2.2

OBJECTIVES

- Classify the relative extrema of a function using the Second-Derivative Test.
- Sketch the graph of a continuous function.

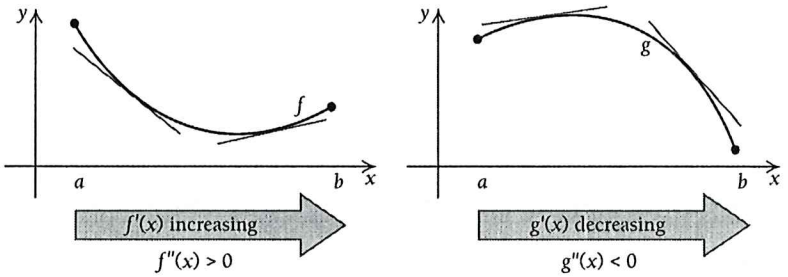
Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

The "turning" behavior of a graph is called its *concavity*. The second derivative plays a pivotal role in analyzing the concavity of a function's graph.

Concavity: Increasing and Decreasing Derivatives

The graphs of two functions are shown below. The graph of f is turning up and the graph of g is turning down. Let's see if we can relate these observations to the functions' derivatives.

Consider first the graph of f. Take a ruler, or straightedge, and draw tangent lines as you move along the curve from left to right. What happens to the slopes of the tangent lines? Do the same for the graph of g. Look at the curvature and decide whether you see a pattern.



For the graph of f, the slopes of the tangent lines are increasing. That is, f' is increasing over the interval. This can be determined by noting that $f''(x)$ is positive, since the relationship between f' and f'' is like the relationship between f and f'. Note also that all the tangent lines for f are below the graph. For the graph of g, the slopes are decreasing. This can be determined by noting that g' is decreasing whenever $g''(x)$ is negative. For g, all tangent lines are above the graph.

DEFINITION

Suppose that f is a function whose derivative f' exists at every point in an open interval I. Then

f is concave up on I if f' is increasing over I.

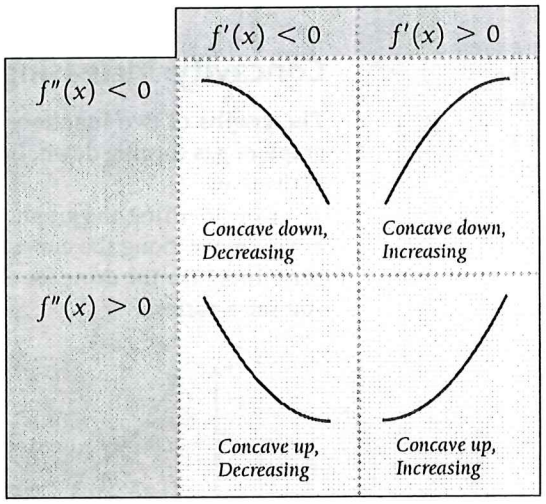
f is concave down on I if f' is decreasing over I.

The following theorem states how the concavity of a function's graph and the second derivative of the function are related.

THEOREM 4 A Test for Concavity

1. If $f''(x) > 0$ on an interval I, then the graph of f is concave up. (f' is increasing, so f is turning up on I.)
2. If $f''(x) < 0$ on an interval I, then the graph of f is concave down. (f' is decreasing, so f is turning down on I.)

Keep in mind that a function can be decreasing and concave up, decreasing and concave down, increasing and concave up, or increasing and concave down. That is, *concavity* and *increasing/decreasing* are independent concepts. It is the increasing or decreasing aspect of the *derivative* that tells us about the function's concavity.



TECHNOLOGY CONNECTION

Exploratory

Graph the function

$$f(x) = -\tfrac{1}{3}x^3 + 6x^2 - 11x - 50$$

and its second derivative,

$$f''(x) = -2x + 12,$$

using the viewing window $[-10, 25, -100, 150]$, with Xscl = 5 and Yscl = 25.

Over what intervals is the graph of f concave up?
Over what intervals is the graph of f concave down?
Over what intervals is the graph of f'' positive?
Over what intervals is the graph of f'' negative?

What can you conjecture?

Now graph the first derivative

$$f'(x) = -x^2 + 12x - 11$$

and the second derivative

$$f''(x) = -2x + 12$$

using the viewing window $[-10, 25, -200, 50]$, with Xscl = 5 and Yscl = 25.

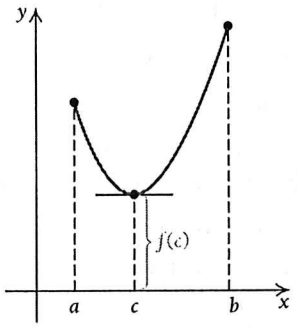
Over what intervals is the first derivative f' increasing?
Over what intervals is the first derivative f' decreasing?
Over what intervals is the graph of f'' positive?
Over what intervals is the graph of f'' negative?

What can you conjecture?

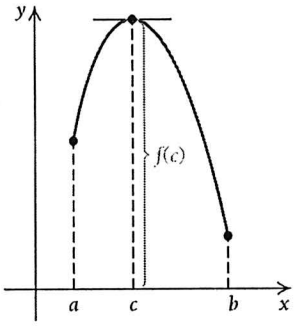
Classifying Relative Extrema Using Second Derivatives

Let's see how we can use second derivatives to determine whether a function has a relative extremum on an open interval.

The following graphs show both types of concavity at a critical value (where $f'(c) = 0$). When the second derivative is positive (graph is concave up) at the critical value, the critical point is a relative minimum point, and when the second derivative is negative (graph is concave down), the critical point is a relative maximum point.



$f'(c) = 0$,
$f''(c) > 0$,
The graph is concave up around c.
Therefore, $f(c)$ is a relative minimum.



$f'(c) = 0$,
$f''(c) < 0$,
The graph is concave down around c.
Therefore, $f(c)$ is a relative maximum.

This analysis is summarized in Theorem 5:

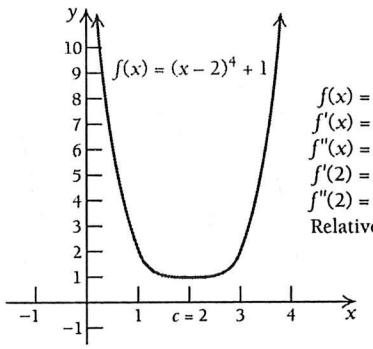
THEOREM 5 The Second-Derivative Test for Relative Extrema

Suppose that f is differentiable for every x in an open interval (a, b) and that there is a critical value c in (a, b) for which $f'(c) = 0$. Then:

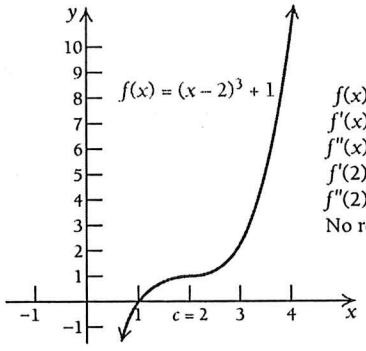
1. $f(c)$ is a relative minimum if $f''(c) > 0$.
2. $f(c)$ is a relative maximum if $f''(c) < 0$.

For $f''(c) = 0$, the First-Derivative Test can be used to determine whether $f(x)$ is a relative extremum.

Consider the following graphs. In each one, f' and f'' are both 0 at $c = 2$, but the first function has an extremum and the second function does not. When c is a critical value and $f''(c) = 0$, an extremum may or may not exist at c. Note too that if $f'(c)$ does not exist and c is a critical value, then $f''(c)$ also does not exist. Again, an approach other than the Second-Derivative Test must be used to determine whether $f(c)$ is an extremum.



$f(x) = (x-2)^4 + 1$
$f'(x) = 4(x-2)^3$
$f''(x) = 12(x-2)^2$
$f'(2) = 0$
$f''(2) = 0$
Relative minimum at $c = 2$



$f(x) = (x-2)^3 + 1$
$f'(x) = 3(x-2)^2$
$f''(x) = 6(x-2)$
$f'(2) = 0$
$f''(2) = 0$
No relative extremum

The second derivative is used to help identify extrema and determine the overall behavior of a graph, as we see in the following examples.

EXAMPLE 1 Find the relative extrema of the function f given by

$$f(x) = x^3 + 3x^2 - 9x - 13,$$

and sketch the graph.

Solution To find any critical values, we determine $f'(x)$. To determine whether any critical values lead to extrema, we also find $f''(x)$:

$$f'(x) = 3x^2 + 6x - 9,$$
$$f''(x) = 6x + 6.$$

Then we solve $f'(x) = 0$:

$$\begin{aligned} 3x^2 + 6x - 9 &= 0 \\ x^2 + 2x - 3 &= 0 && \text{Dividing both sides by 3} \\ (x + 3)(x - 1) &= 0 && \text{Factoring} \\ x + 3 = 0 \quad &\text{or} \quad x - 1 = 0 && \text{Using the Principle of Zero Products} \\ x = -3 \quad &\text{or} \quad x = 1. \end{aligned}$$

TECHNOLOGY CONNECTION

Exploratory

Consider the function given by

$$y = x^3 - 3x^2 - 9x - 1.$$

Use a graph to estimate the relative extrema. Then find the first and second derivatives. Graph both in the same window. Use the ZERO feature to determine where the first derivative is zero. Verify that relative extrema occur at those x-values by checking the sign of the second derivative. Then check your work using the approach of Example 1.

We next find second coordinates by substituting the critical values in the original function:

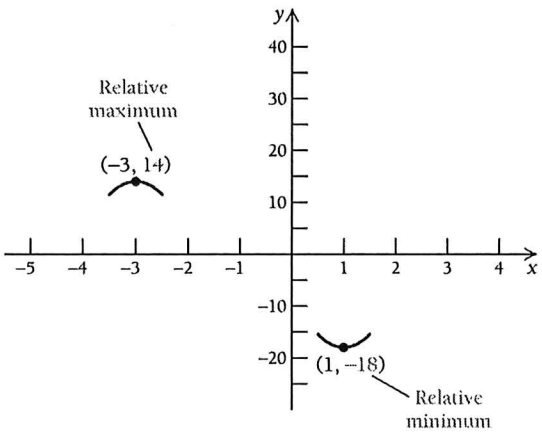
$$f(-3) = (-3)^3 + 3(-3)^2 - 9(-3) - 13 = 14;$$
$$f(1) = (1)^3 + 3(1)^2 - 9(1) - 13 = -18.$$

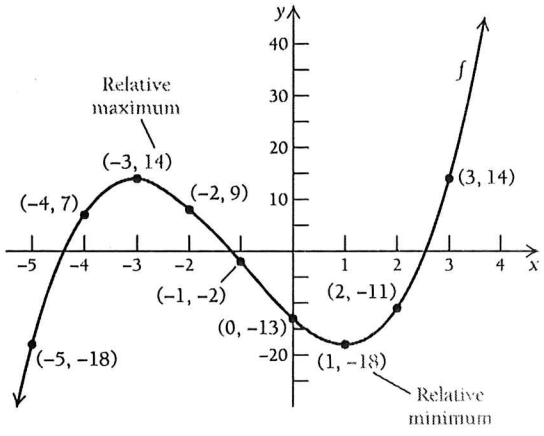
Are the points $(-3, 14)$ and $(1, -18)$ relative extrema? Let's look at the second derivative. We use the Second-Derivative Test with the critical values -3 and 1:

$f''(-3) = 6(-3) + 6 = -12 < 0$; ⟶ Relative maximum

$f''(1) = 6(1) + 6 = 12 > 0$. ⟶ Relative minimum

Thus, $f(-3) = 14$ is a relative maximum and $f(1) = -18$ is a relative minimum. We plot both $(-3, 14)$ and $(1, -18)$, including short arcs at each point to indicate the graph's concavity. Then, by calculating and plotting a few more points, we can make a sketch, as shown below.





EXAMPLE 2 Find the relative extrema of the function f given by

$$f(x) = 3x^5 - 20x^3,$$

and sketch the graph.

APPLICATIONS

Business and Economics

63. Depreciation. Suppose that the value V of the inventory at Fido's Pet Supply decreases, or depreciates, with time t, in months, where

$$V(t) = 50 - \frac{25t^2}{(t+2)^2}.$$

a) Find $V(0)$, $V(5)$, $V(10)$, and $V(70)$.
b) Find the maximum value of the inventory over the interval $[0, \infty)$.
c) Sketch a graph of V.
d) Does there seem to be a value below which $V(t)$ will never fall? Explain.

64. Average cost. The total-cost function for Acme, Inc., to produce x units of a product is given by

$$C(x) = 3x^2 + 80.$$

a) The *average cost* is given by $A(x) = C(x)/x$. Find $A(x)$.
b) Graph the average cost.
c) Find the slant asymptote for the graph of $y = A(x)$, and interpret its significance.

65. Cost of pollution control. Cities and companies find that the cost of pollution control increases along with the percentage of pollutants to be removed in a situation. Suppose that the cost C, in dollars, of removing p% of the pollutants from a chemical spill is given by

$$C(p) = \frac{48{,}000}{100 - p}.$$

a) Find $C(0)$, $C(20)$, $C(80)$, and $C(90)$.
b) Find the domain of C.
c) Sketch a graph of C.
d) Can the company or city afford to remove 100% of the pollutants due to this spill? Explain.

66. Total cost and revenue. The total cost and total revenue, in dollars, from producing x couches are given by

$$C(x) = 5000 + 600x \quad \text{and} \quad R(x) = -\tfrac{1}{2}x^2 + 1000x.$$

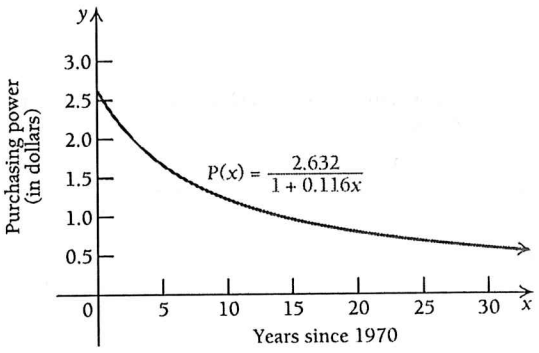
a) Find the total-profit function, $P(x)$.
b) The *average profit* is given by $A(x) = P(x)/x$. Find $A(x)$.
c) Graph the average profit.
d) Find the slant asymptote for the graph of $y = A(x)$.

67. Purchasing power. Since 1970, the purchasing power of the dollar, as measured by consumer prices, can be modeled by the function

$$P(x) = \frac{2.632}{1 + 0.116x},$$

where x is the number of years since 1970. (*Source*: U.S. Bureau of Economic Analysis.)

a) Find $P(10)$, $P(20)$, and $P(40)$.
b) When was the purchasing power \$0.50?
c) Find $\lim_{x\to\infty} P(x)$.



