Applications of Differentiation

2

Chapter Snapshot

What You'll Learn

2.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

2.2 Using Second Derivatives to Find Maximum and Minimum Values and Sketch Graphs

2.3 Graph Sketching: Asymptotes and Rational Functions

2.4 Using Derivatives to Find Absolute Maximum and Minimum Values

2.5 Maximum–Minimum Problems; Business and Economics Applications

2.6 Marginals and Differentials

2.7 Implicit Differentiation and Related Rates

Why It's Important

In this chapter, we explore many applications of differentiation. We learn to find maximum and minimum values of functions, and that skill allows us to solve many kinds of problems in which we need to find the largest and/or smallest value in a real-world situation. We also apply our differentiation skills to graphing and to approximating function values.

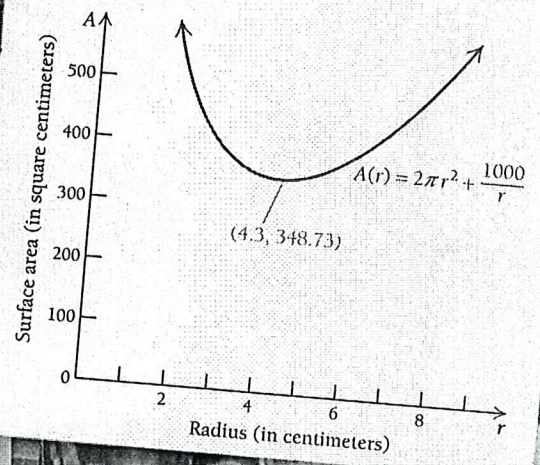
Where It's Used

MINIMIZING MATERIAL USED

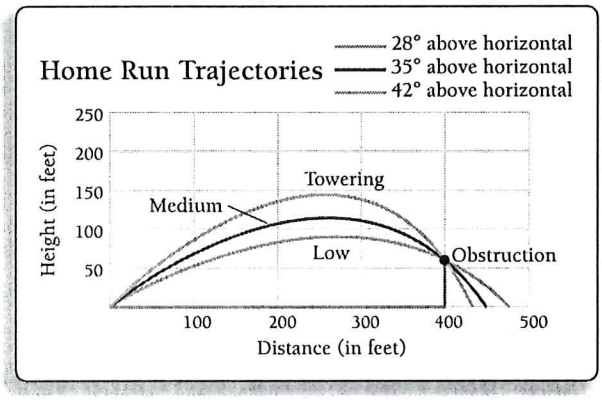
Minimizing the amount of material used is a common goal in manufacturing, as it reduces overall costs as well as increases efficiency. For example, cylindrical food cans come in a variety of sizes. Suppose a can is to have a volume of 500 milliliters. Are there optimal dimensions for the can's height and radius that will minimize the material needed to produce each can? Can you see how minimizing the material used per can translates into minimized costs and conservation of resources?

This problem appears as Example 3 in Section 2.5.

RELATING THE RADIUS OF A CAN TO ITS SURFACE AREA



Tale of the tape. Actually, scoreboard operators in the major leagues use different models to predict the distance that a home run ball would have traveled. The models are linear and are related to the trajectory of the ball, that is, how high the ball is hit. See the following graph.



Suppose that a ball hits an obstruction d feet horizontally from home plate at a height of H feet. Then the estimated horizontal distance D that the ball would have traveled, depending on its trajectory type, is

Low trajectory: $D = 1.1H + d$,

Medium trajectory: $D = 0.7H + d$,

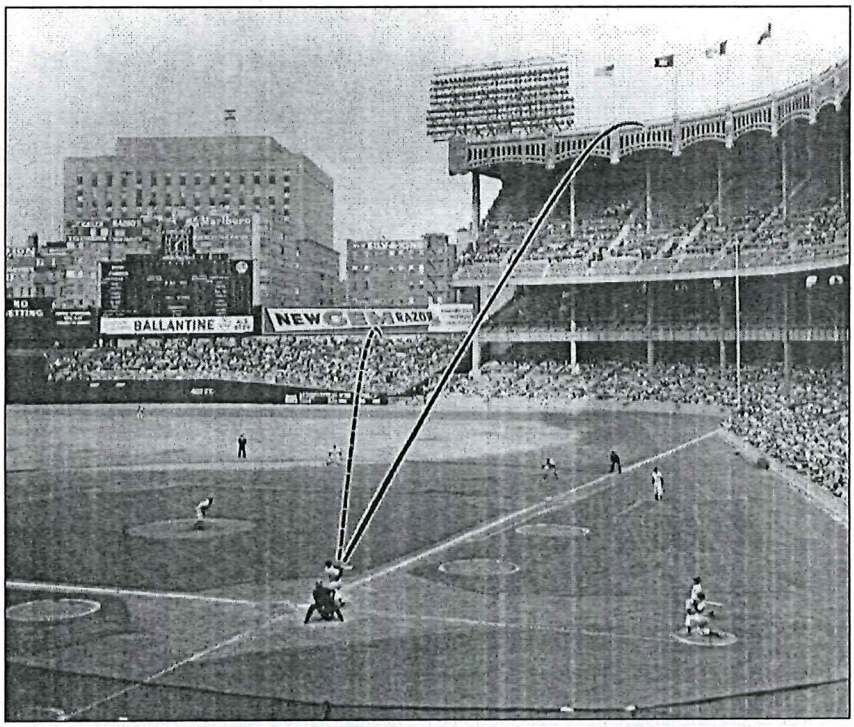
Towering trajectory: $D = 0.5H + d$.

6. For a ball striking an obstacle at $d = 400$ ft and $H = 60$ ft, estimate how far the ball would have traveled if it were following a low trajectory, or a medium trajectory, or a towering trajectory.

7. In 1953, Hall-of-Famer Mickey Mantle hit a towering home run in old Griffith Stadium in Washington, D.C., that hit an obstruction 60 ft high and 460 ft from home plate. Reporters asserted at the time that the ball would have traveled 565 ft. Is this estimate valid?

8. Use the appropriate formula to estimate the distance D for each the following famous long home runs.
   a) Ted Williams (Boston Red Sox, June 9, 1946): Purportedly the longest home run ball ever hit to right field at Boston's Fenway Park, Williams's ball landed in the stands 502 feet from home plate, 30 feet above the ground. Assume a medium trajectory.
   b) Reggie Jackson (Oakland Athletics, July 13, 1971): Jackson's mighty blast hit an electrical transformer on top of the right-field roof at old Tiger Stadium in the 1971 All-Star Game. The transformer was 380 feet from home plate, 100 feet up. Assume a towering trajectory. Jackson's home run was reported to still be on the upward arc when it hit the transformer.
   c) Richie Sexson (Arizona Diamondbacks, April 26, 2004): Sexson hit a drive that caromed off the center-field scoreboard at Bank One Ballpark in Phoenix. The scoreboard is 414 feet from home plate and 75 feet high. Assume a medium trajectory.

