g(x)) = x$

$f^{-1}(\sqrt[3]{x-1}) = (\sqrt[3]{x-1})^3 + 1 = x - 1 + x = x$

$g^{-1}(x^3+1) = \sqrt[3]{x^3+1-1} = x \quad \therefore f^{-1} = g, \; g^{-1} = f$ //

b) $f(x) = 5 - x$, $g(f(x)) = g(5-x) = 5 - x - 5 = -x$

$\neq x$

$\therefore f^{-1} \neq g$

c) $f(x) = 3x - 2$, $g(x) = \frac{1}{3}x + 2$

$f(g(x)) = f\left(\frac{1}{3}x + 2\right) = 3\left(\frac{1}{3}x - 2\right) + 2 = -6 + 2 = -4$

$\neq x$

$\therefore f^{-1} \neq g$

d) $f(x) = \frac{1}{4}x + 3$, $g(x) = 4x - 12$

$f(g(x)) = f(4x - 12) = \frac{1}{4}(4x - 12) + 3 = x$

$g(f(x)) = g\left(\frac{1}{4}x + 3\right) = 4\left(\frac{1}{4}x + 3\right) - 12 = x$ //

$\therefore f = g^{-1}, \; g = f^{-1}$

#2) a) $f(x) = 4x + 2$ (note: This is one-one. Good!)

If f^{-1} is the inverse fcn., then notationally either solving $y = f(x)$ via $x = g(y)$ for y will give $f^{-1}(x)$. More rigorous than this "switching x + y" method is that of p. 82:

Set $f(f^{-1}(x)) = x$ + solve for $f^{-1}(x)$

We seek this

$$f(f^{-1}(x)) = 4f^{-1}(x) + 2 = x$$

Candidate for $f^{-1}(x)$ $\longrightarrow$

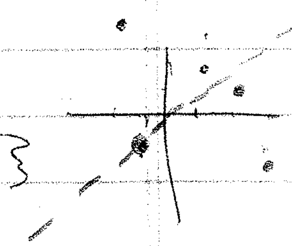
$$f^{-1}(x) = \frac{x-2}{4}$$

found by assuming $f(f^{-1}(x)) = x$. Verify by checking that $f^{-1}(f(x)) = x$

$$f^{-1}(4x+2) = \frac{4x+2-2}{4} = x \quad \text{yes! } //$$

b) $f(x) = \{(1,2), (-2,3), (-1,-1)\}$

$f^{-1}(x) = \{(2,1), (3,-2), (-1,-1)\}$



c) $f(x) = 2x^5 - 3$

Find f^{-1} such that $f(f^{-1}(x)) = x$ + check via $f^{-1}(f(x)) = x$

$$f(f^{-1}(x)) = 2(f^{-1}(x))^5 - 3 = x$$

$$[f^{-1}(x)]^5 = \frac{x+3}{2}$$

$$f^{-1}(x) = \left(\frac{x+3}{2}\right)^{1/5}$$

If this is f^{-1} then $f^{-1}(f(x)) = x$

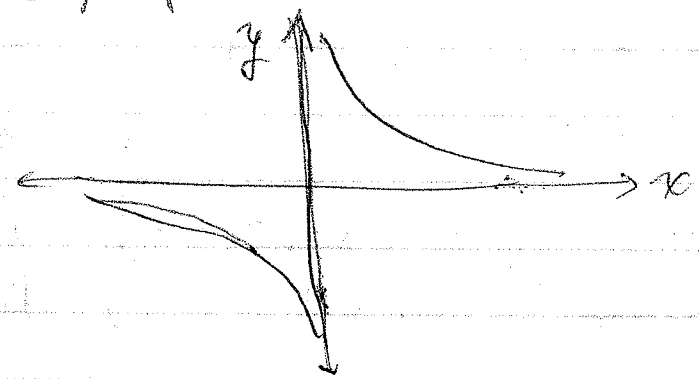
Check: $f^{-1}(2x^5 - 3) = \left(\dfrac{2x^5 - 3 + 3}{2}\right)^{1/5} = x$ //

Sec 3.1

#2d) $f(x) = \frac{1}{x} \Rightarrow f(f^{-1}(x)) = x \Rightarrow \frac{1}{f^{-1}(x)} = x$

$\Rightarrow \frac{1}{x} = f^{-1}(x)$ ~~it~~ works both ways (check).

The graph is shown:



$f(x) = \frac{1}{x}$ is its own inverse. Draw $y = x$ & check reflection to see this.

e) $f(x) = \frac{4+x}{2x-7}$ Like the example – you can find the range of f by taking f^{-1} & naming its domain.

<u>The "switching x & y" method</u>:

$f(x) = y = \frac{4+x}{2x-7} \longrightarrow x = \frac{4+y}{2y-7} \longrightarrow x(2y-7) = 4+y$

$\longrightarrow 2xy - 7x = 4 + y \longrightarrow -7x = 4 + y - 2xy$

$\longrightarrow -7x - 4 = y - 2xy \xrightarrow{\text{common factor}} -7x - 4 = y \cdot (1 - 2x)$

$\longrightarrow \boxed{\frac{-7x-4}{1-2x} = y = f^{-1}}$ Check that this is indeed $f^{-1}(x)$.

