Math 108 Ch 3 Sec 3.2 HW Solns

#1 $f(x) = \frac{1}{x}$, $g(x) = x$ $f \circ f \neq g$ because even though $f \circ f = f\left(\frac{1}{x}\right) = \frac{1}{1/x} = x$, which looks like $g(x)$, the domain of $f \circ f$ is not the same as the domain of g. For 2 fcns to be $=$ their rules and domains must be $=$.

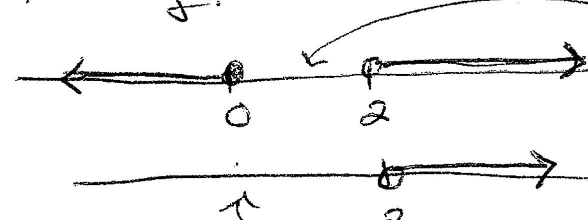
$\text{Dom } f \circ f = \{x \in \mathbb{R} \mid x \neq 0\}$ since the unsimplified fcn. $\frac{1}{1/x}$ is not defined at $x = 0$, whereas $g(x) = x$ is defined at all $x \in \mathbb{R}$.

#2 Show the domains of $f(x) = \sqrt{\frac{x}{x-2}}$ + $g(x) = \frac{\sqrt{x}}{\sqrt{x-2}}$ are not the same + therefore $f \neq g$. This amounts to finding at least one $x \in \text{Dom } f$ that is not in $\text{Dom } g$ (or vice-versa). You may do this by trial + error, since the domains are not asked for, but here's the complete picture of the domains to help you see why $f \neq g$.

Dom f:

$\neq$

Dom g:



Radicand is negative here, so omit it.

Both radicands are negative here, so omit it.

Note that $x \neq 2$ for either fcn.

So, for example, $x = -1$ is not in Dom g but it is in Dom f.

#3. $f = \frac{3}{x+1}$ $g = \frac{x+2}{x-1}$ Dom f: $x \neq -1$
Dom g: $x \neq 1$ ← Shorthand!

a) $f + g = \frac{3}{x+1} + \frac{x+2}{x-1}$

Dom $(f+g)$ is simply intersection of Dom f + Dom g: $\{x \in \mathbb{R} \mid x \neq \pm 1\}$

b) $f - g = \frac{3}{x+1} - \frac{x+2}{x-1}$ Likewise!

c) $f \cdot g = \left(\frac{3}{x+1}\right)\left(\frac{x+2}{x-1}\right)$ Notice - no need to expand these. Same domain.

d) $\frac{f}{g} = \frac{3/(x+1)}{(x+2)/(x-1)}$

Now we throw $x = -2$ out of the domain, since it makes denom $f/g = 0$.

Dom $f/g = \{x \in \mathbb{R} \mid x \neq \pm 1, -2\}$

e) $(f+g)(2) = \frac{3}{3} + 4 = \cancel{4\tfrac{3}{4}}$ $1 + 4 = 5$ //

~~$(f \cdot g)(+2) = \frac{3}{3} \cdot 4 = 4$~~ Wrong x!

$(f \cdot g)(-2) = (-3)(0) = 0$ //

#4 We did this in class, but not the first part, $(f+g)(0)$ and $(f+g)(5)$. These are asked to situate us in the domain of $f+g$, which is the essential feature of a piecewise fcn.

(#4 continued)

$(f+g)(0)$ is evaluated ~~when~~ in the way the individual fcns. are defined at the value $x=0$.

$$f(0) = 0+3 = 3, \quad g(0) = 4-0 = 4$$

so $(f+g)(0) = 3+4 = 7$ //

$$f(5) = 2(5)-5 = 5, \quad g(5) = 4-5 = -1$$

so $(f+g)(5) = 5 + {}^{-}1 = 4$ //

#5) $f = 1-2x^2, \quad g = x+1$

Notice that f & g are both polynomials, so their domains are $\mathbb{R}$. Hence, the compositions $f \circ g$, $g \circ f$, $f \circ f$, $g \circ g$ are all polynomials & hence all their domains are $\mathbb{R}$ as well. (Why?)