   The reported distances these balls would have traveled are 527 feet for Williams's home run, 530 feet for Jackson's, and 469 feet for Sexson's (*Source:* www.hittrackeronline.com). How close are your estimates?



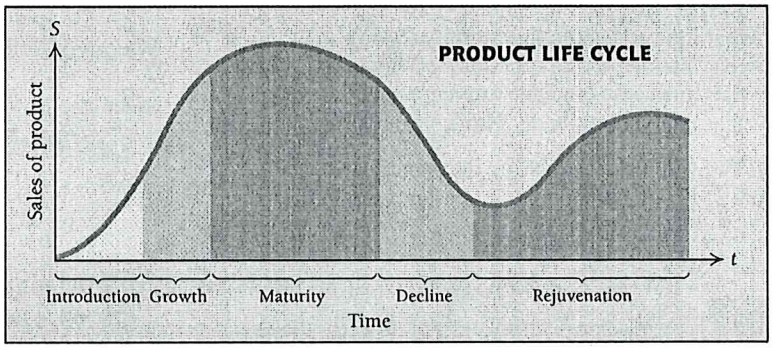
Many thanks to Robert K. Adair, professor of physics at Yale University, for many of the ideas presented in this application.

2.1 Using First Derivatives to Find Maximum and Minimum Values and Sketch Graphs

OBJECTIVES

- Find relative extrema of a continuous function using the First-Derivative Test.
- Sketch graphs of continuous functions.

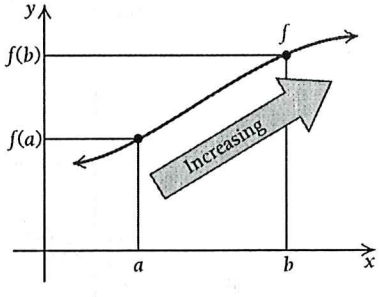
The graph below shows a typical life cycle of a retail product and is similar to graphs we will consider in this chapter. Note that the number of items sold varies with respect to time. Sales begin at a small level and increase to a point of maximum sales, after which they taper off to a low level, where the decline is probably due to the effect of new competitive products. The company then rejuvenates the product by making improvements. Think about versions of certain products: televisions can be traditional, flat-screen, or high-definition; music recordings have been produced as phonograph (vinyl) records, audiotapes, compact discs, and MP3 files. Where might each of these products be in a typical product life cycle? Does the curve seem appropriate for each product?



Finding the largest and smallest values of a function—that is, the maximum and minimum values—has extensive applications. The first and second derivatives of a function are calculus tools that provide information we can use in graphing functions and finding minimum and maximum values. Throughout this section we will assume, unless otherwise noted, that all functions are continuous. However, continuity of a function does not guarantee that its first and second derivatives are continuous.

Increasing and Decreasing Functions

If the graph of a function rises from left to right over an interval I, the function is said to be **increasing** on, or over, I.



f is an increasing function over I:
for all a, b in I, if $a < b$, then $f(a) < f(b)$.

TECHNOLOGY CONNECTION

Exploratory

Graph the function

$$y = -\tfrac{1}{3}x^3 + 6x^2 - 11x - 50$$

and its derivative

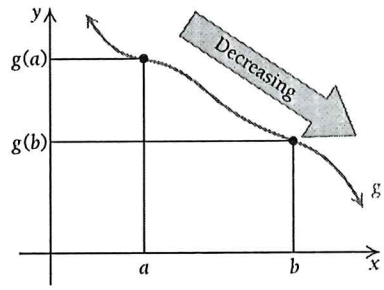
$$y' = -x^2 + 12x - 11$$

using the window $[-10, 25, -100, 150]$, with Xscl = 5 and Yscl = 25. Then TRACE from left to right along each graph. As you move the cursor from left to right, note that the x-coordinate always increases. If a function is increasing over an interval, the y-coordinate will increase as well. If a function is decreasing over an interval, the y-coordinate will decrease.

Over what intervals is the function increasing?
Over what intervals is the function decreasing?
Over what intervals is the derivative positive?
Over what intervals is the derivative negative?

What rules can you propose relating the sign of y' to the behavior of y?

If the graph drops from left to right, the function is said to be **decreasing** on, or over, I.



g is a decreasing function over I:
for all a, b in I, if $a < b$, then $g(a) > g(b)$.

We can describe these phenomena mathematically as follows.

DEFINITIONS strict monotonicity

A function f is **increasing** over I if, for every a and b in I,

$$\text{if } a < b, \quad \text{then } f(a) < f(b).$$

(If the input a is less than the input b, then the output for a is less than the output for b.)

A function f is **decreasing** over I if, for every a and b in I,

$$\text{if } a < b, \quad \text{then } f(a) > f(b).$$

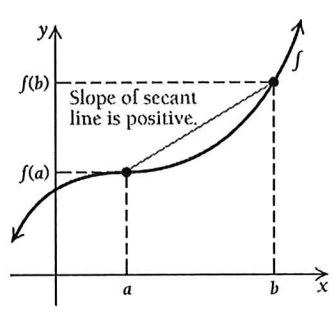
(If the input a is less than the input b, then the output for a is greater than the output for b.)

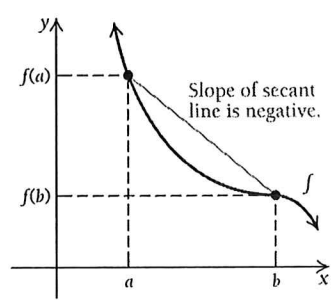
f is monotonic on I if
it is entirely non-increasing
or non-decreasing.

If $a<b$, $f(a) \le f(b)$ mono. ↑
If $a<b$, $f(a) \ge f(b)$
mono ↓

The above definitions can be restated in terms of secant lines. If a graph is increasing over an interval I, then, for all a and b in I such that $a < b$, the slope of the secant line between $x = a$ and $x = b$ is positive. Similarly, if a graph is decreasing over an interval I, then, for all a and b in I such that $a < b$, the slope of the secant line between $x = a$ and $x = b$ is negative:

$$\text{Increasing: } \frac{f(b) - f(a)}{b - a} > 0. \qquad \text{Decreasing: } \frac{f(b) - f(a)}{b - a} < 0.$$



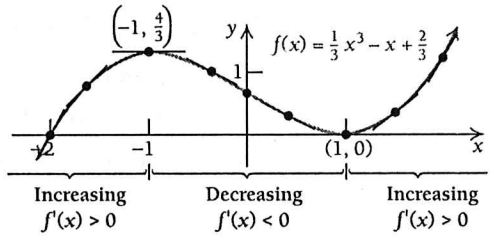


The following theorem shows how we can use the derivative (the slope of a tangent line) to determine whether a function is increasing or decreasing.