Life and Physical Sciences

68. Medication in the bloodstream. After an injection, the amount of a medication A, in cubic centimeters (cc), in the bloodstream decreases with time t, in hours. Suppose that under certain conditions A is given by

$$A(t) = \frac{A_0}{t^2 + 1},$$

where A_0 is the initial amount of the medication. Assume that an initial amount of 100 cc is injected.

a) Find $A(0)$, $A(1)$, $A(2)$, $A(7)$, and $A(10)$.
b) Find the maximum amount of medication in the bloodstream over the interval $[0, \infty)$.
c) Sketch a graph of the function.
d) According to this function, does the medication ever completely leave the bloodstream? Explain your answer.

General Interest

69. Baseball: earned-run average. A pitcher's *earned-run average* (the average number of runs given up every 9 innings, or 1 game) is given by

$$E = 9 \cdot \frac{r}{n},$$

where r is the number of earned runs allowed in n innings. Suppose that we fix the number of earned runs allowed at 4 and let n vary. We get a function given by

$$E(n) = 9 \cdot \frac{4}{n}.$$

a) Complete the following table, rounding to two decimal places:

Innings Pitched, n	9	6	3	1	$\frac{2}{3}$	$\frac{1}{3}$
Earned-Run Average, E						

EXERCISE SET 2.3

Determine the vertical asymptote(s) of each function. If none exists, state that fact.

1. $f(x) = \dfrac{2x - 3}{x - 5}$
2. $f(x) = \dfrac{x + 4}{x - 2}$
3. $f(x) = \dfrac{3x}{x^2 - 9}$
4. $f(x) = \dfrac{5x}{x^2 - 25}$
5. $f(x) = \dfrac{x + 2}{x^3 - 6x^2 + 8x}$
6. $f(x) = \dfrac{x + 3}{x^3 - x}$
7. $f(x) = \dfrac{x + 6}{x^2 + 7x + 6}$
8. $f(x) = \dfrac{x + 2}{x^2 + 6x + 8}$
9. $f(x) = \dfrac{6}{x^2 + 36}$
10. $f(x) = \dfrac{7}{x^2 + 49}$

Determine the horizontal asymptote of each function. If none exists, state that fact.

11. $f(x) = \dfrac{6x}{8x + 3}$
12. $f(x) = \dfrac{3x^2}{6x^2 + x}$
13. $f(x) = \dfrac{4x}{x^2 - 3x}$
14. $f(x) = \dfrac{2x}{3x^3 - x^2}$
15. $f(x) = 5 - \dfrac{3}{x}$
16. $f(x) = 4 + \dfrac{2}{x}$
17. $f(x) = \dfrac{8x^4 - 5x^2}{2x^3 + x^2}$
18. $f(x) = \dfrac{6x^3 + 4x}{3x^2 - x}$
19. $f(x) = \dfrac{6x^4 + 4x^2 - 7}{2x^5 - x + 3}$
20. $f(x) = \dfrac{4x^3 - 3x + 2}{x^3 + 2x - 4}$
21. $f(x) = \dfrac{2x^3 - 4x + 1}{4x^3 + 2x - 3}$
22. $f(x) = \dfrac{5x^4 - 2x^3 + x}{x^5 - x^3 + 8}$

Sketch the graph of each function. Indicate where each function is increasing or decreasing, where any relative extrema occur, where asymptotes occur, where the graph is concave up or concave down, where any points of inflection occur, and where any intercepts occur.

23. $f(x) = -\dfrac{5}{x}$
24. $f(x) = \dfrac{4}{x}$
25. $f(x) = \dfrac{1}{x - 5}$
26. $f(x) = \dfrac{-2}{x - 5}$
27. $f(x) = \dfrac{1}{x + 2}$
28. $f(x) = \dfrac{1}{x - 3}$
29. $f(x) = \dfrac{-3}{x - 3}$
30. $f(x) = \dfrac{-2}{x + 5}$
31. $f(x) = \dfrac{3x - 1}{x}$
32. $f(x) = \dfrac{2x + 1}{x}$
33. $f(x) = x + \dfrac{2}{x}$
34. $f(x) = x + \dfrac{9}{x}$
35. $f(x) = \dfrac{-1}{x^2}$
36. $f(x) = \dfrac{2}{x^2}$
37. $f(x) = \dfrac{x}{x + 2}$
38. $f(x) = \dfrac{x}{x - 3}$
39. $f(x) = \dfrac{-1}{x^2 + 2}$
40. $f(x) = \dfrac{1}{x^2 + 3}$
41. $f(x) = \dfrac{x + 3}{x^2 - 9}$ (*Hint:* Simplify.)
42. $f(x) = \dfrac{x - 1}{x^2 - 1}$
43. $f(x) = \dfrac{x - 1}{x + 2}$
44. $f(x) = \dfrac{x - 2}{x + 1}$
45. $f(x) = \dfrac{x^2 - 4}{x + 3}$
46. $f(x) = \dfrac{x^2 - 9}{x + 1}$
47. $f(x) = \dfrac{x + 1}{x^2 - 2x - 3}$
48. $f(x) = \dfrac{x - 3}{x^2 + 2x - 15}$
49. $f(x) = \dfrac{2x^2}{x^2 - 16}$
50. $f(x) = \dfrac{x^2 + x - 2}{2x^2 - 2}$
51. $f(x) = \dfrac{1}{x^2 - 1}$
52. $f(x) = \dfrac{10}{x^2 + 4}$
53. $f(x) = \dfrac{x^2 + 1}{x}$
54. $f(x) = \dfrac{x^3}{x^2 - 1}$
55. $f(x) = \dfrac{x^2 - 9}{x - 3}$
56. $f(x) = \dfrac{x^2 - 16}{x + 4}$

In Exercises 57–62, determine a rational function that meets the given conditions, and sketch its graph.

57. The function f has a vertical asymptote at $x = 2$, a horizontal asymptote at $y = -2$, and $f(0) = 0$.
58. The function f has a vertical asymptote at $x = 0$, a horizontal asymptote at $y = 3$, and $f(1) = 2$.
59. The function g has vertical asymptotes at $x = -1$ and $x = 1$, a horizontal asymptote at $y = 1$, and $g(0) = 2$.
60. The function g has vertical asymptotes at $x = -2$ and $x = 0$, a horizontal asymptote at $y = -3$, and $g(1) = 4$.
61. The function h has vertical asymptotes at $x = -3$ and $x = 2$, a horizontal asymptote at $y = 0$, and $h(1) = 2$.
62. The function h has vertical asymptotes at $x = -\frac{1}{2}$ and $x = \frac{1}{2}$, a horizontal asymptote at $y = 0$, and $h(0) = -3$.

EXAMPLE 10 Determine a rational function f (in lowest terms) whose graph has vertical asymptotes at $x = -5$ and $x = 2$ and a horizontal asymptote at $y = 2$ and for which $f(1) = 3$.

Solution We know that the graph of f has vertical asymptotes at $x = -5$ and $x = 2$, so we can conclude that the denominator must contain the factors $x + 5$ and $x - 2$. Writing these as a product, we see that the denominator will have degree 2 (although we do not carry out the multiplication). Since the graph has a horizontal asymptote, the function must have a polynomial of degree 2 in the numerator, and the leading coefficients must form a ratio of 2. Therefore, a reasonable first guess for f is given by

$$f(x) = \frac{2x^2}{(x + 5)(x - 2)}. \qquad \text{This is a first guess.}$$

However, this does not satisfy the requirement that $f(1) = 3$. We can add a constant in the numerator and then use the fact that $f(1) = 3$ to solve for this constant:

$$f(x) = \frac{2x^2 + B}{(x + 5)(x - 2)}$$

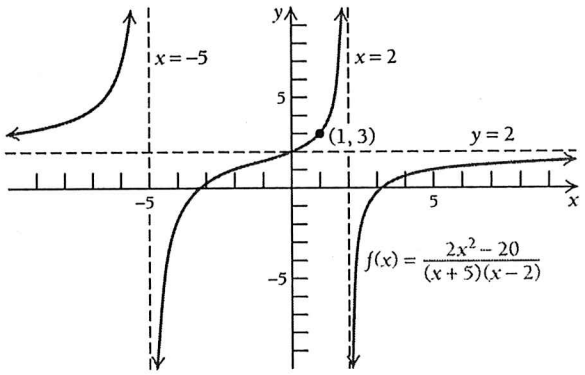
$$3 = \frac{2(1)^2 + B}{(1 + 5)(1 - 2)} \qquad \text{Setting } x = 1 \text{ and } f(x) = 3$$

$$3 = \frac{2 + B}{-6}$$

Multiplying both sides by -6, we have $-18 = 2 + B$. Therefore, $B = -20$. The rational function is given by

$$f(x) = \frac{2x^2 - 20}{(x + 5)(x - 2)}.$$

A sketch of its graph serves as a visual check that all of the initial conditions are met.



Quick Check 7

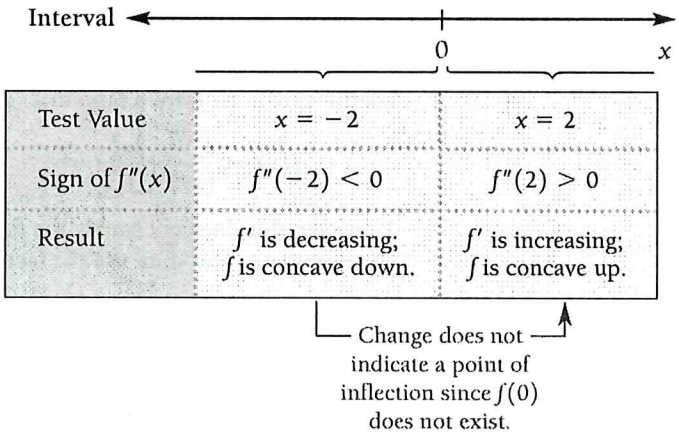
Determine a rational function g that has vertical asymptotes at $x = -2$ and $x = 2$ and a horizontal asymptote at $y = 3$, and for which $g(1) = -3$.

Quick Check 7

Section Summary

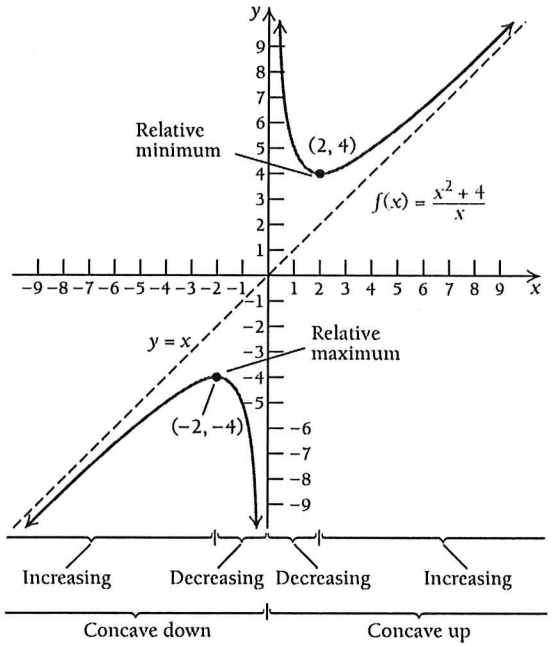
- A line $x = a$ is a *vertical asymptote* if $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.
- A line $y = b$ is a *horizontal asymptote* if $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$.
- A graph may cross a horizontal asymptote but never a vertical asymptote.
- A *slant asymptote* occurs when the degree of the numerator is 1 greater than the degree of the denominator. Long division of polynomials can be used to determine the equation of the slant asymptote.
- Vertical, horizontal, and slant asymptotes can be used as guides for accurate curve sketching. Asymptotes are not a part of a graph but are visual guides only.

g) *Concavity.* Since no values were found in step (f), the only place where concavity could change would be on either side of the vertical asymptote $x = 0$. In step (e), we used the Second-Derivative Test to determine relative extrema. From that work, we know that f is concave down over the interval $(-\infty, 0)$ and concave up over $(0, \infty)$.



x	$f(x)$, approximately
−6	−6.67
−5	−5.8
−4	−5
−3	−4.3
−2	−4
−1	−5
−0.5	−8.5
0.5	8.5
1	5
2	4
3	4.3
4	5
5	5.8
6	6.67

h) *Sketch the graph.* We sketch the graph using the preceding information and additional computed values of f, as needed. The graph follows.



Quick Check 6

Sketch the graph of the function given by

$$f(x) = \frac{x^2 - 9}{x - 1}.$$

Quick Check 6

We can apply our analytic strategy for graphing to "building" a rational function that meets certain initial conditions.

Slant: The degree of the numerator is 1 greater than the degree of the denominator, so there is a slant asymptote. We do the division

$$\begin{array}{r} x \\ x\overline{)x^2+4} \\ \underline{x^2} \\ 4 \end{array}$$

and express the function in the form

$$f(x) = x + \frac{4}{x}.$$

As $|x|$ gets larger, $4/x$ approaches 0, so the line $y = x$ is a slant asymptote.

c) *Derivatives and domain.* We find $f'(x)$ and $f''(x)$:

$$f'(x) = 1 - 4x^{-2} = 1 - \frac{4}{x^2};$$

$$f''(x) = 8x^{-3} = \frac{8}{x^3}.$$

The domain of f is $(-\infty, 0) \cup (0, \infty)$, or all real numbers except 0.

d) *Critical values of f.* We see from step (c) that $f'(x)$ is undefined at $x = 0$, but 0 is not in the domain of f. Thus, to find critical values, we solve $f'(x) = 0$, looking for solutions other than 0:

$$1 - \frac{4}{x^2} = 0 \qquad \text{Setting } f'(x) \text{ equal to } 0$$

$$1 = \frac{4}{x^2}$$

$$x^2 = 4 \qquad \text{Multiplying both sides by } x^2$$

$$x = \pm 2.$$

Thus, -2 and 2 are critical values.

e) *Increasing and/or decreasing; relative extrema.* We use the points found in step (d) to find intervals over which f is increasing and intervals over which f is decreasing. The points to consider are -2, 0, and 2.

Since

$$f''(-2) = \frac{8}{(-2)^3} = -1 < 0,$$

we know that a relative maximum exists at $(-2, f(-2))$, or $(-2, -4)$. Thus, f is increasing on $(-\infty, -2)$ and decreasing on $(-2, 0)$.