$$(f \circ g)(x) = f(g(x)) = 1-2(x+1)^2 = -2x^2-4x-1 \text{ //}$$

$$g \circ f = g(f(x)) = g(1-2x^2) = 1-2x^2+1$$
$$= 2-2x^2 \text{ //}$$

$$f \circ f = f(1-2x^2) = 1-2(1-2x^2)^2$$
$$= 1-2(1-4x^2+4x^4) = -1+8x^2-8x^4 \text{ //}$$

$$g \circ g = g(g(x)) = g(x+1) = (x+1)+1 = x+2 \text{ //}$$

#6 Find $f \circ g$ + $g \circ f$ including domains

a) $f(x) = \sqrt{x}$, $g(x) = x^2$, $f \circ g = \sqrt{x^2}$
Dom = $\mathbb{R}$ //

For $f \circ g$, the dom includes $x < 0$ since we square the values of x before taking their roots. Order is important here, as $g \circ f$ shows:

$g \circ f = g(\sqrt{x}) = (\sqrt{x})^2$
Dom: $\{x \in \mathbb{R} \mid x \geq 0\}$ //

b) $f(x) = x + 2$
$g(x) = \dfrac{1}{x-3}$

$f \circ g = f\left(\frac{1}{x-3}\right) = \dfrac{1}{x-3} + 2$
Dom: $\{x \mid x \neq 3\}$ //

$g \circ f = g(x+2)$
$= \dfrac{1}{x+2-3} = \dfrac{1}{x-1}$
Dom: $\{x \mid x \neq 1\}$ //

#7) Here we go. The key here is, as we've said, the values that $g(x)$ gives (the range) must be present in the domain of f when we describe $f \circ g$. But since the values f produces depend on Dom f, then g has available only these x values, as "seen" by g.

#7 It turns out that this is not too hard. Since the range of g lies in the domain of f when we compose f with g, then what happens to x in domain of f now happens to $g = 1 - x$ in the domain of $f \circ g$. (It took me three attempts to word that so it wasn't confusing ;-) Here goes:

$$f \circ g = f(g(x)) = f(1-x) = \ldots$$

It's still a piecewise fcn. We'll evaluate it first at $1-x < 0$, rather than $x < 0$:

$$f(x) = 2x \quad \text{if} \quad x < 0$$

$$f(g(x)) = 2(1-x) \quad \text{if} \quad 1-x < 0 \text{, that is, } x > 1$$

Now we evaluate it at $1-x > 3$ rather than $x > 3$:

$$f(x) = x^2 \quad \text{if} \quad x > 3$$

$$f(g(x)) = (1-x)^2 \quad \text{if} \quad 1-x > 3 \text{; that is, } x < -2$$

Summarizing:

$$f \circ g = \begin{cases} (1-x)^2 & \text{if } x < -2 \\ 2(1-x) & \text{if } x > 1 \end{cases}$$

(The order of the pieces flipped; we write them in order of the x.)

Finally: $(f \circ g)(-4) = (1 - -4)^2 = 25$

$(f \circ g)(4) = 2(1-4) = -6$

#8 $f = \{(-1,2), (0,7), (2,5)\}$

$g = \{(-1,1), (3,0), (4,-1)\}$

This goes back to the original definition of function as a rule that assigns values in one set (X) to another set ($Y = f(x)$ or $g(x)$ here), with the condition that each element of X ~~take~~ goes to a unique element of Y.

$f \circ g = f(g(x))$ Do these one at a time.

$f(g(-1)) = f(1) =$ not defined b/c $1 \notin \text{Dom } f$

$f(g(3)) = f(0) = 7$

$f(g(4)) = f(-1) = 2$

As a set of ordered pairs:

$f \circ g = \{(3,7), (4,2)\}$

#9 The difference quotient is defined (from calculus) as the following operation on a function f:

$$\frac{f(x+h) - f(x)}{h} = DQ \quad (\text{my abbreviation})$$

(a) $f(x) = \frac{1}{x}$, $f(x+h) = \frac{1}{x+h}$

$$DQ = \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \quad \text{Simplify this}$$

$$\Rightarrow \frac{\left(\frac{1}{x+h} - \frac{1}{x}\right)}{(h)} \frac{(x(x+h))}{(x(x+h))} \leftarrow \text{LCD}$$

$$= \frac{x-(x+h)}{h(x(x+h))} = \frac{-h}{h(x(x+h))} = \frac{-1}{x(x+h)} \quad //$$

#10 There are a few ways to do these. Quite a few if we get creative! But basically...

a) $h(x) = \sqrt{x-4}$ from $f(x) = x-4$
$g(x) = \sqrt{x}$
where $h(x) = g \circ f$

b) $h(x) = f(g(x)) = \cancel{f(\ldots)}$ wait.