THEOREM 1

If $f'(x) > 0$ for all x in an open interval I, then f is increasing over I.
If $f'(x) < 0$ for all x in an open interval I, then f is decreasing over I.

Theorem 1 is illustrated in the following graph.



f is increasing over the intervals $(-\infty, -1)$ and $(1, \infty)$; slopes of tangent lines are positive.
f is decreasing over the interval $(-1, 1)$; slopes of tangent lines are negative.

For determining increasing or decreasing behavior using a derivative, the interval I is an open interval; that is, it does not include its endpoints. Note how the intervals on which f is increasing and decreasing are written in the preceding graph: $x = -1$ and $x = 1$ are not included in any interval over which the function is increasing or decreasing. These values are examples of *critical values*.

Critical Values

Consider the graph of a continuous function f in Fig. 1.

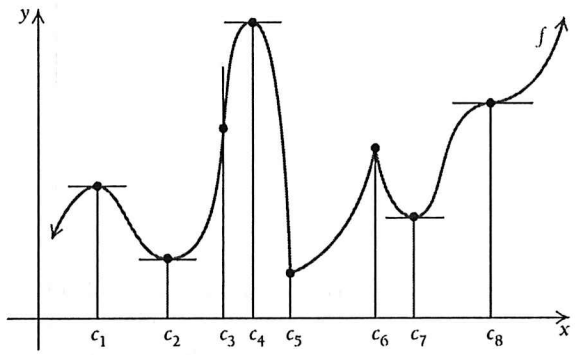


FIGURE 1

Note the following:

1. $f'(c) = 0$ at $x = c_1, c_2, c_4, c_7$, and c_8. That is, the tangent line to the graph is horizontal for these values.
2. $f'(c)$ does not exist at $x = c_3, c_5$, and c_6. The tangent line is vertical at c_3, and there are corner points at both c_5 and c_6. (See also the discussion at the end of Section 1.4.)

DEFINITION

A **critical value** of a function f is any number c in the domain of f for which the tangent line at $(c, f(c))$ is horizontal or for which the derivative does not exist. That is, c is a critical value if $f(c)$ exists and

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

Thus, in the graph of f in Fig. 1:

1. c_1, c_2, c_4, c_7, and c_8 are critical values because $f'(c) = 0$ for each value.
2. c_3, c_5, and c_6 are critical values because $f'(c)$ does not exist for each value.

Also note that a continuous function can change from increasing to decreasing or from decreasing to increasing *only* at a critical value. In the graph in Fig. 1, c_1, c_2, c_4, c_5, c_6, and c_7 separate the intervals over which the function changes from increasing to decreasing or from decreasing to increasing. Although c_3 and c_8 are critical values, they do not separate intervals over which the function changes from increasing to decreasing or from decreasing to increasing.

Finding Relative Maximum and Minimum Values

Now consider the graph in Fig. 2. Note the "peaks" and "valleys" at the interior points c_1, c_2, and c_3.

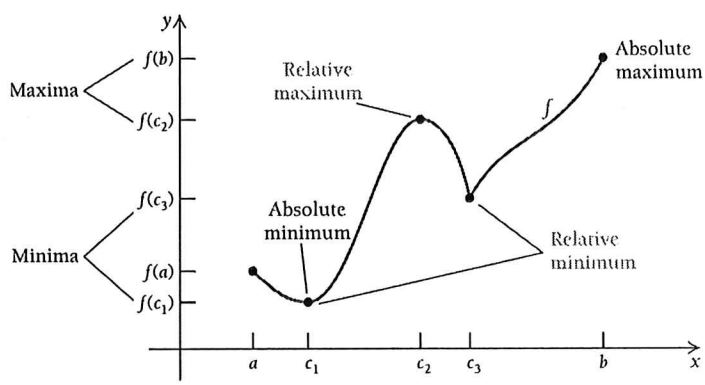


FIGURE 2

Here $f(c_2)$ is an example of a **relative maximum** (plural: **maxima**). Each of $f(c_1)$ and $f(c_3)$ is called a **relative minimum** (plural: **minima**). The terms **local maximum** and **local minimum** are also used.

DEFINITIONS

Let I be the domain of f.

$f(c)$ is a **relative minimum** if there exists within I an open interval I_1 containing c such that $f(c) \leq f(x)$, for all x in I_1;

and

$f(c)$ is a **relative maximum** if there exists within I an open interval I_2 containing c such that $f(c) \geq f(x)$, for all x in I_2.

A relative maximum can be thought of loosely as the second coordinate of a "peak" that may or may not be the highest point over all of I. Similarly, a relative minimum can

be thought of as the second coordinate of a "valley" that may or may not be the lowest point on I. The second coordinates of the points that are the highest and the lowest on the interval are, respectively, the **absolute maximum** and the **absolute minimum**. For now, we focus on finding relative maximum or minimum values, collectively referred to as **relative extrema** (singular: **extremum**).

Look again at the graph in Fig. 2. The x-values at which a continuous function has relative extrema are those values for which the derivative is 0 or for which the derivative does not exist—the critical values.

THEOREM 2

If a function f has a relative extreme value $f(c)$ on an open interval, then c is a critical value, so

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ does not exist.}$$

A *relative extreme point*, $(c, f(c))$, is higher or lower than all other points over some open interval containing c. A relative minimum point will have a y-value that is lower than that of points both to the left and to the right of it, and, similarly, a relative maximum point will have a y-value that is higher than that of points to the left and right of it. Thus, relative extrema cannot be located at the endpoints of a closed interval, since an endpoint lacks "both sides" with which to make the necessary comparisons. However, as we will see in Section 2.4, endpoints *can* be absolute extrema. Note that the right endpoint of the curve in Fig. 2 is the absolute maximum point.