Since

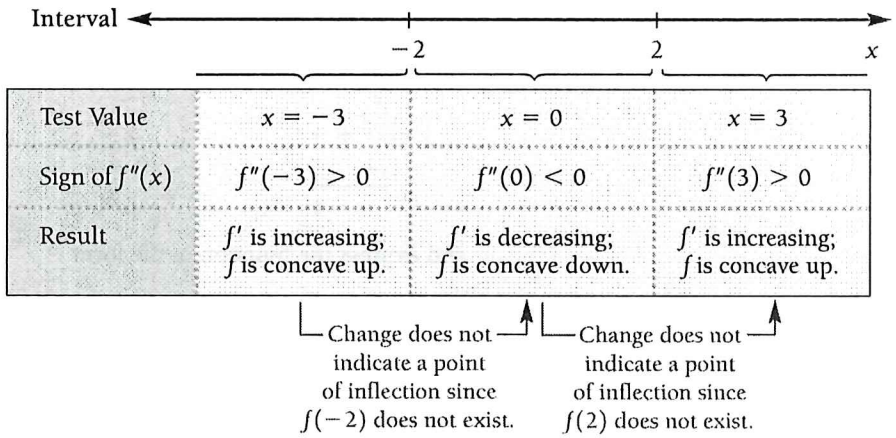
$$f''(2) = \frac{8}{(2)^3} = 1 > 0,$$

we know that a relative minimum exists at $(2, f(2))$, or $(2, 4)$. Thus, f is decreasing on $(0, 2)$ and increasing on $(2, \infty)$.

f) *Inflection points.* We determine candidates for inflection points by finding where $f''(x)$ does not exist or where $f''(x) = 0$. The only value for which $f''(x)$ does not exist is 0, but 0 is not in the domain of f. Thus, the only place an inflection point could occur is where $f''(x) = 0$:

$$\frac{8}{x^3} = 0.$$

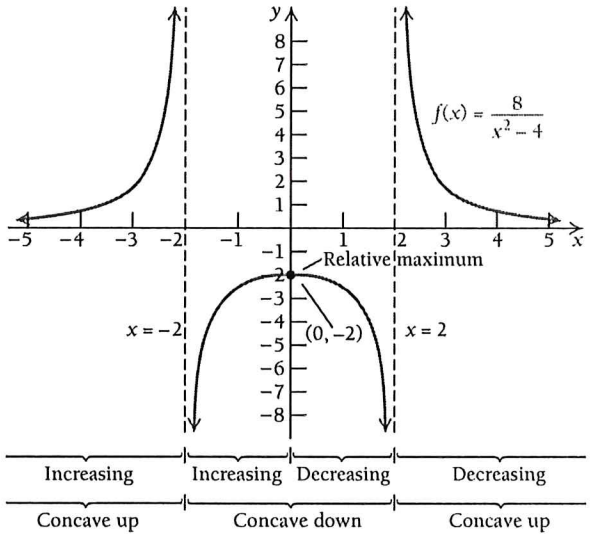
But this equation has no solution. Thus, there are no points of inflection.



The function is concave up over the intervals $(-\infty, -2)$ and $(2, \infty)$. The function is concave down over the interval $(-2, 2)$.

h) *Sketch the graph.* We sketch the graph using the information in the following table, plotting extra points as needed. The graph is shown below.

x	$f(x)$ approximately
−5	0.38
−4	0.67
−3	1.6
−1	−2.67
0	−2
1	−2.67
3	1.6
4	0.67
5	0.38



EXAMPLE 9

Sketch the graph of the function given by $f(x) = \dfrac{x^2 + 4}{x}$.

Solution

a) *Intercepts.* The equation $f(x) = 0$ has no real-number solution. Thus, there are no x-intercepts. The number 0 is not in the domain of the function. Thus, there is no y-intercept.

b) *Asymptotes.*

Vertical: Since replacing x with 0 makes the denominator 0, the line $x = 0$ is a vertical asymptote.

Horizontal: The degree of the numerator is greater than the degree of the denominator, so there is no horizontal asymptote.

Horizontal: The degree of the numerator is less than the degree of the denominator, so the x-axis, $y = 0$, is the horizontal asymptote.

Slant: There is no slant asymptote since the degree of the numerator is not 1 more than the degree of the denominator.

c) *Derivatives and domain.* We find $f'(x)$ and $f''(x)$ using the Quotient Rule:

$$f'(x) = \frac{-16x}{(x^2 - 4)^2} \quad \text{and} \quad f''(x) = \frac{16(3x^2 + 4)}{(x^2 - 4)^3}.$$

The domain of f is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$ as determined in step (b).

d) *Critical values of f.* We look for values of x for which $f'(x) = 0$ or for which $f'(x)$ does not exist. From step (c), we see that $f'(x) = 0$ for values of x for which $-16x = 0$, but the denominator is not 0. The only such number is 0 itself. The derivative $f'(x)$ does not exist at -2 and 2, but neither value is in the domain of f. Thus, the only critical value is 0.

e) *Increasing and/or decreasing; relative extrema.* We use the undefined values and the critical values to determine the intervals over which f is increasing and the intervals over which f is decreasing. The values to consider are $-2, 0$, and 2.

Since

$$f''(0) = \frac{16(3 \cdot 0^2 + 4)}{(0^2 - 4)^3} = \frac{64}{-64} < 0,$$

we know that a relative maximum exists at $(0, f(0))$, or $(0, -2)$. Thus, f is increasing on the interval $(-2, 0)$ and decreasing on $(0, 2)$.

Since $f''(x)$ does not exist for the x-values -2 and 2, we use $f'(x)$ and test values to see if f is increasing or decreasing on $(-\infty, -2)$ and $(2, \infty)$:

$$\text{Test } -3, \quad f'(-3) = \frac{-16(-3)}{[(-3)^2 - 4]^2} = \frac{48}{25} > 0, \text{ so } f \text{ is increasing on } (-\infty, -2);$$

$$\text{Test } 3, \quad f'(3) = \frac{-16(3)}{[(3)^2 - 4]^2} = \frac{-48}{25} < 0, \text{ so } f \text{ is decreasing on } (2, \infty).$$

f) *Inflection points.* We determine candidates for inflection points by finding where $f''(x)$ does not exist and where $f''(x) = 0$. The only values for which $f''(x)$ does not exist are where $x^2 - 4 = 0$, or -2 and 2. Neither value is in the domain of f, so we focus solely on where $f''(x) = 0$, or

$$16(3x^2 + 4) = 0.$$

Since $16(3x^2 + 4) > 0$ for all real numbers x, there are no points of inflection.

g) *Concavity.* Since no values were found in step (f), the only place where concavity could change is on either side of the vertical asymptotes, $x = -2$ and $x = 2$. To determine the concavity, we check to see where $f''(x)$ is positive or negative. The numbers -2 and 2 divide the x-axis into three intervals. We choose test values in each interval and make a substitution into f'':

$$\text{Test } -3, \quad f''(-3) = \frac{16[3(-3)^2 + 4]}{[(-3)^2 - 4]^3} > 0;$$

$$\text{Test } 0, \quad f''(0) = \frac{16[3(0)^2 + 4]}{[(0)^2 - 4]^3} < 0; \quad \text{We already knew this from step (e).}$$

$$\text{Test } 3, \quad f''(3) = \frac{16[3(3)^2 + 4]}{[(3)^2 - 4]^3} > 0.$$

Quick Check 5

Find the intercepts of the function given by

$$h(x) = \frac{x^3 - x}{x^2 - 4}.$$

To find the y-intercept, we let $x = 0$:

$$f(0) = \frac{0^3 - 0^2 - 6(0)}{0^2 - 3(0) + 2} = 0.$$

In this case, the y-intercept is also an x-intercept, $(0, 0)$.

Quick Check 5

Sketching Graphs

We can now refine our analytic strategy for graphing.

Strategy for Sketching Graphs

a) *Intercepts.* Find the x-intercept(s) and the y-intercept of the graph.
b) *Asymptotes.* Find any vertical, horizontal, or slant asymptotes.
c) *Derivatives and domain.* Find $f'(x)$ and $f''(x)$. Find the domain of f.
d) *Critical values of f.* Find any inputs for which $f'(x)$ is not defined or for which $f'(x) = 0$.
e) *Increasing and/or decreasing; relative extrema.* Substitute each critical value, x_0, from step (d) into $f''(x)$. If $f''(x_0) < 0$, then x_0 yields a relative maximum and f is increasing to the left of x_0 and decreasing to the right. If $f''(x_0) > 0$, then x_0 yields a relative minimum and f is decreasing to the left of x_0 and increasing to the right. On intervals where no critical value exists, use f' and test values to find where f is increasing or decreasing.
f) *Inflection points.* Determine candidates for inflection points by finding x-values for which $f''(x)$ does not exist or for which $f''(x) = 0$. Find the function values at these points. If a function value $f(x)$ does not exist, then the function does not have an inflection point at x.
g) *Concavity.* Use the values from step (f) as endpoints of intervals. Determine the concavity over each interval by checking to see where f' is increasing—that is, where $f''(x) > 0$—and where f' is decreasing—that is, where $f''(x) < 0$. Do this by substituting a test value from each interval into $f''(x)$. Use the results of step (d).
h) *Sketch the graph.* Use the information from steps (a)–(g) to sketch the graph, plotting extra points as needed.

EXAMPLE 8 Sketch the graph of $f(x) = \dfrac{8}{x^2 - 4}$.

Solution

a) *Intercepts.* The x-intercepts occur at values for which the numerator is 0 but the denominator is not. Since in this case the numerator is the constant 8, there are no x-intercepts. To find the y-intercept, we compute $f(0)$:

$$f(0) = \frac{8}{0^2 - 4} = \frac{8}{-4} = -2.$$

This gives us one point on the graph, $(0, -2)$.

b) *Asymptotes.*

Vertical: The denominator, $x^2 - 4 = (x + 2)(x - 2)$, is 0 for x-values of -2 and 2. Thus, the graph has the lines $x = -2$ and $x = 2$ as vertical asymptotes. We draw them using dashed lines (they are *not* part of the actual graph, just guidelines).

DEFINITION

A linear asymptote that is neither vertical nor horizontal is called a **slant asymptote**, or an **oblique asymptote**. For any rational function of the form $f(x) = p(x)/q(x)$, a slant asymptote occurs when the degree of $p(x)$ is exactly 1 more than the degree of $q(x)$. A graph can cross a slant asymptote.

How can we find a slant asymptote? One way is by division.

EXAMPLE 6 Find the slant asymptote:

$$f(x) = \frac{x^2 - 4}{x - 1}.$$

Solution When we divide the numerator by the denominator, we obtain a quotient of $x + 1$ and a remainder of -3:

$$\begin{array}{r} x + 1 \\ x - 1\overline{)x^2 \quad - 4} \\ \underline{x^2 - x} \\ \quad x - 4 \\ \quad \underline{x - 1} \\ \quad\quad -3 \end{array} \qquad f(x) = \frac{x^2 - 4}{x - 1} = (x + 1) + \frac{-3}{x - 1}.$$

Now we can see that when $|x|$ gets very large, $-3/(x - 1)$ approaches 0. Thus, for very large $|x|$, the expression $x + 1$ is the dominant part of

$$(x + 1) + \frac{-3}{x - 1}.$$

Thus, $y = x + 1$ is a slant asymptote.

〈 Quick Check 4

TECHNOLOGY CONNECTION

EXERCISES

Graph each of the following. Try to visually locate the slant asymptotes. Then use the method in Example 6 to find each slant asymptote and graph it along with the original function.

1. $f(x) = \dfrac{3x^2 - 7x + 8}{x - 2}$

2. $f(x) = \dfrac{5x^3 + 2x + 1}{x^2 - 4}$

› Quick Check 4

Find the slant asymptote:

$$g(x) = \frac{2x^2 + x - 1}{x - 3}.$$

Intercepts

If they exist, the ***x*-intercepts** of a function occur at those values of x for which $y = f(x) = 0$, and they give us points at which the graph crosses the x-axis. If it exists, the ***y*-intercept** of a function occurs at the value of y for which $x = 0$, and it gives us the point at which the graph crosses the y-axis.

EXAMPLE 7 Find the intercepts of the function given by

$$f(x) = \frac{x^3 - x^2 - 6x}{x^2 - 3x + 2}.$$

Solution We factor the numerator and the denominator:

$$f(x) = \frac{x(x + 2)(x - 3)}{(x - 1)(x - 2)}.$$

To find the x-intercepts, we solve the equation $f(x) = 0$. Such values occur when the numerator is 0 and the denominator is not. Thus, we solve the equation

$$x(x + 2)(x - 3) = 0.$$

The x-values that make the numerator 0 are 0, -2, and 3. Since none of these make the denominator 0, they yield the x-intercepts $(0, 0)$, $(-2, 0)$, and $(3, 0)$.

TECHNOLOGY CONNECTION

EXERCISES

Graph each of the following. Use the ZERO feature and a table in ASK mode to find the x- and y-intercepts.

1. $f(x) = \dfrac{x(x - 3)(x + 5)}{(x + 2)(x - 4)}$

2. $f(x) = \dfrac{x^3 + 2x^2 - 3x}{x^2 + 5}$

TECHNOLOGY CONNECTION

To see why a horizontal asymptote is determined by the leading terms in the expression's numerator and denominator, consider the terms $3x^2$, $2x$, and -4 in Example 4. If $y_1 = 3x^2$ and $y_2 = 3x^2 + 2x - 4$, a table reveals that as x gets larger, the difference between y_2 and y_1 becomes less significant. This is because $3x^2$ grows far more rapidly than does $2x - 4$.

X	Y1	Y2
20	1200	1236
120	43200	43436
220	145200	145636
320	307200	307836
420	529200	530036
520	811200	812236
620	1.15 E6	1.15 E6
X=20		

EXERCISES

1. Let $y_1 = 2x^2$ and $y_2 = 2x^2 - x + 1$. Use a table to show that as x gets large, $y_1 \approx y_2$.
2. Let $y_1 = (3x^2)/(2x^2)$ and $y_2 = (3x^2 + 2x - 1)/(2x^2 - x + 1)$. Show that for large x, we have $y_1 \approx y_2$.

TECHNOLOGY CONNECTION

EXERCISES

Graph each of the following. Try to locate the horizontal asymptotes using the TABLE and TRACE features. Verify your results using the methods of Examples 3–5.

1. $f(x) = \dfrac{x^2 + 5}{x^3 - x^2 - 6x}$
2. $f(x) = \dfrac{9x^4 - 7x^2 - 9}{3x^4 + 7x^2 + 9}$
3. $f(x) = \dfrac{135x^5 - x^2}{x^7}$
4. $f(x) = \dfrac{3x^2 - 4x + 3}{6x^2 + 2x - 5}$

Examples 3 and 4 lead to the following result.