Note that its domain is $\{x \in \mathbb{R} \mid x \neq 1/2\}$

This is also the range of f. (What's the domain of f?)

$\cancel{x = \mathbb{R} \setminus \{7/2\}}$

#2e detail

$(-\infty, 7/2) \cup ($

$$f(x) = \frac{4+x}{2x-7} = y$$

Dom f : $2x - 7 \neq 0$

$x \neq 7/2$

Dom $= \{x \mid x \neq 7/2\}$

- Try to isolate y

$$\frac{4+y}{2y-7} = x$$

cross x + expanded

$$4 + y = 2xy - 7x$$

- Get the terms containing y together on one side

$$4 + 7x = 2xy - y$$

- Factor out y

$$4 + 7x = y \cdot (2x - 1)$$

- Divide by $(2x-1)$

$$\frac{4+7x}{2x-1} = y = f^{-1}$$

Dom: $x \neq 1/2$

Therefore, this is the range of of f

f) $f(x) = 3 - x^2$ for $x \geq 0$

Notice the domain is restricted.

This is b/c $f(x)$ is not one-one otherwise, so an inverse can't be found.

$$y = 3 - x^2 \rightarrow x = 3 - y^2 \rightarrow y^2 = 3 - x$$

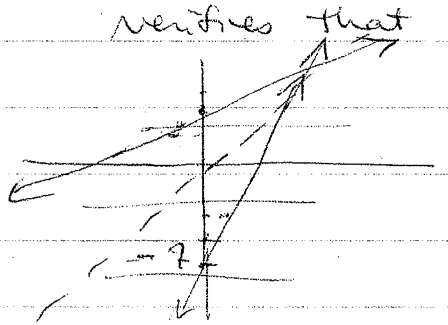
$$\rightarrow \overset{(f^{-1})}{y} = \sqrt{3-x} \geq 0 \text{ for } x \leq 3$$

Thus, Dom $f^{-1} = \{x \mid x \leq 3\}$ Range $f^{-1} = \{y \mid y \geq 0\}$

Range $f = \{x \mid x \leq 3\}$ Dom $f = \{x \mid x \geq 0\}$

As given. //

#3) Showing $f(x) = 2x - 7$ has an inverse verifies that it is one-one. So does its graph.



$$f(f^{-1}(x)) = x$$

$$2f^{-1}(x) - 7 = x$$

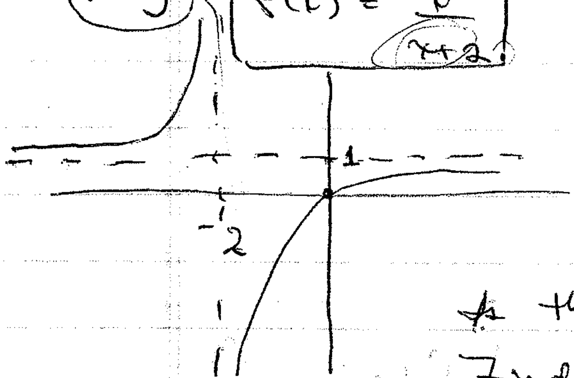
$$f^{-1}(x) = \frac{x}{2} + \frac{7}{2}. \quad //$$

#4 The y-intercept of a fcn f is the value it takes at $x = 0$. Thus, $f(0)$ is the y-intercept. Since $f^{-1}(x) = g(x) = 0$ when $x \in$ Dom g is the y-intercept of f, and since the x-intercept of a fcn g is the value it takes when set $= 0$, then apparently $f^{-1} = g$ has an x-intercept at $(x, 0) = (f(0), 0)$.

$$\boxed{x\text{-intercept} \equiv \text{root(s)} \equiv \text{zero(s)}}$$

Sec 3.4 con'd

#5 $\boxed{f(x) = \dfrac{x}{x+2}}$



Incidentally, the graph of this — you don't need to sketch it yet, but a picture helps w̄ inverse understanding — is seen left.

Is this one-one? Sure! Odd? No.

Find $f^{-1}(x)$. If it exists (and it does, as Idea 3.4.2 justifies), then

$$f \circ f^{-1} = f(f^{-1}(x)) = x \text{, that is,}$$

$$f(f^{-1}(x)) = \frac{f^{-1}(x)}{f^{-1}(x)+2} \equiv x \equiv \frac{f^{-1}}{f^{-1}+2}$$

Solving for $f^{-1}(x)$, we get:

$$f^{-1}(x) = x f^{-1}(x) + 2x$$

$$f^{-1}(x) - x f^{-1}(x) = 2x$$

$$f^{-1}(x)\,(1-x) = 2x$$

$$\boxed{f^{-1}(x) = \frac{2x}{1-x}}$$

$$f^{-1} = x f^{-1} + 2x$$
$$f^{-1} - x f^{-1} = 2x$$
$$f^{-1}(1-x) = 2x$$

(Check using $f^{-1}(f(x)) = x$ that this works.)

