Chapter 7 Abs. Value Functions, Abs. Value Equations, Abs. Value Inequalities, Rational Inequalities, Quadratic Inequalities

7.1 Linear, quadratic, and rational inequalities

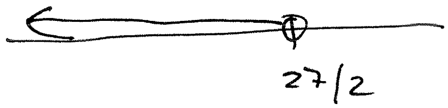
7.2 Absolute Value functions, equations, graphs

7.3 Absolute Value Inequalities

$\boxed{7.1}$

Linear inequalities - these are solved like linear equations, but instead of a unique soln they have soln. intervals.

Ex $2x-9 < 18 \rightarrow 2x < 27 \rightarrow x < 27/2$



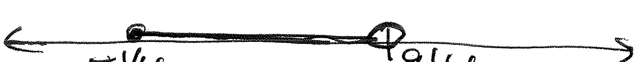
$(-\infty, 27/2)$ $\qquad$ $x < 27/2$

graph on number line $\qquad$ interval notation $\qquad$ algebraic notation

Ex

$$0 \le 4x+1 < 10$$
$$-1 \qquad -1 \qquad -1$$
$$-1 \le 4x < 9$$
$$-\frac{1}{4} \le x < \frac{9}{4}$$

Balance all parts by moving constant (+ or −) and then dividing by coeff.



Graph $\qquad$ $[-1/4, 9/4)$ Interval

Ex $\dfrac{x+7}{3} > -2$

$x + 7 > -6$

$x > -13$ algebraic

 graph

-13

$(-13, \infty)$ interval

Ex $8 - 2x < 15$

$-2x < 7$

$x > -7/2$

flip the $<$ when $\div$ or $\times$ by a negative

$-7/2$ graph

$(-7/2, \infty)$ interval

Ex $\dfrac{x}{3} + \dfrac{1}{4} \leq \dfrac{2x}{5}$

Multiply through by the LCD, $3 \cdot 4 \cdot 5 = 60$

$60\left(\dfrac{x}{3} + \dfrac{1}{4}\right) \leq \left(\dfrac{2x}{5}\right)60$

$20x + 15 \leq 24x$

$-4x \leq -15$

$x \geq 15/4$

$15/4$

$[15/4, \infty)$

Quadratic inequalities – This is the first type that you cannot solve directly. You must look at roots of the associated quadratic equation and inspect intervals on the number line.

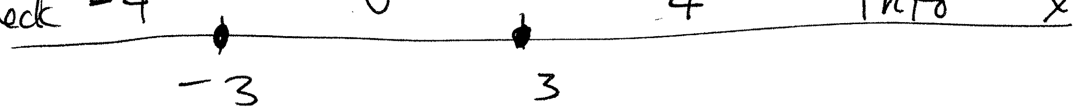
Ex $\quad x^2 - 9 \le 0 \;\rightarrow\;$ Write $\quad x^2 - 9 = 0$

$$(x-3)(x+3) = 0$$

$$x = 3, -3$$

Inspect values in each interval that results from graphing $3, -3$

Check $-4 \qquad 0 \qquad 4$ into $x^2 - 9 \le 0$



$(-4)^2 - 9 = 16 - 9 > 0$, so $(-\infty, -3)$ is not in the soln. set.

$0^2 - 9 = -9 < 0$, so $[-3, 3]$ is in the ~~soln~~ soln. set.

$4^2 - 9 = 16 - 9 > 0$, so $(3, \infty)$ is not in soln.

Thus, $\quad -3 \le x \le 3 \quad$ or $\quad [-3, 3]$

Notice we include the endpts because the original problem is $\le$ (not a strict inequality).

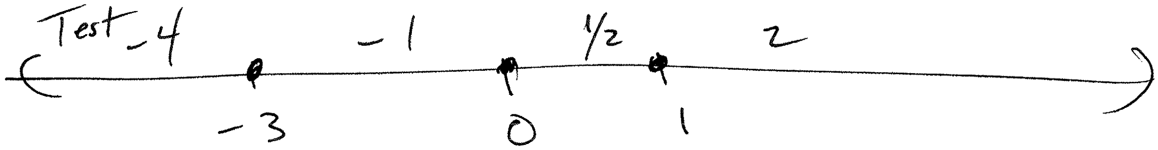
$\underline{\text{Ex}}$ $\quad x^3 + 2x^2 - 3x \geq 0$

This is a cubic, not a quadratic, but the process is the same: factor, plot roots on number line, test values in each interval. Remember, you just need to discern the <u>sign</u> of the expression at the value you're testing. (the single

$$x^3 + 2x^2 - 3x \geq 0$$

$$x(x^2 + 2x - 3) = x(x+3)(x-1) \geq 0$$

Roots : $x = 0, -3, 1$



Note: If you substitute test values into the <u>factored</u> polynomial, it's quicker:

$x = -4$:	$(-)(-)(-) = (-) < 0$	no
$x = -1$:	$(-)(+)(-) = (+) > 0$	yes
$x = 1/2$:	$(+)(+)(-) = (-) < 0$	no
$x = 2$:	$(+)(+)(+) = (+) > 0$	yes

Thus, $[-3, 0] \cup [1, \infty)$ is the soln.