When the degree of the numerator is the same as the degree of the denominator, the line $y = a/b$ is a horizontal asymptote, where a is the leading coefficient of the numerator and b is the leading coefficient of the denominator.

EXAMPLE 5 Determine the horizontal asymptote:

$$f(x) = \frac{2x + 3}{x^3 - 2x^2 + 4}.$$

Solution Since the degree of the numerator is less than the degree of the denominator, there is a horizontal asymptote. To identify that asymptote, we divide both the numerator and denominator by the highest power of x in the denominator, just as in Examples 3 and 4, and find the limits as $|x| \to \infty$:

$$f(x) = \frac{2x + 3}{x^3 - 2x^2 + 4} = \frac{\dfrac{2}{x^2} + \dfrac{3}{x^3}}{1 - \dfrac{2}{x} + \dfrac{4}{x^3}}.$$

As x gets smaller and smaller negatively, $|x|$ gets larger and larger. Similarly, as x gets larger and larger positively, $|x|$ gets larger and larger. Thus, as $|x|$ becomes very large, every expression with a denominator that is a power of x gets ever closer to 0. Thus, the numerator of $f(x)$ approaches 0 as its denominator approaches 1; hence, the entire expression takes on values ever closer to 0. That is, for $x \to -\infty$ or $x \to \infty$, we have

$$f(x) \approx \frac{0 + 0}{1 - 0 + 0},$$

so $\lim_{x \to -\infty} f(x) = 0$ and $\lim_{x \to \infty} f(x) = 0$,

and the x-axis, the line $y = 0$, is a horizontal asymptote.

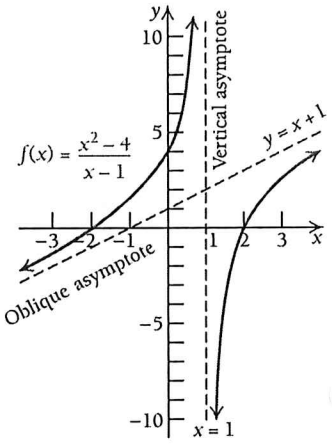
When the degree of the numerator is less than the degree of the denominator, the x-axis, or the line $y = 0$, is a horizontal asymptote.

Slant Asymptotes

Some asymptotes are neither vertical nor horizontal. For example, in the graph of

$$f(x) = \frac{x^2 - 4}{x - 1},$$

shown at right, as $|x|$ gets larger and larger, the curve gets closer and closer to $y = x + 1$. The line $y = x + 1$ is called a *slant asymptote*, or *oblique asymptote*. In Example 6, we will see how the line $y = x + 1$ was determined.



TECHNOLOGY CONNECTION

EXERCISES

1. Verify the limit

$$\lim_{x\to\infty}\frac{3x-4}{x}=3$$

by using the TABLE feature with larger and larger x-values.

X	Y1
50	2.92
150	2.9733
250	2.984
350	2.9886
450	2.9911
550	2.9927
650	2.9938

X = 50

X	Y1
500	2.992
1500	2.9973
2500	2.9984
3500	2.9989
4500	2.9991
5500	2.9993
6500	2.9994

X = 500

2. Graph the function

$$f(x)=\frac{3x-4}{x}$$

in DOT mode. Then use TRACE, moving the cursor along the graph from left to right, and observe the behavior of the y-coordinates.

For Exercises 3 and 4, consider $\lim_{x\to\infty}\frac{2x+5}{x}$.

3. Use the TABLE feature to find the limit.

4. Graph the function in DOT mode, and use TRACE to find the limit.

As the inputs get larger and larger without bound, the outputs get closer and closer to 3. Thus,

$$\lim_{x\to\infty}\frac{3x-4}{x}=3.$$

Another way to find this limit is to use some algebra and the fact that

$$\text{as } x\to\infty, \text{ we have } \frac{1}{x}\to 0, \text{ and more generally, } \frac{b}{ax^n}\to 0,$$

for any positive integer n and any constants a and b, $a\neq 0$. We multiply by 1, using $(1/x)\div(1/x)$. This amounts to dividing both the numerator and the denominator by x:

$$\begin{aligned}\lim_{x\to\infty}\frac{3x-4}{x}&=\lim_{x\to\infty}\frac{3x-4}{x}\cdot\frac{(1/x)}{(1/x)}\\&=\lim_{x\to\infty}\frac{\frac{3x}{x}-\frac{4}{x}}{\frac{x}{x}}\\&=\lim_{x\to\infty}\frac{3-\frac{4}{x}}{1}\\&=\lim_{x\to\infty}\left(3-\frac{4}{x}\right)\\&=3-0=3.\end{aligned}$$

In a similar manner, it can be shown that

$$\lim_{x\to-\infty}f(x)=3.$$

The horizontal asymptote is the line $y=3$.

EXAMPLE 4 Determine the horizontal asymptote of the function given by

$$f(x)=\frac{3x^2+2x-4}{2x^2-x+1}.$$

Solution As in Example 3, the degree of the numerator is the same as the degree of the denominator. Let's adapt the algebraic approach used in that example.

To do so, we divide the numerator and the denominator by x^2 and find the limit as $|x|$ gets larger and larger:

$$f(x)=\frac{3x^2+2x-4}{2x^2-x+1}=\frac{3+\frac{2}{x}-\frac{4}{x^2}}{2-\frac{1}{x}+\frac{1}{x^2}}.$$

As $|x|$ gets very large, the numerator approaches 3 and the denominator approaches 2. Therefore, the value of the function gets very close to $\frac{3}{2}$. Thus,

$$\lim_{x\to-\infty}f(x)=\frac{3}{2}\quad\text{and}\quad\lim_{x\to\infty}f(x)=\frac{3}{2}.$$

The line $y=\frac{3}{2}$ is a horizontal asymptote.

Quick Check 3

Determine the horizontal asymptote of the function given by

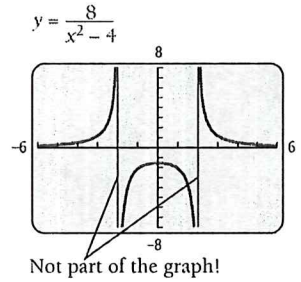
$$f(x)=\frac{(2x-1)(x+1)}{(3x+2)(5x+6)}.$$

Quick Check 3

TECHNOLOGY CONNECTION

Asymptotes

Our discussion here allows us to attach the term "vertical asymptote" to those mysterious vertical lines that appear when graphing rational functions in the CONNECTED mode. For example, consider the graph of $f(x) = 8/(x^2 - 4)$, using the window $[-6, 6, -8, 8]$. Vertical asymptotes occur at $x = -2$ and $x = 2$. These lines are not part of the graph.



EXERCISES

Graph each of the following in both DOT and CONNECTED modes. Try to locate the vertical asymptotes visually. Then verify your results using the method of Examples 1 and 2. You may need to try different viewing windows.

1. $f(x) = \dfrac{x^2 + 7x + 10}{x^2 + 3x - 28}$ **2.** $f(x) = \dfrac{x^2 + 5}{x^3 - x^2 - 6x}$

Look again at the graph in Fig. 1. Note that function values get closer and closer to 1 as x approaches $-\infty$, meaning that $f(x) \to 1$ as $x \to -\infty$. Also, function values get closer and closer to 1 as x approaches ∞, meaning that $f(x) \to 1$ as $x \to \infty$. Thus,

$$\lim_{x \to -\infty} f(x) = 1 \quad \text{and} \quad \lim_{x \to \infty} f(x) = 1.$$

The line $y = 1$ is called a *horizontal asymptote.*

DEFINITION

The line $y = b$ is a **horizontal asymptote** if either or both of the following limit statements is true:

$$\lim_{x \to -\infty} f(x) = b \quad \text{or} \quad \lim_{x \to \infty} f(x) = b.$$

The graph of a rational function may or may not cross a horizontal asymptote. Horizontal asymptotes occur when the degree of the numerator is less than or equal to the degree of the denominator. (The degree of a polynomial in one variable is the highest power of that variable.)

In Figs. 3–5, we see three ways in which horizontal asymptotes can occur.

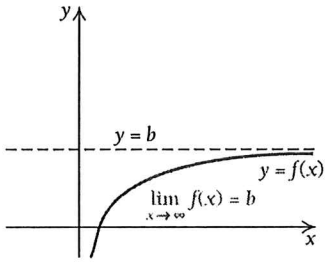


FIGURE 3

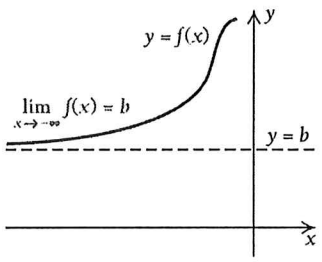


FIGURE 4

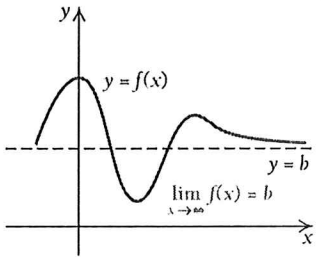


FIGURE 5

Horizontal asymptotes are found by determining the limit of a rational function as inputs approach $-\infty$ or ∞.

EXAMPLE 3 Determine the horizontal asymptote of the function given by

$$f(x) = \frac{3x - 4}{x}.$$

Solution To find the horizontal asymptote, we consider

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{3x - 4}{x}.$$

One way to find such a limit is to use an input–output table, as follows, using progressively larger x-values.

Inputs, x	1	10	50	100	2000
Outputs, $\dfrac{3x - 4}{x}$	−1	2.6	2.92	2.96	2.998

For example,

$$f(x) = \frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3}$$

does not have a vertical asymptote at $x = 3$, even though 3 is an input that makes the denominator 0. This is because when $(x^2 - 9)/(x - 3)$ is simplified, it has $x - 3$ as a common factor of the numerator and the denominator. In contrast,

$$g(x) = \frac{x^2 - 4}{x^2 + x - 12} = \frac{(x + 2)(x - 2)}{(x - 3)(x + 4)}$$

is simplified and has $x = 3$ and $x = -4$ as vertical asymptotes.

Figure 2 shows the four ways in which a vertical asymptote can occur. The dashed lines represent the asymptotes. They are sketched in for visual assistance only; they are not part of the graphs of the functions.

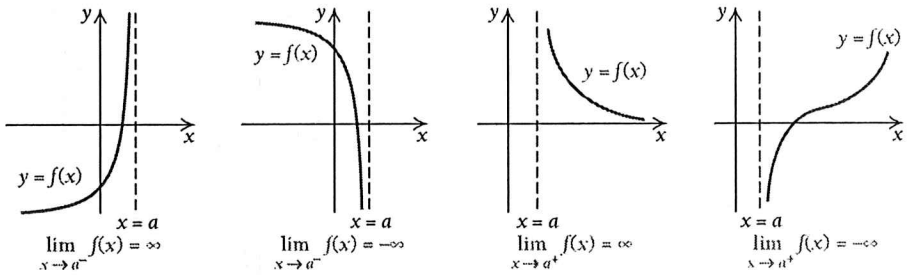


FIGURE 2

EXAMPLE 1 Determine the vertical asymptotes: $f(x) = \dfrac{3x - 2}{x(x - 5)(x + 3)}$.

Solution The expression is in simplified form. The vertical asymptotes are the lines $x = 0, x = 5$, and $x = -3$.

〈 Quick Check 1

› **Quick Check 1**

Determine the vertical asymptotes:

$$f(x) = \frac{1}{x(x^2 - 16)}.$$

EXAMPLE 2 Determine the vertical asymptotes of the function given by

$$f(x) = \frac{x^2 - 2x}{x^3 - x}.$$

Solution We write the expression in simplified form:

$$f(x) = \frac{x^2 - 2x}{x^3 - x} = \frac{x(x - 2)}{x(x - 1)(x + 1)}$$

$$= \frac{x - 2}{(x - 1)(x + 1)}, \quad x \neq 0.$$

The expression is now in simplified form. The vertical asymptotes are the lines $x = -1$ and $x = 1$.

〈 Quick Check 2

› **Quick Check 2**

For the function in Example 2, explain why $x = 0$ does not correspond to a vertical asymptote. What kind of discontinuity occurs at $x = 0$?

Polynomials are themselves a special kind of rational function, since $Q(x)$ can be 1. Here we are considering graphs of rational functions in which the denominator is not a constant. Before we do so, however, we need to reconsider limits.

Vertical and Horizontal Asymptotes

Figure 1 shows the graph of the rational function

$$f(x) = \frac{x^2 - 1}{x^2 + x - 6} = \frac{(x - 1)(x + 1)}{(x - 2)(x + 3)}.$$

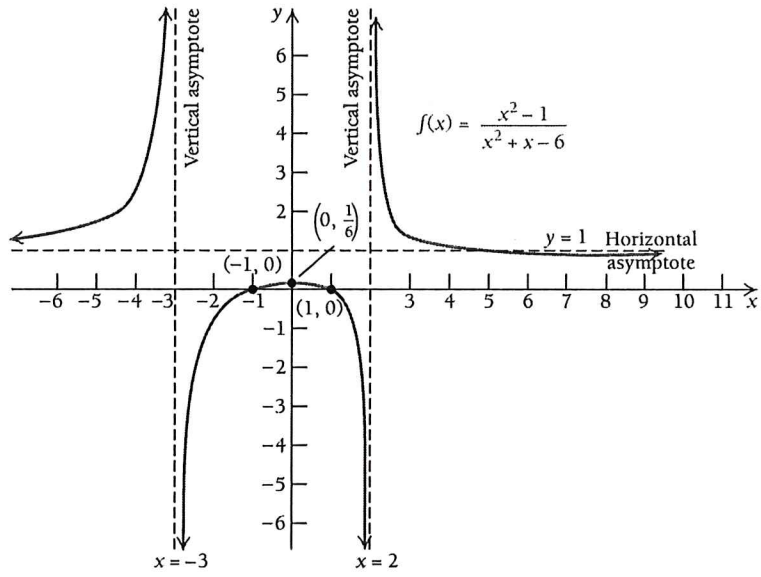


FIGURE 1

Note that as x gets closer to 2 from the left, the function values get smaller and smaller negatively, approaching $-\infty$. As x gets closer to 2 from the right, the function values get larger and larger positively. Thus,

$$\lim_{x \to 2^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to 2^+} f(x) = \infty.$$

For this graph, we can think of the line $x = 2$ as a "limiting line" called a *vertical asymptote*. Similarly, the line $x = -3$ is another vertical asymptote.