Theorem 2 is very useful, but it is important to understand it precisely. What it says is that to find relative extrema, we need only consider those inputs for which the derivative is 0 or for which it does not exist. We can think of a critical value as a *candidate* for a value where a relative extremum *might* occur. That is, Theorem 2 does not say that every critical value will yield a relative maximum or minimum. Consider, for example, the graph of

$$f(x) = (x - 1)^3 + 2,$$

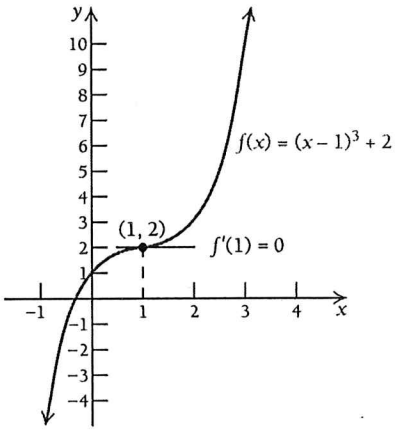
shown at the right. Note that

$$f'(x) = 3(x - 1)^2,$$

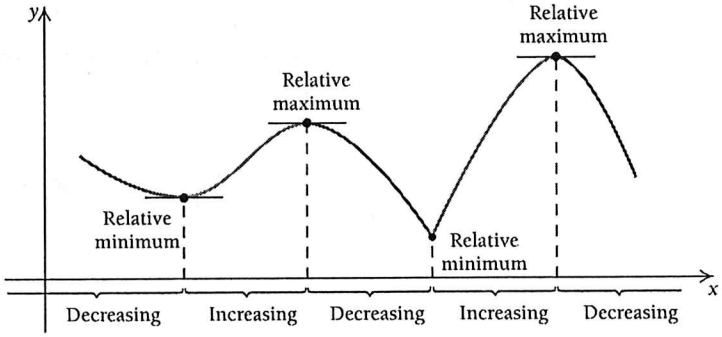
and

$$f'(1) = 3(1 - 1)^2 = 0.$$

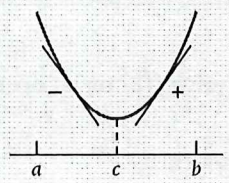
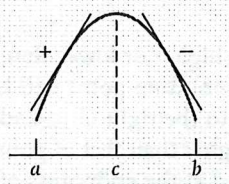
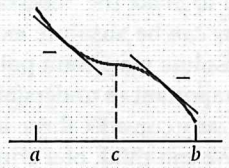
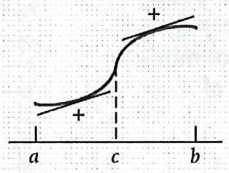
The function has $c = 1$ as a critical value, but has no relative maximum or minimum at that value.



Theorem 2 does say that if a relative maximum or minimum occurs, then the first coordinate of that extremum will be a critical value. How can we tell when the existence of a critical value leads us to a relative extremum? The following graph leads us to a test.



Note that at a critical value where there is a relative minimum, the function is decreasing to the left of the critical value and increasing to the right. At a critical value where there is a relative maximum, the function is increasing to the left of the critical value and decreasing to the right. In both cases, the derivative changes signs on either side of the critical value.

Graph over the interval (a, b)	$f(c)$	Sign of $f'(x)$ for x in (a, c)	Sign of $f'(x)$ for x in (c, b)	Increasing or decreasing
	Relative minimum	−	+	Decreasing on (a, c); increasing on (c, b)
	Relative maximum	+	−	Increasing on (a, c); decreasing on (c, b)
	No relative maxima or minima	−	−	Decreasing on (a, b)
	No relative maxima or minima	+	+	Increasing on (a, b)

Derivatives tell us when a function is increasing or decreasing. This leads us to the First-Derivative Test.

THEOREM 3 The First-Derivative Test for Relative Extrema

For any continuous function f that has exactly one critical value c in an open interval (a, b):

F1. f has a relative minimum at c if $f'(x) < 0$ on (a, c) and $f'(x) > 0$ on (c, b). That is, f is decreasing to the left of c and increasing to the right of c.

F2. f has a relative maximum at c if $f'(x) > 0$ on (a, c) and $f'(x) < 0$ on (c, b). That is, f is increasing to the left of c and decreasing to the right of c.

F3. f has neither a relative maximum nor a relative minimum at c if $f'(x)$ has the same sign on (a, c) as on (c, b).

Now let's see how we can use the First-Derivative Test to find relative extrema and create accurate graphs.

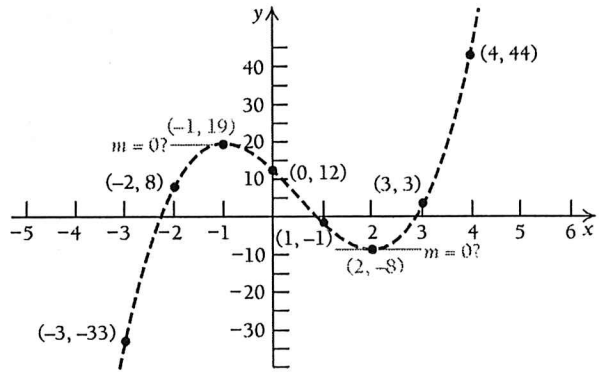
EXAMPLE 1 Graph the function f given by

$$f(x) = 2x^3 - 3x^2 - 12x + 12,$$

and find the relative extrema.

Solution Suppose that we are trying to graph this function but don't know any calculus. What can we do? We could plot several points to determine in which direction the graph seems to be turning. Let's pick some x-values and see what happens.

x	$f(x)$
-3	-33
-2	8
-1	19
0	12
1	-1
2	-8
3	3
4	44



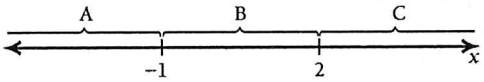
We plot the points and use them to sketch a "best guess" of the graph, shown as the dashed line in the figure above. According to this rough sketch, it appears that the graph has a tangent line with slope 0 somewhere around $x = -1$ and $x = 2$. But how do we know for sure? We use calculus to support our observations. We begin by finding a general expression for the derivative:

$$f'(x) = 6x^2 - 6x - 12.$$

We next determine where $f'(x)$ does not exist or where $f'(x) = 0$. Since we can evaluate $f'(x) = 6x^2 - 6x - 12$ for any real number, there is no value for which $f'(x)$ does not exist. So the only possibilities for critical values are those where $f'(x) = 0$, locations at which there are horizontal tangents. To find such values, we solve $f'(x) = 0$:

$$\begin{aligned}
6x^2 - 6x - 12 &= 0 \\
x^2 - x - 2 &= 0 && \text{Dividing both sides by 6} \\
(x + 1)(x - 2) &= 0 && \text{Factoring} \\
x + 1 = 0 \quad &\text{or} \quad x - 2 = 0 && \text{Using the Principle of Zero Products} \\
x = -1 \quad &\text{or} \quad x = 2.
\end{aligned}$$

The critical values are -1 and 2. Since it is at these values that a relative maximum or minimum might exist, we examine the intervals on each side of the critical values: A is $(-\infty, -1)$, B is $(-1, 2)$, and C is $(2, \infty)$, as shown below.

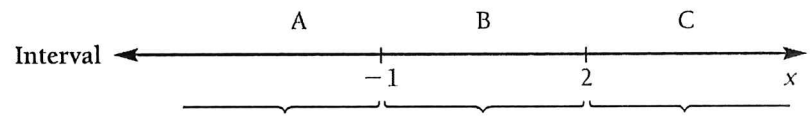


Next, we analyze the sign of the derivative on each interval. If $f'(x)$ is positive for one value in the interval, then it will be positive for all values in the interval. Similarly, if it is negative for one value, it will be negative for all values in the interval. Thus, we choose a test value in each interval and make a substitution. The test values we choose are -2, 0, and 4.