Name $g(x) = x^5 - 2x^3 + 1$
$f(x) = \frac{1}{x}$

c) $h(x) = \sqrt{\sqrt{x}+1}$ Remember, you name g so that when f operates on x normally, it now operates on $g(x)$. So, if we work backwards:

$$h(x) = f(g(x)) = f(\sqrt{x}+1) = \sqrt{\sqrt{x}+1}$$

then $f(x) = \sqrt{x}$.

<u>Chapter 3 HW solns</u>

<u>Sec. 3.2</u>

#1 To find y-intercept, let $x = 0$
To find x-intercept, $f(x) = 0$
Another name for the x-intercepts is <u>roots</u> of f. Also, the <u>zeroes</u>.

a) $f(x) = (2x-1)(5+x)$

$f(0) = (-1)(5) = -5$ //

$f(x) = 0 = (2x-1)(5+x)$ when

$2x-1 = 0$ or $5+x = 0$

$x = 1/2, -5$ //

b) $f(x) = \dfrac{2x-3}{5+x}$, $f(0) = -3/5$ //

$f(x) = 0 = \dfrac{2x-3}{5+x}$ Denominator doesn't matter. $2x-3 = 0$ gives $x = 3/2$ //

c) $f(x) = -2x^3 - 3x^2 + 2x - 3$

$f(0) = -3$ //

#1c continued. - Factor $f(x) = 2x^3 - 3x^2 + 2x - 3$ using techniques of Ch. 1: Set $= 0$.

Factor this by <u>grouping</u>:

$$x^2(2x-3) + (2x-3) = (2x-3)(x^2+1) = 0$$

$$2x - 3 = 0 \quad \text{when} \quad x = 3/2$$

$$x^2 + 1 \neq 0 \quad \text{for} \quad x \in \mathbb{R}$$

So there's only <u>one</u> real root.

#1d) $f(x) = \sqrt{x+9} - 5$, $f(0) = -2$ //

$0 = \sqrt{x+9} - 5$

$5 = \sqrt{x+9}$

$25 = x + 9$, $x = 16$ (check it) //

#1(e) $f(x) = -\frac{1}{x} + \frac{x}{6-x}$

$f(0)$ doesn't exist //

$$0 = -\frac{1}{x} + \frac{x}{6-x} = \frac{x - 6 + x^2}{x(6-x)}$$

Only the numerator matters.

$x^2 + x - 6 = 0 \longrightarrow (x+3)(x-2) = 0$

$x = -3, 2$ //

#2 An odd fcn. f is defined as follows:

- For all $x \in \text{Dom } f$, $f(-x) = -f(x)$

An even fcn f is defined as follows:

- For all $x \in \text{Dom } f$, $f(-x) = f(x)$

#2 continued – You just see what happens when you express $f(-x)$ as usual and compare it to $f(x)$).

(a) $f(-x) = 3(-x)^4 - 2(-x)^2 - 5$

$= 3x^4 - 2x^2 - 5$ (why?)

$= f(x)$, so f is even. //

Helpful fact: All even order polynomial fcns. are even.

(b) $f(-x) = \underbrace{(-x)^3}_{(-1x)^3 \,=\, (-1)^3 x^3 \,=\, -x^3} + 5 = -x^3 + 5$

How does $-x^3 + 5$ compare to $x^3 + 5$?

It isn't equal, so f is not even.