Rational inequalities — These are probably the trickiest; if you have x terms top + bottom, you cannot just cross-multiply! If you do you are ignoring all negative solutions, since multiplying by the negative switches the direction of the inequality.

What we do, then, is similar to quadratic inequality approach: We find the roots of the top + bottom and inspect values in the resulting intervals. Again, we'll be checking for the sign.

Ex $\frac{8}{x} < 3$. Before we start, we first must convert this to one that sets the rational expression against 0, not 3. How else could we inspect the sign?!

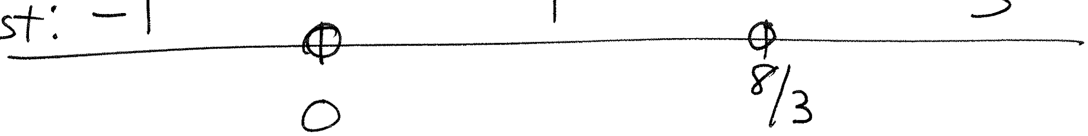
$\frac{8}{x} < 3$ - rewrite as $\frac{8}{x} - 3 < 0$

* combine using LCD $\boxed{\frac{8-3x}{x} < 0}$

- The roots are $8/3$ and 0.

- We need x values such that the quotient $\frac{8-3x}{x}$ is negative. That is, either the top will be negative or the bottom: $\frac{-}{+}$ or $\frac{+}{-} < 0$

Test: -1 $\quad 1 \quad 3$



$x = -1$: $\frac{8-3(-1)}{-1} = \frac{11}{-1} < 0$ yes

$X = 1$: $\frac{8-3(1)}{1} = \frac{5}{1} > 0$ no

$x = 3$: $\frac{8-3(3)}{3} = \frac{-1}{3} < 0$ yes

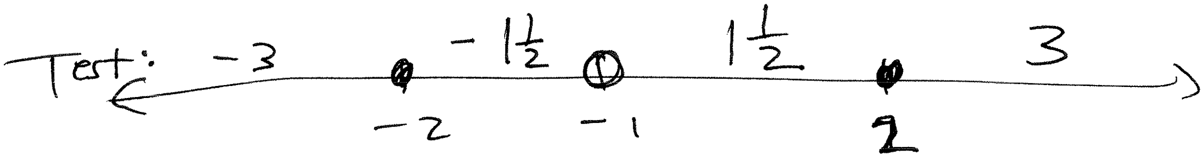
Soln: $(-\infty, 0) \cup (8/3, \infty)$

(Notice we don't include endpts since the original is a strict inequality.)

Ex $\quad \dfrac{4-x^2}{1+x} \geq 0$

This is already in the form we need to check the signs of top + bottom. Also, it's not a strict inequality, as 0 is allowed. But – the denominator $\neq$ zero, so you still have to leave that endpt out of soln.

Roots of all factors. $\quad \dfrac{(2-x)(2+x)}{(1+x)}$ are $x = 2, -2, -1$



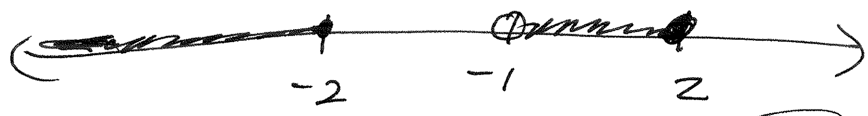
$x = -3$: $\quad \dfrac{(+)(-)}{(-)} = \dfrac{+}{-} > 0$ yes!

$x = -1\frac{1}{2}$: $\quad \dfrac{(+)(+)}{(-)} = \dfrac{+}{-} < 0$ no

$x = 1\frac{1}{2}$: $\quad \dfrac{(+)(+)}{(+)} = \dfrac{+}{+} > 0$ yes!

$x = 3$: $\quad (-)(+)/(+) = - < 0$ no.

soln.