A: Test -2, $f'(-2) = 6(-2)^2 - 6(-2) - 12$
$= 24 + 12 - 12 = 24 > 0$;

B: Test 0, $f'(0) = 6(0)^2 - 6(0) - 12 = -12 < 0$;

C: Test 4, $f'(4) = 6(4)^2 - 6(4) - 12$
$= 96 - 24 - 12 = 60 > 0$.



Test Value	$x = -2$	$x = 0$	$x = 4$
Sign of $f'(x)$	$f'(-2) > 0$	$f'(0) < 0$	$f'(4) > 0$
Result	f is increasing on $(-\infty, -1)$	f is decreasing on $(-1, 2)$	f is increasing on $(2, \infty)$

Change indicates a relative maximum.

Change indicates a relative minimum.

Therefore, by the First-Derivative Test,

f has a relative maximum at $x = -1$ given by

$$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 12 \qquad \text{Substituting into the } \textit{original} \text{ function}$$

$$= 19 \qquad \text{This is a relative maximum.}$$

and f has a relative minimum at $x = 2$ given by

$$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 12 = -8. \qquad \text{This is a relative minimum.}$$

Thus, there is a relative maximum at $(-1, 19)$ and a relative minimum at $(2, -8)$, as we suspected from the sketch of the graph.

The information we have obtained from the first derivative can be very useful in sketching a graph of the function. We know that this polynomial is continuous, and we know where the function is increasing, where it is decreasing, and where it has relative extrema. We complete the graph by using a calculator to generate some additional function values. The graph of the function, shown below in red, has been scaled to clearly show its curving nature.

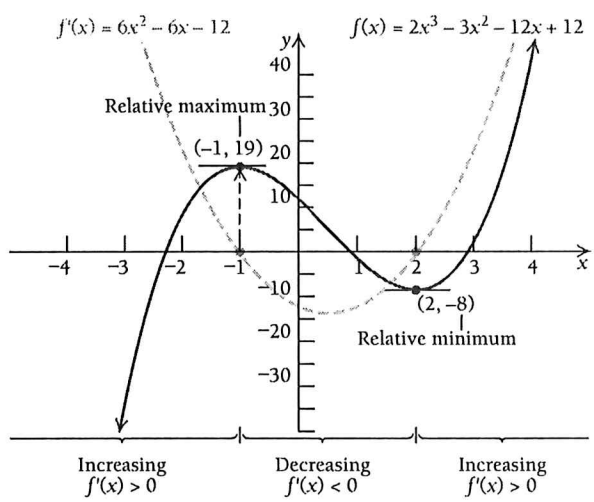
TECHNOLOGY CONNECTION

Exploratory

Consider the function f given by

$$f(x) = x^3 - 3x + 2.$$

Graph both f and f' using the same set of axes. Examine the graphs using the TABLE and TRACE features. Where do you think the relative extrema of $f(x)$ occur? Where is the derivative equal to 0? Where does $f(x)$ have critical values?



For reference, the graph of the derivative is shown in blue. Note that $f'(x) = 0$ where $f(x)$ has relative extrema. We summarize the behavior of this function as follows, by noting where it is increasing or decreasing, and by characterizing its critical points:

- The function f is increasing over the interval $(-\infty, -1)$.
- The function f has a relative maximum at the point $(-1, 19)$.
- The function f is decreasing over the interval $(-1, 2)$.
- The function f has a relative minimum at the point $(2, -8)$.
- The function f is increasing over the interval $(2, \infty)$.

Quick Check 1

Graph the function g given by $g(x) = x^3 - 27x - 6$, and find the relative extrema.

Quick Check 1

Interval notation and point notation look alike. Be clear when stating your answers whether you are identifying an interval or a point.

To use the first derivative for graphing a function f:

1. Find all critical values by determining where $f'(x)$ is 0 and where $f'(x)$ is undefined (but $f(x)$ is defined). Find $f(x)$ for each critical value.
2. Use the critical values to divide the x-axis into intervals and choose a test value in each interval.
3. Find the sign of $f'(x)$ for each test value chosen in step 2, and use this information to determine where $f(x)$ is increasing or decreasing and to classify any extrema as relative maxima or minima.
4. Plot some additional points and sketch the graph.

The *derivative* f' is used to find the critical values of f. The test values are substituted into the *derivative* f', and the function values are found using the *original* function f.

EXAMPLE 2 Find the relative extrema of the function f given by

$$f(x) = 2x^3 - x^4.$$

Then sketch the graph.

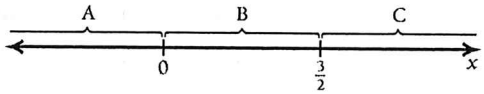
Solution First, we must determine the critical values. To do so, we find $f'(x)$:

$$f'(x) = 6x^2 - 4x^3.$$

Next, we find where $f'(x)$ does not exist or where $f'(x) = 0$. Since $f'(x) = 6x^2 - 4x^3$ is a polynomial, it exists for all real numbers x. Therefore, the only candidates for critical values are where $f'(x) = 0$, that is, where the tangent line is horizontal:

$$\begin{aligned} 6x^2 - 4x^3 &= 0 && \text{Setting } f'(x) \text{ equal to } 0 \\ 2x^2(3 - 2x) &= 0 && \text{Factoring} \\ 2x^2 = 0 \quad &\text{or} \quad 3 - 2x = 0 \\ x^2 = 0 \quad &\text{or} \quad 3 = 2x \\ x = 0 \quad &\text{or} \quad x = \tfrac{3}{2}. \end{aligned}$$

The critical values are 0 and $\frac{3}{2}$. We use these values to divide the x-axis into three intervals as shown below: A is $(-\infty, 0)$; B is $\left(0, \frac{3}{2}\right)$; and C is $\left(\frac{3}{2}, \infty\right)$.