What is the dom of f^{-1}? $\{x \mid x \neq \}$

This is the same as the range of $f(x)$ (see the graph). If we graph f^{-1} we find its asymptotes are reversed + it reflects f across the line $y = x$. (This is where you used to pull out your calculator — now (this semester) we'll learn how to create the graph from our brain!)

#5 Summary

$$\boxed{f^{-1}(x) = \frac{2x}{1-x}} \qquad \boxed{f(x) = \frac{x}{x+2}}$$

Checking: $f^{-1}(f(x)) \stackrel{?}{=} x$

$$f^{-1}\left(\frac{x}{x+2}\right) \stackrel{?}{=} x$$

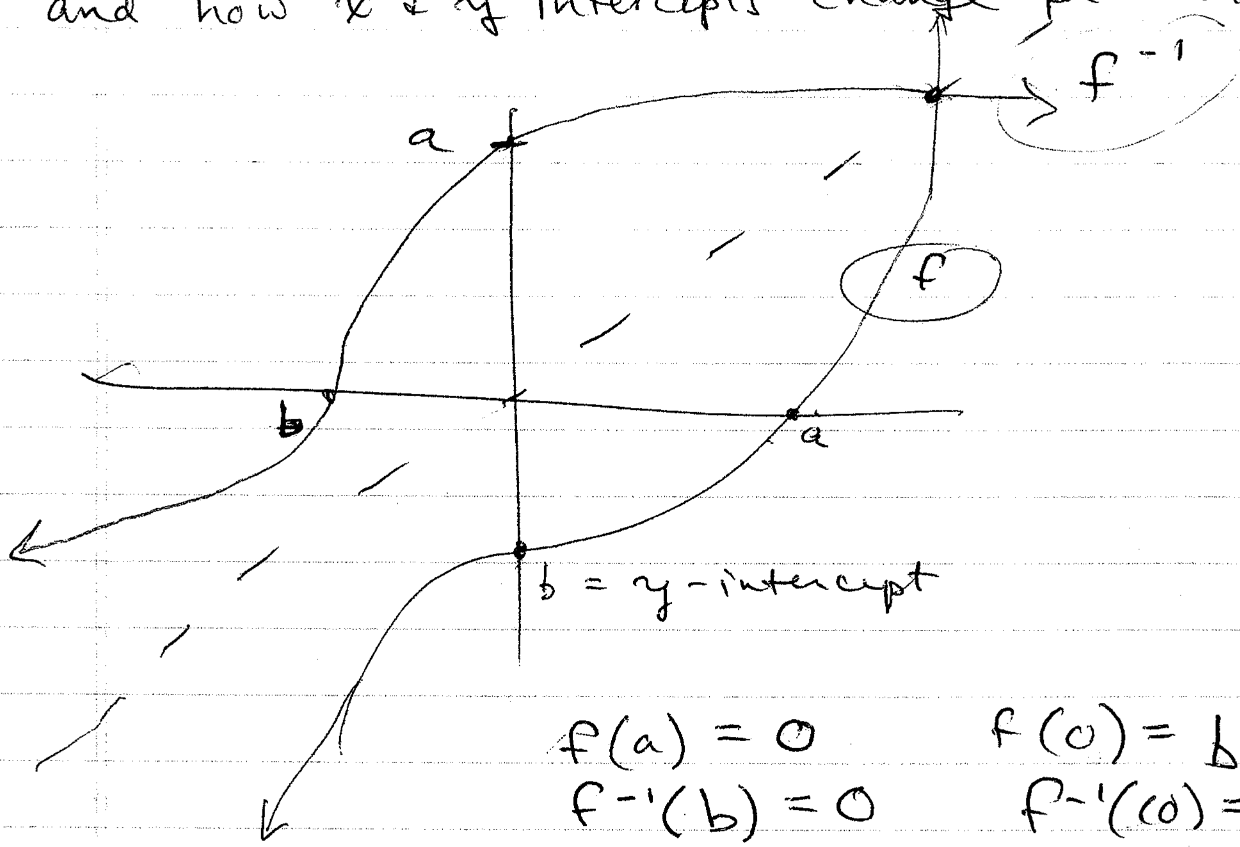
$$\boxed{\text{LCD } x+2 \qquad \frac{x+2}{x+2} \cdot \frac{2\left(\frac{x}{x+2}\right)}{\frac{1}{1} - \left(\frac{x}{x+2}\right)} \stackrel{?}{=} x}$$

Simplifying a complex rational expression

$$\frac{2x}{\cancel{x} + 2 - \cancel{x}} = x$$

So $f^{-1} \circ f = x$

Example of a function f, its inverse f^{-1}, and how x & y intercepts change places.



$f(a) = 0$ $\qquad$ $f(0) = b$

$f^{-1}(b) = 0$ $\qquad$ $f^{-1}(0) = a$

The y-intercept of f = The x-intercept of f^{-1}

and vice-versa.

(What are 2 other names for the x-intercept?
Answer: root(s), zero(s))