$(-\infty, -2] \cup (-1, 2]$

Leaving out $x = -1$ since it's not in dom. of $\frac{4-x^2}{1+x}$

7.2 Absolute Value Fcn. and Equations

Def: We know "absolute value" of a number is always positive. So $|8| = 8$, but so is $|-8| = 8$. We formalize abs value by considering the function

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

This is a piecewise fcn, with consideration of the domain being critical to its graph.

But first, let's look at how to solve abs. value equations.

Ex $|6x-2| = 3+2x$ one-sided abs value

Either $6x-2 = 3+2x$ or $-(6x-2) = 3+2x$

$4x = 5$

$\boxed{x = 5/4}$

$-6x+2 = 3+2x$

$-1 = 8x$

$\boxed{-1/8 = x}$

There are two ways to check which answers are correct.
The best way is to plug them into the original and see if they satisfy the eqn. The other way - known as the tedious way - is to see if the answers satisfy the domain assumptions according the the def $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Ex. 7.2.8 in the book does just that, even while admitting is crazy! I'll just show you the best way and expect you will do the same:

Check $x = 5/4$: left $\left|6 \cdot \frac{5}{4} - 2\right| = \left|\frac{30}{4} - \frac{8}{4}\right| = \frac{22}{4}$

✓ right $3 + 2 \cdot \frac{5}{4} = \frac{3 \cdot 4 + 10}{4} = \frac{22}{4}$

Check $x = -1/8$ left: $\left|6 \cdot \frac{-1}{8} - 2\right| = \left|\frac{-6-16}{8}\right| = \left|\frac{-22}{8}\right| = \frac{22}{8}$

✓ right: $3 + 2 \cdot \frac{-1}{8} = 3 - \frac{2}{8} = \frac{22}{8}$

Both solns work. $\boxed{x = 5/4, -1/8}$

Ex $|3x-2| = |2x+7|$ Double-sided abs value

We could solve this by looking at both cases:

$$3x-2 = 2x+7 \quad \text{or} \quad 3x-2 = -(2x+7)$$

(Their are at first glance four cases, including

$$-(3x-2) = 2x+7 \quad \text{or} \quad -(3x-2) = -(2x+7),$$

but these reduce to the first two cases, so it's not so bad). But wait! There's a property of abs. value we can use here:

$$|a|^2 = |a||a| = a \cdot a = a^2$$

In other words, squaring an abs value removes the fences. Then we'd get a quadratic eqn:

$$|3x-2|^2 = |2x+7|^2$$

$$(3x-2)^2 = (2x+7)^2$$

$$9x^2 - 12x + 4 = 4x^2 + 28x + 49$$

$$5x^2 - 40x - 45 = 0 \rightarrow x^2 - 8x - 9 = 0$$

$$\rightarrow (x-9)(x+1) = 0 \rightarrow x = 9 \text{ or } -1$$

Check $|3 \cdot 9 - 2| \stackrel{?}{=} |2 \cdot 9 + 7| \rightarrow |25| = |25|$ ✓

$|3 \cdot -1 - 2| \stackrel{?}{=} |2 \cdot -1 + 7| \rightarrow |-5| = |5|$ ✓

$$\boxed{x = 9, -1}$$

Ex (from HW) $|1-2x| = 3 + |x+5|$

This is a more complicated problem because there really are four cases. And we can't square both sides because that does not clear the abs. values. So we must consider:

① Either $1-2x = 3 + x + 5$

② or $1-2x = 3 + -(x+5)$

③ or $-(1-2x) = 3 + (x+5)$

④ or $-(1-2x) = 3 + -(x+5)$

Solving:

① $1-2x = 8+x \rightarrow -7 = 3x \rightarrow x = -7/3$

② $1-2x = 3-x-5 \rightarrow 3 = x$

③ $-1+2x = 8+x \rightarrow x = 9$

④ $-1+2x = 3-x-5 \rightarrow 3x = -1 \rightarrow x = -1/3$

Checking each: ① $\left|1+\frac{14}{3}\right| \overset{?}{=} 3 + \left|-\frac{7}{3}+\frac{15}{3}\right|$

$\frac{17}{3} \overset{?}{=} 3 + \frac{8}{3} = \frac{9+8}{3} = \frac{17}{3}$ ✓

$\boxed{x = -7/3 \text{ checks}}$

② $|1-6| \overset{?}{=} 3 + |8| \Rightarrow |-5| \neq 3+8$ $\boxed{\text{discard } x = 3}$

③ $|1-18| \overset{?}{=} 3 + |14| \Rightarrow |-17| = 3 + |14|$ ✓

$\boxed{x = 9 \text{ checks}}$

④ $\left|1+\frac{2}{3}\right| \overset{?}{=} 3 + \left|4\frac{2}{3}\right|$ nope $\boxed{\text{discard } x = -1/3}$