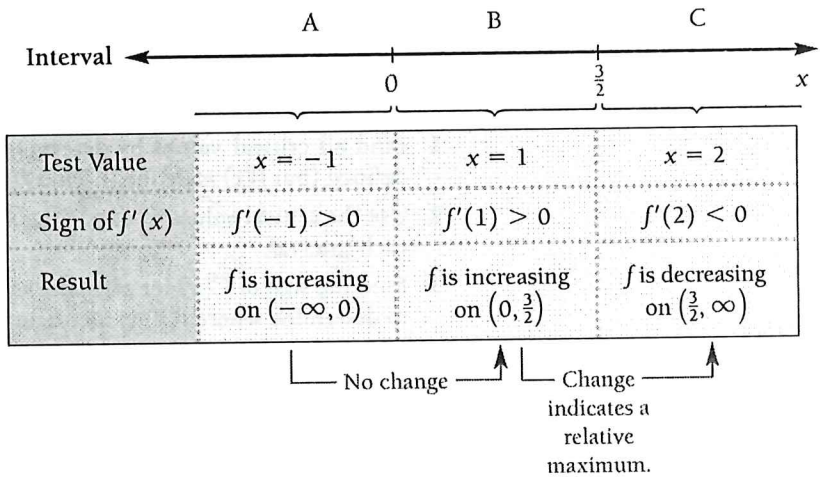
Note that $f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^4 = \frac{27}{16}$ and $f(0) = 2 \cdot 0^3 - 0^4 = 0$ are possible extrema.

We now determine the sign of the derivative on each interval by choosing a test value in each interval and substituting. We generally choose test values for which it is easy to compute $f'(x)$.

A: Test -1, $f'(-1) = 6(-1)^2 - 4(-1)^3$
$= 6 + 4 = 10 > 0;$

B: Test 1, $f'(1) = 6(1)^2 - 4(1)^3$
$= 6 - 4 = 2 > 0;$

C: Test 2, $f'(2) = 6(2)^2 - 4(2)^3$
$= 24 - 32 = -8 < 0.$

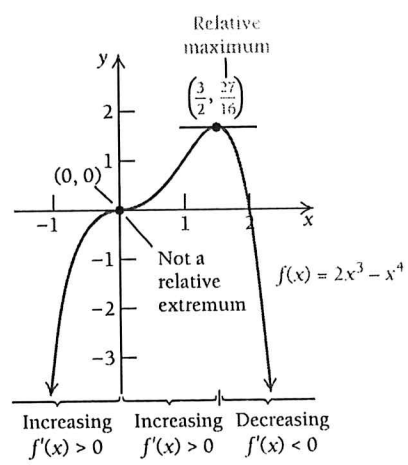


Test Value	$x = -1$	$x = 1$	$x = 2$
Sign of $f'(x)$	$f'(-1) > 0$	$f'(1) > 0$	$f'(2) < 0$
Result	f is increasing on $(-\infty, 0)$	f is increasing on $\left(0, \frac{3}{2}\right)$	f is decreasing on $\left(\frac{3}{2}, \infty\right)$

Therefore, by the First-Derivative Test, f has no extremum at $x = 0$ (since $f(x)$ is increasing on both sides of 0) and has a relative maximum at $x = \frac{3}{2}$. Thus, $f\left(\frac{3}{2}\right)$, or $\frac{27}{16}$, is a relative maximum.

We use the information obtained to sketch the graph below. Other function values are listed in the table.

x	$f(x)$, approximately
-1	-3
-0.5	-0.31
0	0
0.5	0.19
1	1
1.25	1.46
2	0



We summarize the behavior of f:

- The function f is increasing over the interval $(-\infty, 0)$.
- The function f has a critical point at $(0, 0)$, which is neither a minimum nor a maximum.

- The function f is increasing over the interval $\left(0, \frac{3}{2}\right)$.
- The function f has relative maximum at the point $\left(\frac{3}{2}, \frac{27}{16}\right)$.
- The function f is decreasing over the interval $\left(\frac{3}{2}, \infty\right)$.

Since f is increasing over the intervals $(-\infty, 0)$ and $\left(0, \frac{3}{2}\right)$, we can say that f is increasing over $\left(-\infty, \frac{3}{2}\right)$ despite the fact that $f'(0) = 0$ within this interval. In this case, we can observe that any secant line connecting two points within this interval will have a positive slope.

Quick Check 2

Find the relative extrema of the function h given by $h(x) = x^4 - \frac{8}{3}x^3$. Then sketch the graph.

EXAMPLE 3 Find the relative extrema of the function f given by

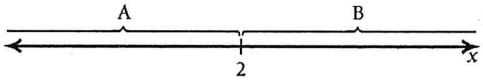
$$f(x) = (x - 2)^{2/3} + 1.$$

Then sketch the graph.

Solution First, we determine the critical values. To do so, we find $f'(x)$:

$$f'(x) = \frac{2}{3}(x - 2)^{-1/3}$$
$$= \frac{2}{3\sqrt[3]{x - 2}}.$$

Next, we find where $f'(x)$ does not exist or where $f'(x) = 0$. Note that $f'(x)$ does not exist at 2, although $f(x)$ does. Thus, 2 is a critical value. Since the only way for a fraction to be 0 is if its numerator is 0, we see that $f'(x) = 0$ has no solution. Thus, 2 is the only critical value. We use 2 to divide the x-axis into the intervals A, which is $(-\infty, 2)$, and B, which is $(2, \infty)$. Note that $f(2) = (2 - 2)^{2/3} + 1 = 1$.



To determine the sign of the derivative, we choose a test value in each interval and substitute each value into the derivative. We choose test values 0 and 3. It is not necessary to find an exact value of the derivative; we need only determine the sign. Sometimes we can do this by just examining the formula for the derivative:

$$\text{A:} \quad \text{Test } 0, \quad f'(0) = \frac{2}{3\sqrt[3]{0 - 2}} < 0;$$

$$\text{B:} \quad \text{Test } 3, \quad f'(3) = \frac{2}{3\sqrt[3]{3 - 2}} > 0.$$

Interval ← 2 → x

Test Value	$x = 0$	$x = 3$
Sign of $f'(x)$	$f'(0) < 0$	$f'(3) > 0$
Result	f is decreasing on $(-\infty, 2)$	f is increasing on $(2, \infty)$

Change indicates a relative minimum.

TECHNOLOGY CONNECTION

EXERCISES

In Exercises 1 and 2, consider the function f given by

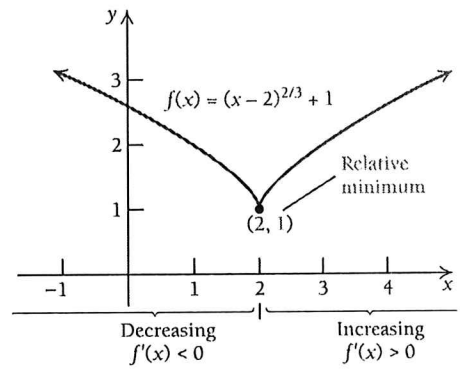
$$f(x) = 2 - (x - 1)^{2/3}.$$

1. Graph the function using the viewing window $[-4, 6, -2, 4]$.
2. Graph the first derivative. What happens to the graph of the derivative at the critical values?