DEFINITION

The line $x = a$ is a **vertical asymptote** if any of the following limit statements is true:

$$\lim_{x \to a^-} f(x) = \infty, \quad \lim_{x \to a^-} f(x) = -\infty, \quad \lim_{x \to a^+} f(x) = \infty, \quad \text{or} \quad \lim_{x \to a^+} f(x) = -\infty.$$

The graph of a rational function *never* crosses a vertical asymptote. If the expression that defines the rational function f is simplified, meaning that it has no common factor other than -1 or 1, then if a is an input that makes the denominator 0, the line $x = a$ is a vertical asymptote.

TECHNOLOGY CONNECTION

Graph each function. Then estimate any relative extrema. Where appropriate, round to three decimal places.

122. $f(x) = 3x^{2/3} - 2x$ **123.** $f(x) = 4x - 6x^{2/3}$

124. $f(x) = x^2(x - 2)^3$ **125.** $f(x) = x^2(1 - x)^3$

126. $f(x) = x - \sqrt{x}$

127. $f(x) = (x - 1)^{2/3} - (x + 1)^{2/3}$

128. Social sciences: time spent on home computer. The following data relate the average number of minutes spent per month on a home computer to a person's age.

Age (in years)	Average Use (in minutes per month)
6.5	363
14.5	645
21	1377
29.5	1727
39.5	1696
49.5	2052
55 and up	2299

(Source: *Media Matrix; The PC Meter Company.*)

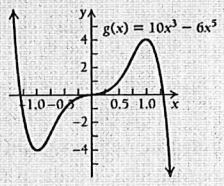
a) Use the regression procedures of Section R.6 to fit linear, cubic, and quartic functions $y = f(x)$ to the data, where x is age and y is average use per month. Decide which function best fits the data. Explain.

b) What is the domain of the function?

c) Does the function have any relative extrema? Explain.

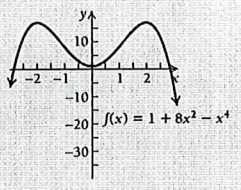
Answers to Quick Checks

1. Relative minimum: -4 at $x = -1$; relative maximum: 4 at $x = 1$



2. $(0, 0)$, $\left(\frac{\sqrt{2}}{2}, 2.475\right)$, and $\left(-\frac{\sqrt{2}}{2}, -2.475\right)$

3. Relative maxima: 17 at $x = -2$ and $x = 2$; relative minimum: 1 at $x = 0$



2.3 Graph Sketching: Asymptotes and Rational Functions

OBJECTIVES

- Find limits involving infinity.
- Determine the asymptotes of a function's graph.
- Graph rational functions.

Thus far we have considered a strategy for graphing a continuous function using the tools of calculus. We now want to consider some discontinuous functions, most of which are rational functions. Our graphing skills must now allow for discontinuities as well as certain lines called *asymptotes*.

Let's review the definition of a rational function.

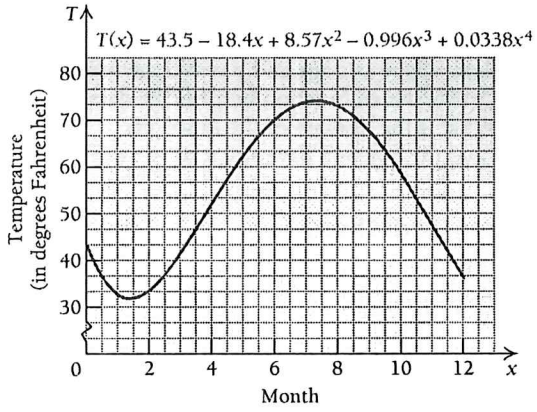
Rational Functions

DEFINITION

A **rational function** is a function f that can be described by

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials, with $Q(x)$ not the zero polynomial. The domain of f consists of all inputs x for which $Q(x) \neq 0$.

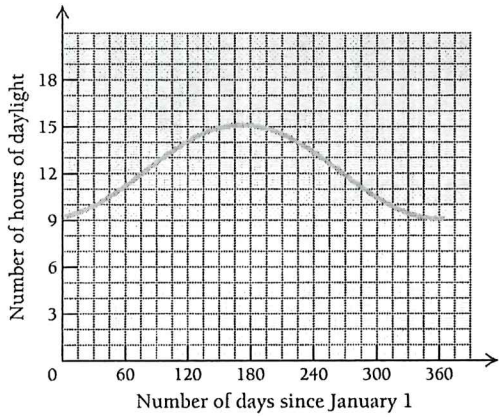


a) Based on the graph, when would you expect the highest temperature to occur in New York?

b) Based on the graph, when would you expect the lowest temperature to occur?

c) Use the Second-Derivative Test to estimate the points of inflection for the function $T(x)$. What is the significance of these points?

109. Hours of daylight. The number of hours of daylight in Chicago is represented in the graph below. On what dates is the number of hours of daylight changing most rapidly? How can you tell?

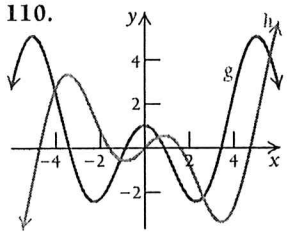


(*Source*: Astronomical Applications Dept., U.S. Naval Observatory.)

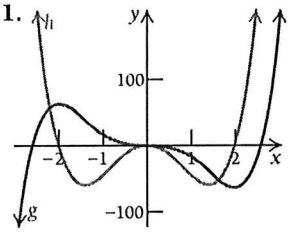
SYNTHESIS

In each of Exercises 110 and 111, determine which graph is the derivative of the other and explain why.

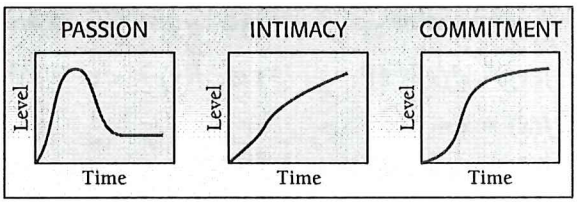
110.



111.



112. Social sciences: three aspects of love. Researchers at Yale University have suggested that the following graphs may represent three different aspects of love.



(*Source*: From "A Triangular Theory of Love," by R. J. Sternberg, 1986, *Psychological Review*, 93(2), 119–135. Copyright 1986 by the American Psychological Association, Inc. Reprinted by permission.)

Analyze each of these graphs in terms of the concepts you have learned: relative extrema, concavity, increasing, decreasing, and so on. Do you agree with the researchers regarding the shapes of these graphs? Why or why not?

113. Use calculus to prove that the relative minimum or maximum for any function f for which

$f(x) = ax^2 + bx + c, a \neq 0,$

occurs at $x = -b/(2a)$.

114. Use calculus to prove that the point of inflection for any function g given by

$g(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0,$

occurs at $x = -b/(3a)$.

For Exercises 115–121, assume the function f is differentiable over the interval $(-\infty, \infty)$; that is, it is smooth and continuous for all real numbers x and has no corners or vertical tangents. Classify each of the following statements as either true or false. If you choose false, explain why.

115. If f has exactly two critical values at $x = a$ and $x = b$, where $a < b$, then there must exist exactly one point of inflection at $x = c$ such that $a < c < b$. In other words, exactly one point of inflection must exist between any two critical points.

116. If f has exactly two critical values at $x = a$ and $x = b$, where $a < b$, then there must exist at least one point of inflection at $x = c$ such that $a < c < b$. In other words, at least one point of inflection must exist between any two critical points.

117. The function f can have no extrema but can have at least one point of inflection.

118. If the function f has two points of inflection, then there must be a critical value located between those points of inflection.

119. The function f can have a point of inflection at a critical value.

120. The function f can have a point of inflection at an extreme value.

121. The function f can have exactly one extreme value but no points of inflection.

30. $f(x) = x^3 + 3x + 1$

31. $f(x) = 5x^3 - 3x^5$

32. $f(x) = 20x^3 - 3x^5$
(Round results to three decimal places.)

33. $f(x) = x^2(3 - x)^2$
(Round results to three decimal places.)

34. $f(x) = x^2(1 - x)^2$
(Round results to three decimal places.)

35. $f(x) = (x + 1)^{2/3}$

36. $f(x) = (x - 1)^{2/3}$

37. $f(x) = (x - 3)^{1/3} - 1$

38. $f(x) = (x - 2)^{1/3} + 3$

39. $f(x) = -2(x - 4)^{2/3} + 5$

40. $f(x) = -3(x - 2)^{2/3} + 3$

41. $f(x) = x\sqrt{4 - x^2}$

42. $f(x) = -x\sqrt{1 - x^2}$

43. $f(x) = \dfrac{x}{x^2 + 1}$

44. $f(x) = \dfrac{8x}{x^2 + 1}$

45. $f(x) = \dfrac{3}{x^2 + 1}$

46. $f(x) = \dfrac{-4}{x^2 + 1}$

For Exercises 47–56, sketch a graph that possesses the characteristics listed. Answers may vary.

47. f is increasing and concave up on $(-\infty, 4)$,
f is increasing and concave down on $(4, \infty)$.

48. f is decreasing and concave up on $(-\infty, 2)$,
f is decreasing and concave down on$(2, \infty)$.

49. f is increasing and concave down on $(-\infty, 1)$,
f is increasing and concave up on $(1, \infty)$.

50. f is decreasing and concave down on $(-\infty, 3)$,
f is decreasing and concave up on $(3, \infty)$.

51. f is concave down at $(1, 5)$, concave up at $(7, -2)$, and has an inflection point at $(4, 1)$.

52. f is concave up at $(1, -3)$, concave down at $(8, 7)$, and has an inflection point at $(5, 4)$.

53. $f'(-1) = 0, f''(-1) > 0, f(-1) = -5$;
$f'(7) = 0, f''(7) < 0, f(7) = 10; f''(3) = 0$, and
$f(3) = 2$

54. $f'(-3) = 0, f''(-3) < 0, f(-3) = 8; f'(9) = 0$,
$f''(9) > 0, f(9) = -6; f''(2) = 0$, and $f(2) = 1$

55. $f'(-1) = 0, f''(-1) > 0, f(-1) = -2; f'(1) = 0$,
$f''(1) > 0, f(1) = -2; f'(0) = 0, f''(0) < 0$,
and $f(0) = 0$

56. $f'(0) = 0, f''(0) < 0, f(0) = 5; f'(2) = 0, f''(2) > 0$,
$f(2) = 2; f'(4) = 0, f''(4) < 0$, and $f(4) = 3$

57–102. Check the results of Exercises 1–46 with a graphing calculator.

APPLICATIONS

Business and Economics

Total revenue, cost, and profit. *Using the same set of axes, sketch the graphs of the total-revenue, total-cost, and total-profit functions.*

103. $R(x) = 50x - 0.5x^2, \quad C(x) = 4x + 10$

104. $R(x) = 50x - 0.5x^2, \quad C(x) = 10x + 3$

105. Small business. The percentage of the U.S. national income generated by nonfarm proprietors may be modeled by the function

$$p(x) = \frac{13x^3 - 240x^2 - 2460x + 585{,}000}{75{,}000},$$

where x is the number of years since 1970. Sketch the graph of this function for $0 \le x \le 40$.

106. Labor force. The percentage of the U.S. civilian labor force aged 45–54 may be modeled by the function

$f(x) = 0.025x^2 - 0.71x + 20.44,$

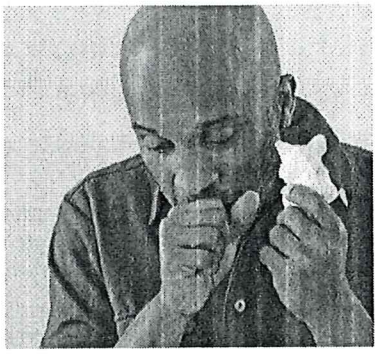
where x is the number of years after 1970. Sketch the graph of this function for $0 \le x \le 30$.

Life and Physical Sciences

107. Coughing velocity. A person coughs when a foreign object is in the windpipe. The velocity of the cough depends on the size of the object. Suppose a person has a windpipe with a 20-mm radius. If a foreign object has a radius r, in millimeters, then the velocity V, in millimeters per second, needed to remove the object by a cough is given by

$V(r) = k(20r^2 - r^3), \quad 0 \le r \le 20,$

where k is some positive constant. For what size object is the maximum velocity required to remove the object?



108. New York temperatures. The average temperature in New York can be approximated by the function

$T(x) = 43.5 - 18.4x + 8.57x^2 - 0.996x^3 + 0.0338x^4,$

where T represents the temperature, in degrees Fahrenheit, $x = 1$ represents the middle of January, $x = 2$ represents the middle of February, and so on. (*Source*: www.WorldClimate.com.)

TECHNOLOGY CONNECTION

EXERCISES

Graph the following:

$$f(x) = 3x^5 - 5x^3, \quad f'(x) = 15x^4 - 15x^2,$$

and

$$f''(x) = 60x^3 - 30x,$$

using the window $[-3, 3, -10, 10]$.

1. From the graph of f', estimate the critical values of f.
2. From the graph of f'', estimate the x-values of any inflection points of f.

Section Summary

- The second derivative f'' determines the *concavity* of the graph of function f.
- If $f''(x) > 0$ for all x in an open interval I, then the graph of f is *concave up* over I.
- If $f''(x) < 0$ for all x in an open interval I, then the graph of f is *concave down* over I.
- If c is a critical value and $f''(c) > 0$, then $f(c)$ is a relative minimum.
- If c is a critical value and $f''(c) < 0$, then $f(c)$ is a relative maximum.
- If c is a critical value and $f''(c) = 0$, the First-Derivative Test must be used to classify $f(c)$.
- If $f''(x_0) = 0$ or $f''(x_0)$ does not exist, and there is a change in concavity to the left and to the right of x_0, then the point $(x_0, f(x_0))$ is called a *point of inflection*.
- Finding relative extrema, intervals over which a function is increasing or decreasing, intervals of upward or downward concavity, and points of inflection is all part of a strategy for accurate curve sketching.

EXERCISE SET 2.2

For each function, find all relative extrema and classify each as a maximum or minimum. Use the Second-Derivative Test where possible.

1. $f(x) = 5 - x^2$
2. $f(x) = 4 - x^2$
3. $f(x) = x^2 - x$
4. $f(x) = x^2 + x - 1$
5. $f(x) = -5x^2 + 8x - 7$
6. $f(x) = -4x^2 + 3x - 1$
7. $f(x) = 8x^3 - 6x + 1$
8. $f(x) = x^3 - 12x - 1$

Sketch the graph of each function. List the coordinates of where extrema or points of inflection occur. State where the function is increasing or decreasing, as well as where it is concave up or concave down.