And, since $-f(x) = -x^3 - 5$,

$f(-x) \neq -f(x)$, so it isn't odd. //

Helpful fact: A function may be neither even nor odd. //

(c) $f(-x) = \dfrac{(-x)^2 + 1}{-x} = \dfrac{x^2+1}{-x} = -\left(\dfrac{x^2+1}{x}\right)$

So $f(-x) = -f(x) = -\left(\dfrac{x^2+1}{x}\right)$

Hence, f is odd. //

Sec 3.3 continued

#3 Given f, g both odd, show $f \cdot g$ is even. — We use definitions of $f \cdot g$ (product of fcns) & odd & even fcn ~~to show this~~.

We need to show that $(f \cdot g)(x)$ is even.

- By definition, $f \cdot g$ even would mean

$$(f \cdot g)(-x) = (f \cdot g)(x)$$

- Also, by definition, the product of f & g means:

$$f \cdot g = f(x)\,g(x)$$

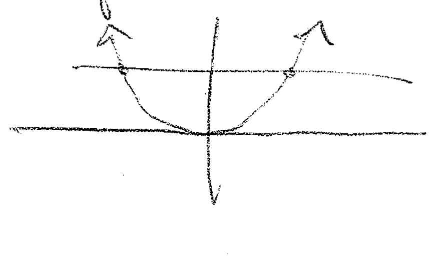
- And by definition, if f & g are each odd, then:

$$f(-x) = -f(x), \quad g(-x) = -g(x).$$

Putting all these together,
$$\begin{aligned}(f \cdot g)(-x) &= f(-x)\,g(-x) \\ &= \big(-f(x)\big)\big(-g(-x)\big) \\ &= (f \cdot g)(x)\end{aligned}$$

#4 A one-to-one function is a fcn. that has a unique y for every x <u>and</u> a unique x for every y. One-to-one fcns pass a horizontal line test the same way fcns. pass a vertical line test.

$y = f(x) = x^2$ is not one-to-one since for any value of $y = x^2$ there are two values of x (except $y = 0$, of course). Its graph fails the horizontal line test.



#5 f contains the pts $(-1,3)$, $(6,2)$, & $(-2,-3)$.

(a) If f is even then for all $x \in \operatorname{dom} f$, $f(-x) = f(x)$. So, since $f(-1)=3$, then $f(1)=3$; since $f(6)=2$, then $f(-6)=2$, and since $f(-2)=-3$ then $f(2)=-3$. Thus f also contains $(1,3)$, $(-6,2)$, & $(2,-3)$.

See Below ~~(b) Likewise, f odd implies that f(-x) = -f(x) for all x ∈ dom f. Therefore, -f(-1) = -f(1) =~~

(c) To see that neither fcn. described in (a) & (b) could be one-one we can graph the pairs & see how they fail the horizontal line test. Or we could use the shoelace digram for one-one fcns.

(a)

x	y
-1	3
1	
6	2
-6	
-2	3
2	

(This one is obvious, since even fcns. reflect across the y-axis)

neither are

x	y

(b)

x	y
-1	3
2	
6	2
-6	-2
-2	-3
1	

(b) If f is odd then $f(-x) = -f(x)$. Following the same idea as in (a), we get

$-f(-1) = -3 = f(1)$

$-f(6) = -2 = f(-6)$

$-f(-2) = 3 = f(+2)$

so the other points are $(1,-3)$, $(-6,-2)$, $(+2,3)$

#7) By definition, f is one-one if for every element in the range there is a unique element in the dom.

This is the same as saying that $f(x_1) = f(x_2)$ implies $x_1 = x_2$. (In other words, x_1 + x_2 cannot both go to the same y; if two x values (say x_1, x_2) produce the same y value ($f(x_1) = f(x_2)$) then x_1 + x_2 must have been just one value. This reasoning may sound a lot like what Alice encountered when talking to the residents of Wonderland but keep in mind that Lewis Carroll was a mathematics tutor (professor) at Oxford!)

Show $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

(a) For $f(x) = 2x-7$:

$$2x_1 - \cancel{7} = 2x_2 - \cancel{7}$$
$$2x_1 = 2x_2$$
$$x_1 = x_2$$

(b) $$\frac{x_1}{x_1+2} = \frac{x_2}{x_2+2}$$

$$\cancel{x_1x_2} + 2x_1 = \cancel{x_2x_1} + 2x_2$$
$$\cancel{2}x_1 = \cancel{2}x_2$$