Since we have a change from decreasing to increasing, we conclude from the First-Derivative Test that a relative minimum occurs at $(2, f(2))$, or $(2, 1)$. The graph has *no* tangent line at $(2, 1)$ since $f'(2)$ does not exist.

We use the information obtained to sketch the graph. Other function values are listed in the table.

x	$f(x)$, approximately
-1	3.08
-0.5	2.84
0	2.59
0.5	2.31
1	2
1.5	1.63
2	1
2.5	1.63
3	2
3.5	2.31
4	2.59



We summarize the behavior of f:

- The function f is decreasing over the interval $(-\infty, 2)$.
- The function f has a relative minimum at the point $(2, 1)$.
- The function f is increasing over the interval $(2, \infty)$.

Quick Check 3

Find the relative extrema of the function g given by $g(x) = 3 - x^{1/3}$. Then sketch the graph.

Quick Check 3

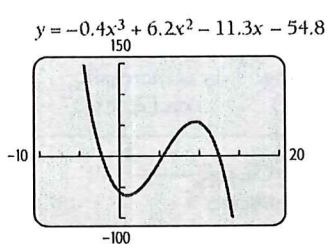
TECHNOLOGY CONNECTION

Finding Relative Extrema

To explore some methods for approximating relative extrema, let's find the relative extrema of

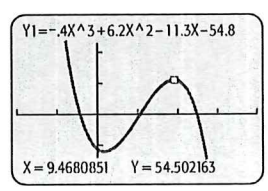
$$f(x) = -0.4x^3 + 6.2x^2 - 11.3x - 54.8.$$

We first graph the function, using a window that reveals the curvature.



Method 1: TRACE

Beginning with the window shown at left, we press TRACE and move the cursor along the curve, noting where relative extrema might occur.

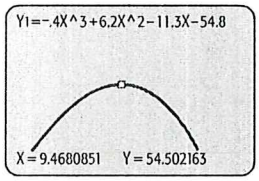


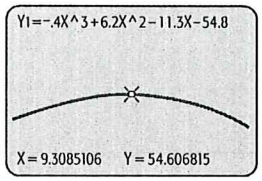
A relative maximum seems to be about $y = 54.5$ at $x = 9.47$. We can refine the approximation by zooming in to obtain the following window. We press TRACE and move

(continued)

Finding Relative Extrema (*continued*)

the cursor along the curve, again noting where the y-value is largest. The approximation is about $y = 54.61$ at $x = 9.31$.





We can continue in this manner until the desired accuracy is achieved.

Method 2: TABLE

We can also use the TABLE feature, adjusting starting points and step values to improve accuracy:

TblStart = 9.3 ΔTbl = .01

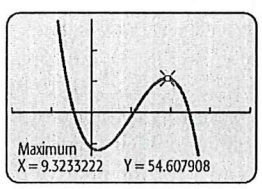
X	Y1
9.3	54.605
9.31	54.607
9.32	54.608
9.33	54.608
9.34	54.607
9.35	54.604
9.36	54.601

X = 9.32

The approximation seems to be nearly $y = 54.61$ at an x-value between 9.32 and 9.33. We could next set up a new table showing function values between $f(9.32)$ and $f(9.33)$ to refine the approximation.

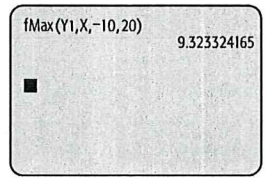
Method 3: MAXIMUM, MINIMUM

Using the MAXIMUM option from the CALC menu, we find that a relative maximum of about 54.61 occurs at $x \approx 9.32$.

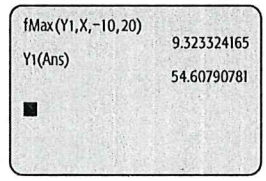


Method 4: fMax *or* fMin

This feature calculates a relative maximum or minimum value over any specified closed interval. We see from the initial graph that a relative maximum occurs in the interval $[-10, 20]$. Using the fMax option from the MATH menu, we see that a relative maximum occurs on $[-10, 20]$ when $x \approx 9.32$.

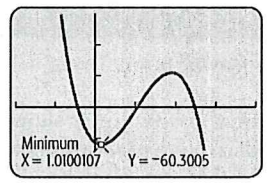


To obtain the maximum value, we evaluate the function at the given x-value, obtaining the following.



The approximation is about $y = 54.61$ at $x = 9.32$.

Using any of these methods, we find the relative minimum to be about $y = -60.30$ at $x = 1.01$.



EXERCISE

1. Using one of the methods just described, approximate the relative extrema of the function in Example 1.

TECHNOLOGY CONNECTION

Finding Relative Extrema with iPlot

We can use iPlot to graph a function and its derivative and then find relative extrema.

iPlot has the capability of graphing a function and its derivative on the same set of axes, though it does not give a formula for the derivative but merely draws the graph. As an example, let's consider the function given by $f(x) = x^3 - 3x + 4$.

To graph a function and its derivative, first open the iPlot app on your iPhone or iPad. You will get a screen like the one in Fig. 1. Notice the four icons at the bottom. The Functions icon is highlighted. Press [+] in the upper right; then enter $f(x) = x^3 - 3x + 4$ using the notation x^3–3*x+4. Press Done at the upper right and then Plot at the lower right (Fig. 2).

(*continued*)