9. $f(x) = x^3 - 12x$
10. $f(x) = x^3 - 27x$
11. $f(x) = 3x^3 - 36x - 3$
12. $f(x) = 2x^3 - 3x^2 - 36x + 28$
13. $f(x) = \frac{8}{3}x^3 - 2x + \frac{1}{3}$
14. $f(x) = 80 - 9x^2 - x^3$
15. $f(x) = -x^3 + 3x^2 - 4$
16. $f(x) = -x^3 + 3x - 2$
17. $f(x) = 3x^4 - 16x^3 + 18x^2$
    (Round results to three decimal places.)
18. $f(x) = 3x^4 + 4x^3 - 12x^2 + 5$
    (Round results to three decimal places.)
19. $f(x) = x^4 - 6x^2$
20. $f(x) = 2x^2 - x^4$
21. $f(x) = x^3 - 2x^2 - 4x + 3$
22. $f(x) = x^3 - 6x^2 + 9x + 1$
23. $f(x) = 3x^4 + 4x^3$
24. $f(x) = x^4 - 2x^3$
25. $f(x) = x^3 - 6x^2 - 135x$
26. $f(x) = x^3 - 3x^2 - 144x - 140$
27. $f(x) = x^4 - 4x^3 + 10$
28. $f(x) = \frac{4}{3}x^3 - 2x^2 + x$
29. $f(x) = x^3 - 6x^2 + 12x - 6$

Graphing a Function and Finding Its Roots, Relative Extrema, and Points of Inflection (continued)

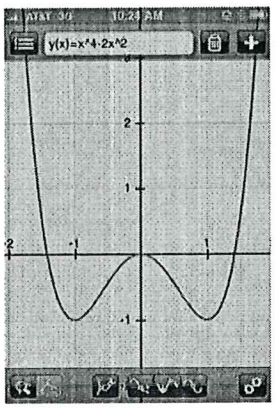


FIGURE 1

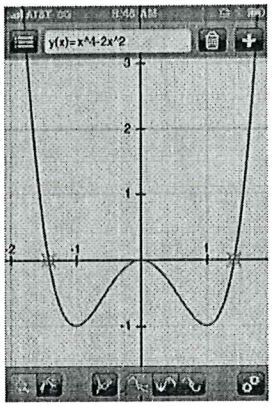


FIGURE 2

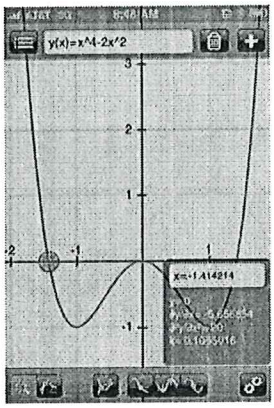


FIGURE 3

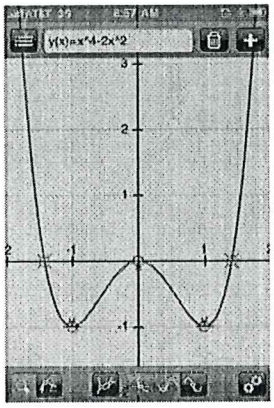


FIGURE 4

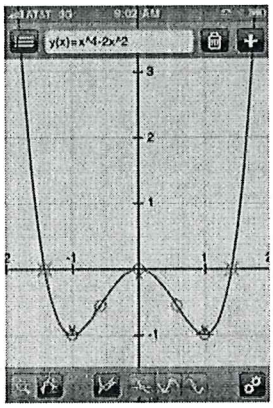


FIGURE 5

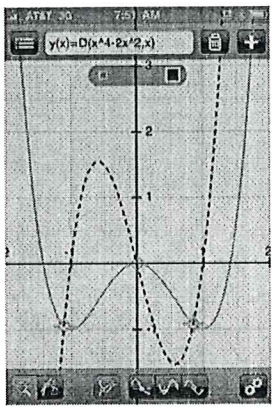


FIGURE 6

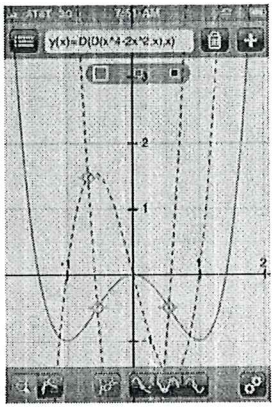


FIGURE 7

Then touch the fourth icon from the left at the bottom, and you will see the roots highlighted on the graph (Fig. 2). Touch the left-hand root symbol, and a value for one root, in this case, an approximation, -1.414214, is displayed (Fig. 3).

To find the relative extrema, touch the fifth icon from the left; the relative extrema will be highlighted on the graph in a different color. Touch each point to identify the relative minima at $(-1, -1)$, and $(1, -1)$, and a relative maximum at $(0, 0)$ (Fig. 4).

To find points of inflection, touch the sixth icon from the left, and the points of inflection will be highlighted as shown in Fig. 5. Touch these points and the approximations $(-0.577, -0.556)$ and $(0.577, -0.556)$ will be displayed.

Graphing a Function and Its Derivatives

Graphicus can graph derivatives. Go back to the original graph of f (Fig. 1). Touch [+]; then press Add derivative, and you will see the original graph and the graph of the first derivative as a dashed line of a different color (Fig. 6). Touch [+] and Add derivative again, and you will see the graph of the original function, with dashed graphs of the first and second derivatives in different colors (Fig. 7). You can toggle between the derivatives by pressing the colored squares above the graphs.

EXERCISES

Use Graphicus to graph each function and its first and second derivatives. Then find approximations for roots, relative extrema, and points of inflection.

1. $f(x) = 2x^3 - x^4$
2. $f(x) = x(200 - x)$
3. $f(x) = x^3 - 6x^2$
4. $f(x) = -4.32 + 1.44x + 3x^2 - x^3$
5. $g(x) = x\sqrt{4 - x^2}$
6. $g(x) = \dfrac{4x}{x^2 + 1}$
7. $f(x) = \dfrac{x^2 - 3x}{x - 1}$
8. $f(x) = |x + 2| - 3$

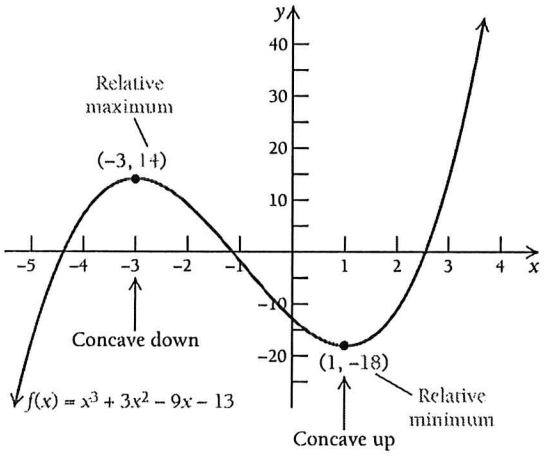


FIGURE 4

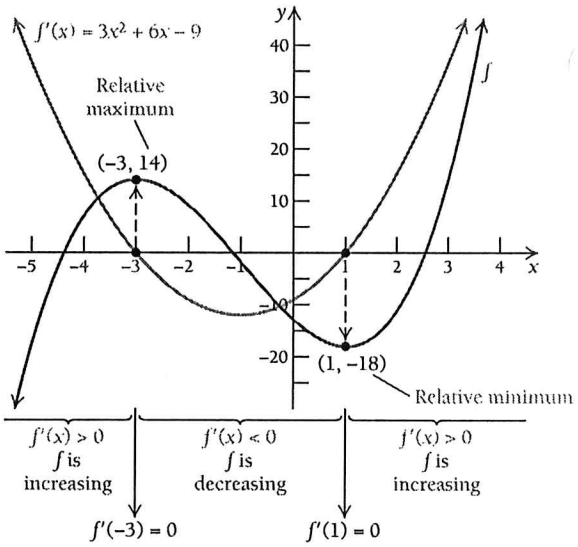


FIGURE 5

In Fig. 6, the intervals over which f' is increasing or decreasing are, respectively, those intervals over which f'' is positive or negative. And finally, in Fig. 7, we note that when $f''(x) < 0$, the graph of f is concave down, and when $f''(x) > 0$, the graph of f is concave up.

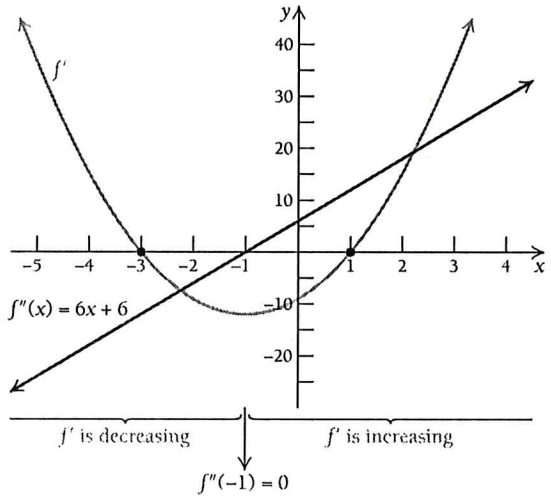


FIGURE 6

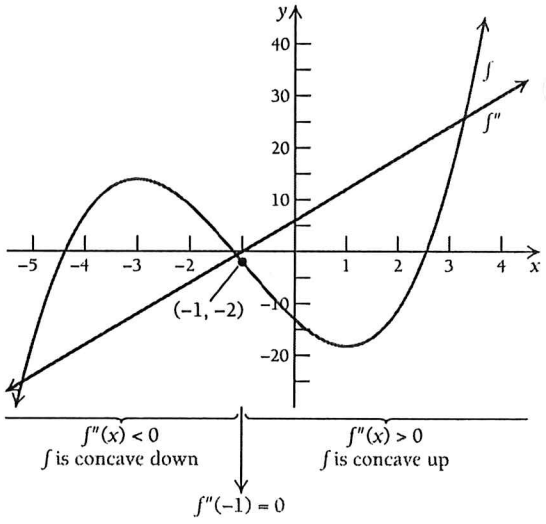


FIGURE 7

TECHNOLOGY CONNECTION

Using Graphicus to Find Roots, Extrema, and Inflection Points

Graphicus has the capability of finding roots, relative extrema, and points of inflection. Let's consider the function of Example 5, given by $f(x) = x^4 - 2x^2$.

Graphing a Function and Finding Its Roots, Relative Extrema, and Points of Inflection

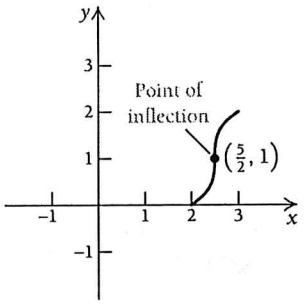
After opening Graphicus, touch the blank rectangle at the top of the screen and enter the function as y(x)=x^4-2x^2. Press + in the upper right. You will see the graph (Fig. 1).

(*continued*)

so f is concave up on $\left(-\infty, \frac{5}{2}\right)$.

$$f''(3) = \frac{-8}{9(2 \cdot 3 - 5)^{5/3}} = \frac{-8}{9 \cdot 1} < 0,$$

so f is concave down on $\left(\frac{5}{2}, \infty\right)$.



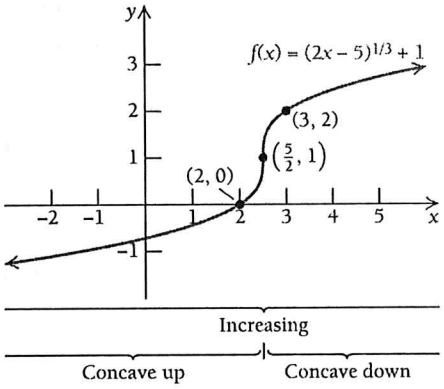
Interval $\longleftarrow$ $\frac{5}{2}$ $\longrightarrow$ x

Test Value	$x = 2$	$x = 3$
Sign of $f''(x)$	$f''(2) > 0$	$f''(3) < 0$
Result	f' is increasing; f is concave up.	f' is decreasing; f is concave down.

Change indicates a point of inflection.

f) *Sketch the graph.* Sketch the graph using the information in steps (a)–(e). By solving $(2x - 5)^{1/3} + 1 = 0$, we can find the x-intercept—it is $(2, 0)$. Extra function values can be calculated, if desired. The graph is shown below.

x	$f(x)$, approximately
0	−0.71
1	−0.44
2	0
$\frac{5}{2}$	1
3	2
4	2.44
5	2.71



The following figures illustrate some information concerning the function in Example 1 that can be found from the first and second derivatives of f. The relative extrema are shown in Figs. 4 and 5. In Fig. 5, we see that the x-coordinates of the x-intercepts of f' are the critical values of f. Note that the intervals over which f is increasing or decreasing are those intervals for which f' is positive or negative, respectively.

Solution

a) *Derivatives and domain.* Find $f'(x)$ and $f''(x)$:

$$f'(x) = \tfrac{1}{3}(2x-5)^{-2/3}\cdot 2 = \tfrac{2}{3}(2x-5)^{-2/3}, \text{ or } \frac{2}{3(2x-5)^{2/3}};$$

$$f''(x) = -\tfrac{4}{9}(2x-5)^{-5/3}\cdot 2 = -\tfrac{8}{9}(2x-5)^{-5/3}, \text{ or } \frac{-8}{9(2x-5)^{5/3}}.$$

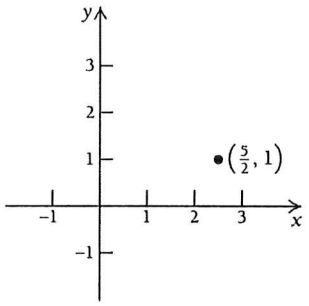
The domain of f is $\mathbb{R}$.

b) *Critical values.* Since

$$f'(x) = \frac{2}{3(2x-5)^{2/3}}$$

is never 0 (a fraction equals 0 only when its numerator is 0), the only critical value is when $f'(x)$ does not exist. The only time $f'(x)$ does not exist is when its denominator is 0:

$$\begin{aligned} 3(2x-5)^{2/3} &= 0 \\ (2x-5)^{2/3} &= 0 \qquad \text{Dividing both sides by 3} \\ (2x-5)^{2} &= 0 \qquad \text{Cubing both sides} \\ 2x-5 &= 0 \\ 2x &= 5 \\ x &= \tfrac{5}{2} \end{aligned}$$



We now have $f\left(\tfrac{5}{2}\right) = \left(2\cdot\tfrac{5}{2} - 5\right)^{1/3} + 1 = 0 + 1 = 1$, so the point $\left(\tfrac{5}{2}, 1\right)$ is on the graph.

c) *Increasing and/or decreasing; relative extrema.* Substitute the critical value into $f''(x)$:

$$f''\left(\tfrac{5}{2}\right) = \frac{-8}{9\left(2\cdot\tfrac{5}{2} - 5\right)^{5/3}} = \frac{-8}{9\cdot 0} = \frac{-8}{0}.$$

Since $f''\left(\tfrac{5}{2}\right)$ does not exist, the Second-Derivative Test cannot be used at $x = \tfrac{5}{2}$. Instead, we use the First-Derivative Test, selecting 2 and 3 as test values on either side of $\tfrac{5}{2}$:

$$f'(2) = \frac{2}{3(2\cdot 2-5)^{2/3}} = \frac{2}{3(-1)^{2/3}} = \frac{2}{3\cdot 1} = \frac{2}{3},$$

$$\text{and } f'(3) = \frac{2}{3(2\cdot 3-5)^{2/3}} = \frac{2}{3\cdot 1^{2/3}} = \frac{2}{3\cdot 1} = \frac{2}{3}.$$

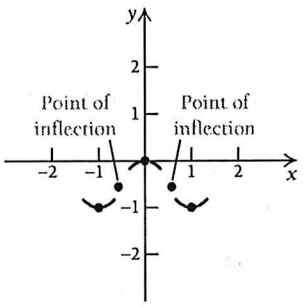
Since $f'(x) > 0$ on either side of $x = \tfrac{5}{2}$, we know that f is increasing on both $\left(-\infty, \tfrac{5}{2}\right)$ and $\left(\tfrac{5}{2}, \infty\right)$; thus, $f\left(\tfrac{5}{2}\right) = 1$ is not an extremum.

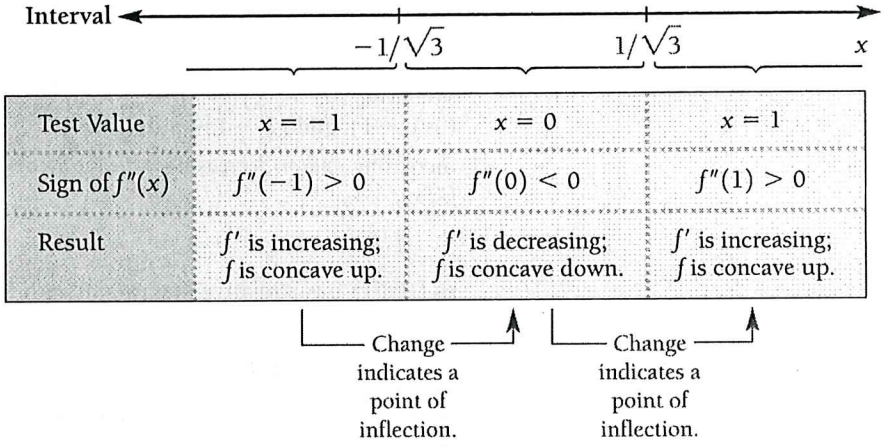
d) *Inflection points.* Find where $f''(x)$ does not exist and where $f''(x) = 0$. Since $f''(x)$ is never 0 (why?), we only need to find where $f''(x)$ does not exist. Since $f''(x)$ cannot exist where $f'(x)$ does not exist, we know from step (b) that a possible inflection point is $\left(\tfrac{5}{2}, 1\right)$.

e) *Concavity.* We check the concavity on either side of $x = \tfrac{5}{2}$. We choose $x = 2$ and $x = 3$ as our test values.

$$f''(2) = \frac{-8}{9(2\cdot 2-5)^{5/3}} = \frac{-8}{9(-1)} > 0,$$

e) *Concavity.* Find the intervals on which f is concave up or concave down, using the points $\left(-\frac{1}{\sqrt{3}}, -\frac{5}{9}\right)$ and $\left(\frac{1}{\sqrt{3}}, -\frac{5}{9}\right)$, from step (d). From step (c), we can conclude that f is concave up over the intervals $\left(-\infty, -1/\sqrt{3}\right)$ and $\left(1/\sqrt{3}, \infty\right)$ and concave down over the interval $\left(-1/\sqrt{3}, 1/\sqrt{3}\right)$.





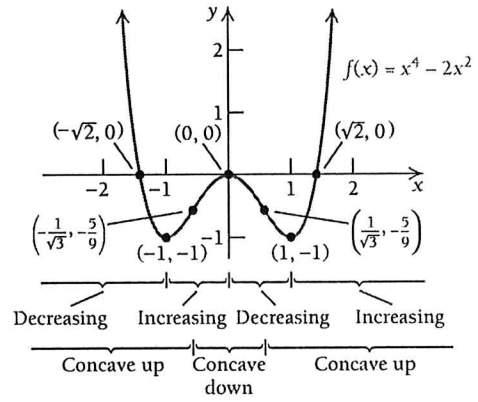
Test Value	$x = -1$	$x = 0$	$x = 1$
Sign of $f''(x)$	$f''(-1) > 0$	$f''(0) < 0$	$f''(1) > 0$
Result	f' is increasing; f is concave up.	f' is decreasing; f is concave down.	f' is increasing; f is concave up.

Change indicates a point of inflection.

Change indicates a point of inflection.

f) *Sketch the graph.* Sketch the graph using the information in steps (a)–(e). By solving $x^4 - 2x^2 = 0$, we can find the x-intercepts easily. They are $\left(-\sqrt{2}, 0\right)$, $(0, 0)$, and $\left(\sqrt{2}, 0\right)$. This also aids with graphing. Extra function values can be calculated if desired. The graph is shown below.

x	$f(x)$, approximately
-2	8
-1.5	0.56
-1	-1
-0.5	-0.44
0	0
0.5	-0.44
1	-1
1.5	0.56
2	8



TECHNOLOGY CONNECTION

EXERCISE

1. Consider

$$f(x) = x^3(x - 2)^3.$$

How many relative extrema do you anticipate finding? Where do you think they will be?

Graph f, f', and f'' using $[-1, 3, -2, 6]$ as a viewing window. Estimate the relative extrema and the inflection points of f. Then check your work using the analytic methods of Examples 4 and 5.

› Quick Check 3

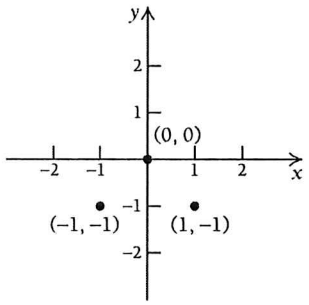
Find the relative maxima and minima of the function f given by $f(x) = 1 + 8x^2 - x^4$, and sketch the graph.

‹ Quick Check 3

EXAMPLE 6 Graph the function f given by

$$f(x) = (2x - 5)^{1/3} + 1.$$

List the coordinates of any extrema and points of inflection. State where the function is increasing or decreasing, as well as where it is concave up or concave down.



b) *Critical values.* Since $f'(x) = 4x^3 - 4x$ exists for all values of x, the only critical values are where $f'(x) = 0$:

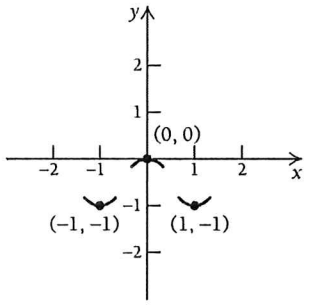
$$\begin{aligned} 4x^3 - 4x &= 0 \qquad \text{Setting } f'(x) \text{ equal to } 0 \\ 4x(x^2 - 1) &= 0 \\ 4x = 0 \quad &\text{or} \quad x^2 - 1 = 0 \\ x = 0 \quad &\text{or} \quad x^2 = 1 \\ &\qquad x = \pm 1. \end{aligned}$$

We have $f(0) = 0, f(-1) = -1$, and $f(1) = -1$, which gives the points $(0, 0)$, $(-1, -1)$, and $(1, -1)$ on the graph.

c) *Increasing and/or decreasing; relative extrema.* Substitute the critical values into $f''(x)$:

$$f''(0) = 12 \cdot 0^2 - 4 = -4 < 0,$$

so $f(0) = 0$ is a relative maximum, with f increasing on $(-1, 0)$ and decreasing on $(0, 1)$.



$$f''(-1) = 12(-1)^2 - 4 = 8 > 0,$$

so $f(-1) = -1$ is a relative minimum, with f decreasing on $(-\infty, -1)$ and increasing on $(-1, 0)$.

$$f''(1) = 12 \cdot 1^2 - 4 = 8 > 0,$$

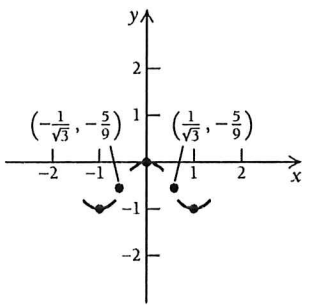
so $f(1) = -1$ is also a relative minimum, with f decreasing on $(0, 1)$ and increasing on $(1, \infty)$.

d) *Inflection points.* Find where $f''(x)$ does not exist and where $f''(x) = 0$. Since $f''(x)$ exists for all real numbers, we just solve $f''(x) = 0$:

$$\begin{aligned} 12x^2 - 4 &= 0 \qquad \text{Setting } f''(x) \text{ equal to } 0 \\ 4(3x^2 - 1) &= 0 \\ 3x^2 - 1 &= 0 \\ 3x^2 &= 1 \\ x^2 &= \frac{1}{3} \\ x &= \pm\sqrt{\frac{1}{3}} \\ &= \pm\frac{1}{\sqrt{3}}. \end{aligned}$$

We have

$$\begin{aligned} f\left(\frac{1}{\sqrt{3}}\right) &= \left(\frac{1}{\sqrt{3}}\right)^4 - 2\left(\frac{1}{\sqrt{3}}\right)^2 \\ &= \frac{1}{9} - \frac{2}{3} = -\frac{5}{9} \end{aligned}$$



and

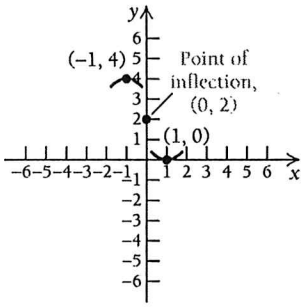
$$f\left(-\frac{1}{\sqrt{3}}\right) = -\frac{5}{9}.$$

These values give

$$\left(-\frac{1}{\sqrt{3}}, -\frac{5}{9}\right) \text{ and } \left(\frac{1}{\sqrt{3}}, -\frac{5}{9}\right)$$

as possible inflection points.

e) *Concavity.* Find the intervals on which f is concave up or concave down, using the point $(0, 2)$ from step (d). From step (c), we can conclude that f is concave down over the interval $(-\infty, 0)$ and concave up over $(0, \infty)$.



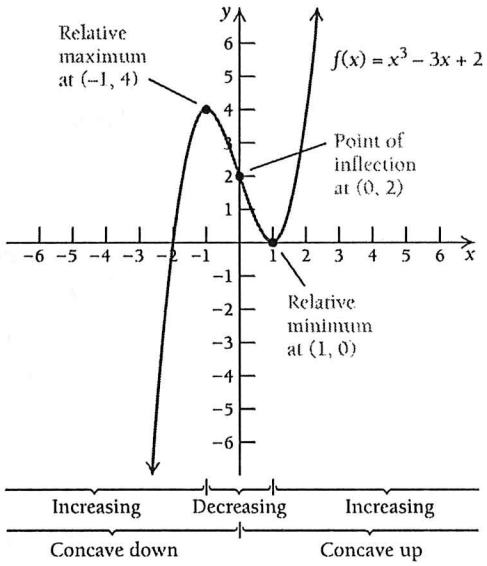
Interval ← 0 x

Test Value	$x = -1$	$x = 1$
Sign of $f''(x)$	$f''(-1) < 0$	$f''(1) > 0$
Result	f' is decreasing; f is concave down.	f' is increasing; f is concave up.

Change indicates a point of inflection.

f) *Sketch the graph.* Sketch the graph using the information in steps (a)–(e). Calculate some extra function values if desired. The graph follows.

x	$f(x)$
−3	−16
−2	0
−1	4
0	2
1	0
2	4
3	20



TECHNOLOGY CONNECTION

Check the results of Example 4 using a calculator.

EXAMPLE 5 Find the relative maxima and minima of the function f given by

$$f(x) = x^4 - 2x^2,$$

and sketch the graph.

Solution

a) *Derivatives and domain.* Find $f'(x)$ and $f''(x)$:

$$f'(x) = 4x^3 - 4x,$$
$$f''(x) = 12x^2 - 4.$$

The domain of f is $\mathbb{R}$.

The examples that follow apply this step-by-step strategy to sketch the graphs of several functions.

EXAMPLE 4 Find the relative extrema of the function f given by

$$f(x) = x^3 - 3x + 2,$$

and sketch the graph.

Solution

a) *Derivatives and domain.* Find $f'(x)$ and $f''(x)$:

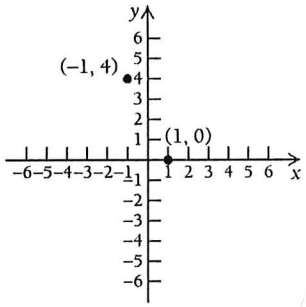
$$f'(x) = 3x^2 - 3,$$
$$f''(x) = 6x.$$

The domain of f (and of any polynomial function) is $(-\infty, \infty)$, or the set of all real numbers, which is also written as $\mathbb{R}$.

b) *Critical values of f.* Find the critical values by determining where $f'(x)$ does not exist and by solving $f'(x) = 0$. We know that $f'(x) = 3x^2 - 3$ exists for all values of x, so the only critical values are where $f'(x)$ is 0:

$$3x^2 - 3 = 0 \qquad \text{Setting } f'(x) \text{ equal to } 0$$
$$3x^2 = 3$$
$$x^2 = 1$$
$$x = \pm 1.$$

We have $f(-1) = 4$ and $f(1) = 0$, so $(-1, 4)$ and $(1, 0)$ are on the graph.



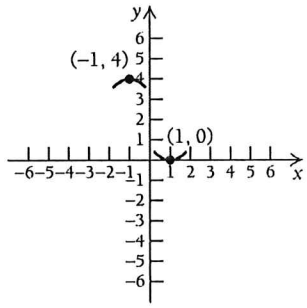
c) *Increasing and/or decreasing; relative extrema.* Substitute the critical values into $f''(x)$:

$$f''(-1) = 6(-1) = -6 < 0,$$

so $f(-1) = 4$ is a relative maximum, with f increasing on $(-\infty, -1)$ and decreasing on $(-1, 1)$.

$$f''(1) = 6 \cdot 1 = 6 > 0,$$

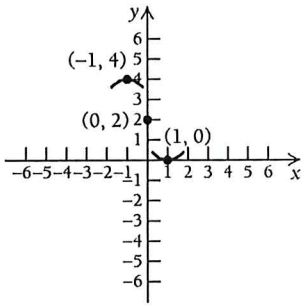
so $f(1) = 0$ is a relative minimum, with f decreasing on $(-1, 1)$ and increasing on $(1, \infty)$.



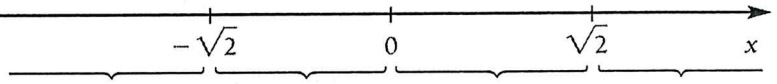
d) *Inflection points.* Find possible inflection points by finding where $f''(x)$ does not exist and by solving $f''(x) = 0$. We know that $f''(x) = 6x$ exists for all values of x, so we try to solve $f''(x) = 0$:

$$6x = 0 \qquad \text{Setting } f''(x) \text{ equal to } 0$$
$$x = 0. \qquad \text{Dividing both sides by } 6$$

We have $f(0) = 2$, which gives us another point, $(0, 2)$, that lies on the graph.



Interval



Test Value	$x = -2$	$x = -1$	$x = 1$	$x = 2$
Sign of $f''(x)$	$f''(-2) < 0$	$f''(-1) > 0$	$f''(1) < 0$	$f''(2) > 0$
Result	Concave down	Concave up	Concave down	Concave up

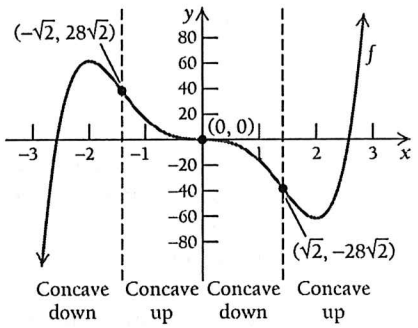
The graph changes from concave down to concave up at $x = -\sqrt{2}$, from concave up to concave down at $x = 0$, and from concave down to concave up at $x = \sqrt{2}$. Therefore, $\left(-\sqrt{2}, f\left(-\sqrt{2}\right)\right)$, $(0, f(0))$, and $\left(\sqrt{2}, f\left(\sqrt{2}\right)\right)$ are points of inflection. Since

$$f\left(-\sqrt{2}\right) = 3\left(-\sqrt{2}\right)^5 - 20\left(-\sqrt{2}\right)^3 = 28\sqrt{2},$$

$$f(0) = 3(0)^5 - 20(0)^3 = 0,$$

and $$f\left(\sqrt{2}\right) = 3\left(\sqrt{2}\right)^5 - 20\left(\sqrt{2}\right)^3 = -28\sqrt{2},$$

these points are $\left(-\sqrt{2}, 28\sqrt{2}\right)$, $(0, 0)$, and $\left(\sqrt{2}, -28\sqrt{2}\right)$, shown in the graph below.



Quick Check 2

Determine the points of inflection for the function given by $g(x) = 10x^3 - 6x^5$.

Quick Check 2

Curve Sketching

The first and second derivatives enhance our ability to sketch curves. We use the following strategy:

Strategy for Sketching Graphs*

a) *Derivatives and domain.* Find $f'(x)$ and $f''(x)$. Note the domain of f.

b) *Critical values of f.* Find the critical values by solving $f'(x) = 0$ and finding where $f'(x)$ does not exist. These numbers yield candidates for relative maxima or minima. Find the function values at these points.

c) *Increasing and/or decreasing; relative extrema.* Substitute each critical value, x_0, from step (b) into $f''(x)$. If $f''(x_0) < 0$, then $f(x_0)$ is a relative maximum and f is increasing to the left of x_0 and decreasing to the right. If $f''(x_0) > 0$, then $f(x_0)$ is a relative minimum and f is decreasing to the left of x_0 and increasing to the right.

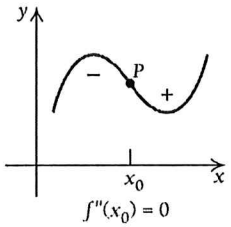
d) *Inflection points.* Determine candidates for inflection points by finding where $f''(x) = 0$ or where $f''(x)$ does not exist. Find the function values at these points.

e) *Concavity.* Use the candidates for inflection points from step (d) to define intervals. Substitute test values into $f''(x)$ to determine where the graph is concave up ($f''(x) > 0$) and where it is concave down ($f''(x) < 0$).

f) *Sketch the graph.* Sketch the graph using the information from steps (a)–(e), calculating and plotting extra points as needed.

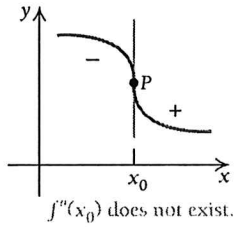
*This strategy is refined further, for rational functions, in Section 2.3.

A **point of inflection**, or an **inflection point**, is a point across which the direction of concavity changes. For example, in Figs. 1–3, point P is an inflection point. The figures display the sign of $f''(x)$ to indicate the concavity on either side of P.



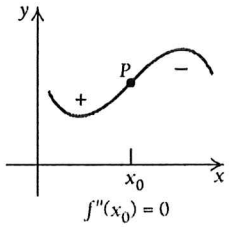
$f''(x_0) = 0$

FIGURE 1



$f''(x_0)$ does not exist.

FIGURE 2



$f''(x_0) = 0$

FIGURE 3

As we move to the right along each curve, the concavity changes at P. Since, as we move through P, the sign of $f''(x)$ changes, either the value of $f''(x_0)$ at P must be 0, as in Figs. 1 and 3, or $f''(x_0)$ must not exist, as in Fig. 2.

THEOREM 6 Finding Points of Inflection

If a function f has a point of inflection, it must occur at a point x_0, where

$$f''(x_0) = 0 \quad \text{or} \quad f''(x_0) \text{ does not exist.}$$

The converse of Theorem 6 is not necessarily true. That is, if $f''(x_0)$ is 0 or does not exist, then there is not necessarily a point of inflection at x_0. There must be a change in the direction of concavity on either side of x_0 for $(x_0, f(x_0))$ to be a point of inflection. For example, for $f(x) = (x - 2)^4 + 1$ (see the graph on p. 218), we have $f''(2) = 0$, but $(2, f(2))$ is not a point of inflection since the graph of f is concave up to the left and to the right of $x = 2$.

To find candidates for points of inflection, we look for numbers x_0 for which $f''(x_0) = 0$ or for which $f''(x_0)$ does not exist. Then, if $f''(x)$ changes sign as x moves through x_0 (see Figs. 1–3), we have a point of inflection at x_0.

Theorem 6, about points of inflection, is completely analogous to Theorem 2 about relative extrema. Theorem 2 tells us that relative extrema occur when $f'(x) = 0$ or $f'(x)$ does not exist. Theorem 6 tells us that points of inflection occur when $f''(x) = 0$ or $f''(x)$ does not exist.

■ **EXAMPLE 3** Use the second derivative to determine the point(s) of inflection for the function in Example 2.

Solution The function is $f(x) = 3x^5 - 20x^3$, and its second derivative is $f''(x) = 60x^3 - 120x$. We set the second derivative equal to 0 and solve for x:

$$60x^3 - 120x = 0$$

$$60x(x^2 - 2) = 0 \qquad \text{Factoring out } 60x$$

$$60x\left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right) = 0 \qquad \text{Factoring } x^2 - 2 \text{ as a difference of squares}$$

$$x = 0 \quad \text{or} \quad x = -\sqrt{2} \quad \text{or} \quad x = \sqrt{2}. \qquad \text{Using the Principle of Zero Products}$$

Next, we check the sign of $f''(x)$ over the intervals bounded by these three x-values. We are looking for a change in sign from one interval to the next:

Solution We find both the first and second derivatives:

$$f'(x) = 15x^4 - 60x^2,$$
$$f''(x) = 60x^3 - 120x.$$

Then we solve $f'(x) = 0$ to find any critical values:

$$15x^4 - 60x^2 = 0$$
$$15x^2(x^2 - 4) = 0$$
$$15x^2(x + 2)(x - 2) = 0 \qquad \text{Factoring}$$
$$15x^2 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{or} \quad x - 2 = 0 \qquad \text{Using the Principle of Zero Products}$$
$$x = 0 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = 2.$$

We next find second coordinates by substituting in the original function:

$$f(-2) = 3(-2)^5 - 20(-2)^3 = 64;$$
$$f(0) = 3(0)^5 - 20(0)^3 = 0.$$
$$f(2) = 3(2)^5 - 20(2)^3 = -64;$$

All three of these y-values are candidates for relative extrema.

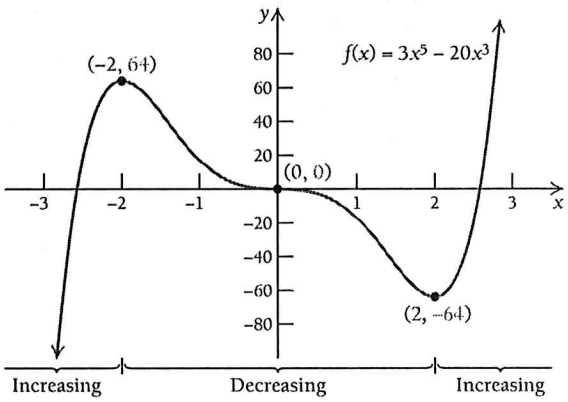
We now use the Second-Derivative Test with the numbers -2, 2, and 0:

$f''(-2) = 60(-2)^3 - 120(-2) = -240 < 0;$ → Relative maximum

$f''(0) = 60(0)^3 - 120(0) = 0.$ → The Second-Derivative Test fails. Use the First-Derivative Test.

$f''(2) = 60(2)^3 - 120(2) = 240 > 0;$ → Relative minimum

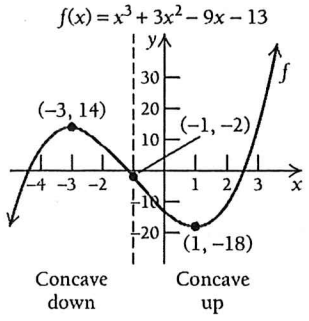
Thus, $f(-2) = 64$ is a relative maximum, and $f(2) = -64$ is a relative minimum. Since $f'(-1) < 0$ and $f'(1) < 0$, we know that f is decreasing on both $(-2, 0)$ and $(0, 2)$. Thus, we know by the First-Derivative Test that f has no relative extremum at $(0, 0)$. We complete the graph, plotting other points as needed. The extrema are shown in the graph at right.



› Quick Check 1

Find the relative extrema of the function g given by $g(x) = 10x^3 - 6x^5$, and sketch the graph.

‹ Quick Check 1



Points of Inflection

Look again at the graphs in Examples 1 and 2. The concavity changes from down to up at the point $(-1, -2)$ in Example 1, and the concavity changes from up to down at the point $(0, 0)$ in Example 2. (In fact, the graph in Example 2 has other points where the concavity changes direction. This is addressed in Example 3.)

EXAMPLE 1 Find the relative extrema of the function f given by

$$f(x) = x^3 + 3x^2 - 9x - 13,$$

and sketch the graph.

Solution To find any critical values, we determine $f'(x)$. To determine whether any critical values lead to extrema, we also find $f''(x)$:

$$f'(x) = 3x^2 + 6x - 9,$$
$$f''(x) = 6x + 6.$$

Then we solve $f'(x) = 0$:

$$3x^2 + 6x - 9 = 0$$
$$x^2 + 2x - 3 = 0 \quad \text{Dividing both sides by 3}$$
$$(x + 3)(x - 1) = 0 \quad \text{Factoring}$$
$$x + 3 = 0 \quad \text{or} \quad x - 1 = 0 \quad \text{Using the Principle of Zero Products}$$
$$x = -3 \quad \text{or} \quad x = 1.$$

TECHNOLOGY CONNECTION

Exploratory

Consider the function given by

$$y = x^3 - 3x^2 - 9x - 1.$$

Use a graph to estimate the relative extrema. Then find the first and second derivatives. Graph both in the same window. Use the ZERO feature to determine where the first derivative is zero. Verify that relative extrema occur at those x-values by checking the sign of the second derivative. Then check your work using the approach of Example 1.

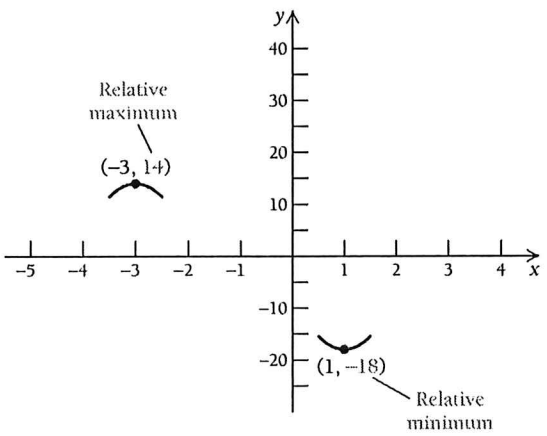
We next find second coordinates by substituting the critical values in the original function:

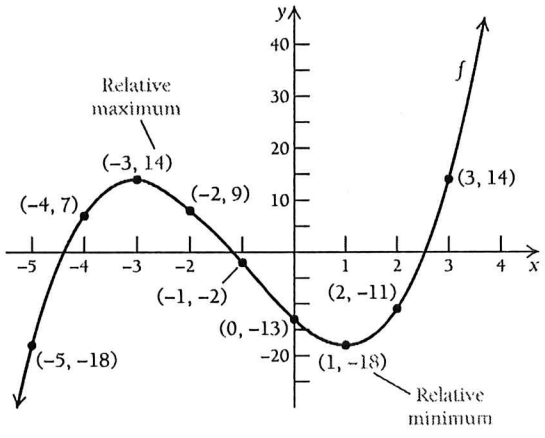
$$f(-3) = (-3)^3 + 3(-3)^2 - 9(-3) - 13 = 14;$$
$$f(1) = (1)^3 + 3(1)^2 - 9(1) - 13 = -18.$$

Are the points $(-3, 14)$ and $(1, -18)$ relative extrema? Let's look at the second derivative. We use the Second-Derivative Test with the critical values -3 and 1:

$$f''(-3) = 6(-3) + 6 = -12 < 0; \longrightarrow \text{Relative maximum}$$
$$f''(1) = 6(1) + 6 = 12 > 0. \longrightarrow \text{Relative minimum}$$

Thus, $f(-3) = 14$ is a relative maximum and $f(1) = -18$ is a relative minimum. We plot both $(-3, 14)$ and $(1, -18)$, including short arcs at each point to indicate the graph's concavity. Then, by calculating and plotting a few more points, we can make a sketch, as shown below.





EXAMPLE 2 Find the relative extrema of the function f given by

$$f(x) = 3x^5 - 20x^3,$$

and sketch the graph.