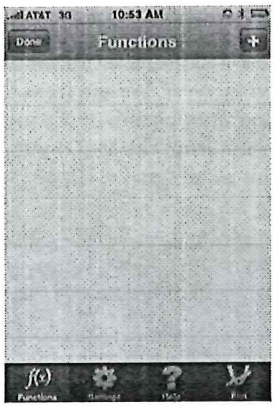


FIGURE 1

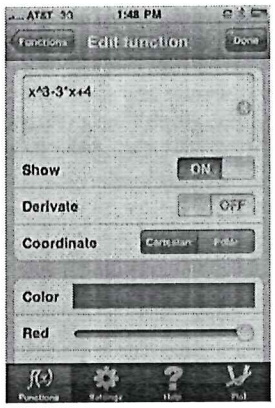


FIGURE 2

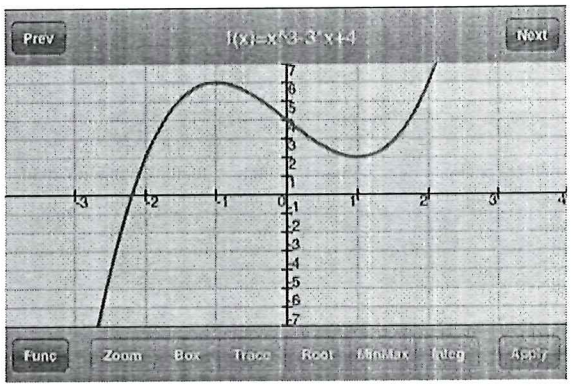


FIGURE 3

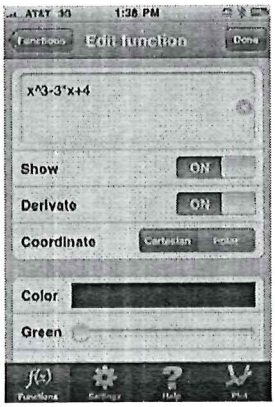


FIGURE 4

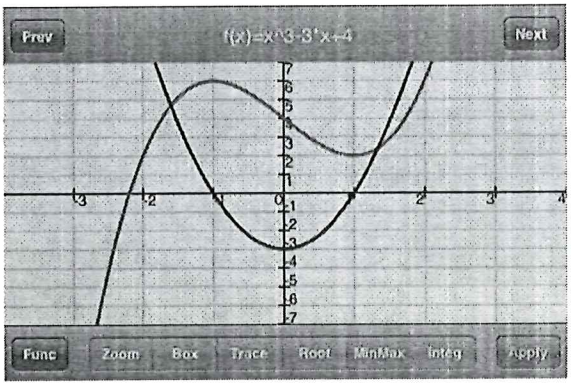


FIGURE 5

The graph of $f(x) = x^3 - 3x + 4$ is shown in red in Fig. 3. To graph the derivative of f, first click on the Functions icon again, and then press $\boxed{+}$. You will get the screen shown in Fig. 4.

Next, slide the Derivate button to the left. ("Derivate" means "Differentiate.") Then enter the same function as before, x^3–3*x+4, and press Done. D(x^3–3*x+4) will appear in the second line. Press Plot, and you will see both functions plotted, as shown in Fig. 5. Look over the two graphs, and use Trace to find various function values. Press Prev to jump between the function and its derivative. Look for x-values where the derivative is 0. What happens at these values of the original function? Examining the graphs in this way reveals that the graph of $f(x) = x^3 - 3x + 4$ has a relative maximum point at $(-1, 6)$ and a relative minimum point at $(1, 2)$.

iPlot has an additional feature that allows us to be more certain about these relative extrema. Go back to the original plot of $f(x) = x^3 - 3x + 4$ (Fig. 3), and press Settings. Change the window to $[-3, 3, 12, -10]$ to better see the graph. Press the MinMax button at the bottom. Touch the screen as closely as possible to what might be a relative extremum. See Figs. 6 and 7 for the relative maximum. The relative minimum can be found similarly.

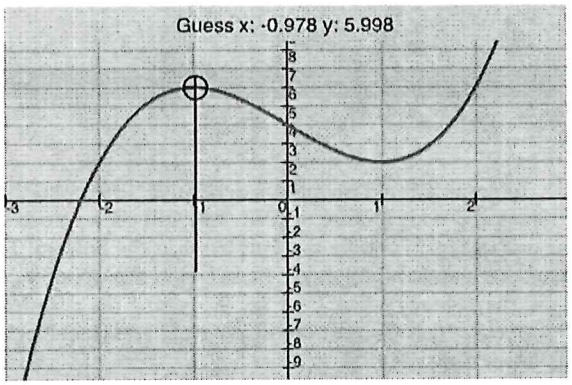


FIGURE 6

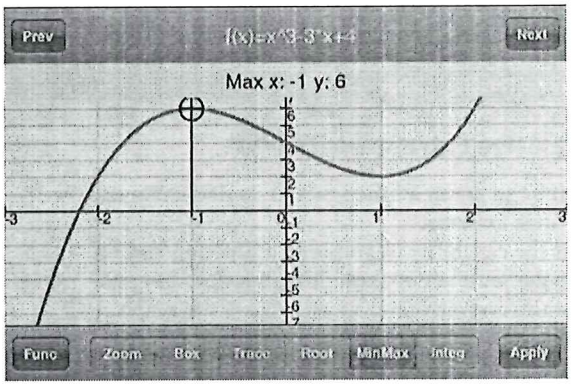


FIGURE 7

(continued)

EXERCISES

For each function, use iPlot to create the graph and find the derivative. Then explore each graph to look for possible relative extrema. Use MinMax to determine the relative extrema.

1. $f(x) = 2x^3 - x^4$
2. $f(x) = x(200 - x)$
3. $f(x) = x^3 - 6x^2$
4. $f(x) = -4.32 + 1.44x + 3x^2 - x^3$
5. $g(x) = x\sqrt{4 - x^2}$
6. $g(x) = \dfrac{4x}{x^2 + 1}$
7. $f(x) = \dfrac{x^2 - 3x}{x - 1}$
8. $f(x) = |x + 2| - 3$

Section Summary

- A function f is *increasing* over an interval I if, for all a and b in I such that $a < b, f(a) < f(b)$. Equivalently, the slope of the secant line connecting a and b is positive:
$$\frac{f(b) - f(a)}{b - a} > 0.$$
- A function f is *decreasing* over an interval I if, for all a and b in I such that $a < b, f(a) > f(b)$. Equivalently, the slope of the secant line connecting a and b is negative:
$$\frac{f(b) - f(a)}{b - a} < 0.$$
- Using the first derivative, a function is *increasing* over an open interval I if, for all x in I, the slope of the tangent line at x is positive; that is, $f'(x) > 0$. Similarly, a function is *decreasing* over an open interval I if, for all x in I, the slope of the tangent line is negative; that is, $f'(x) < 0$.
- A *critical value* is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist. The point $(c, f(c))$ is called a *critical point.*
- A relative maximum point is higher than all other points in some interval containing it. Similarly, a relative minimum point is lower than all other points in some interval containing it. The y-value of such a point is called a relative maximum (or minimum) *value* of the function.
- Minimum and maximum points are collectively called *extrema.*
- Critical values are candidates for possible relative extrema. The *First-Derivative Test* is used to classify a critical value as a relative minimum, a relative maximum, or neither.

EXERCISE SET 2.1

Find the relative extrema of each function, if they exist. List each extremum along with the x-value at which it occurs. Then sketch a graph of the function.

1. $f(x) = x^2 + 4x + 5$
2. $f(x) = x^2 + 6x - 3$
3. $f(x) = 5 - x - x^2$
4. $f(x) = 2 - 3x - 2x^2$
5. $g(x) = 1 + 6x + 3x^2$
6. $F(x) = 0.5x^2 + 2x - 11$
7. $G(x) = x^3 - x^2 - x + 2$
8. $g(x) = x^3 + \frac{1}{2}x^2 - 2x + 5$
9. $f(x) = x^3 - 3x + 6$
10. $f(x) = x^3 - 3x^2$
11. $f(x) = 3x^2 + 2x^3$
12. $f(x) = x^3 + 3x$
13. $g(x) = 2x^3 - 16$
14. $F(x) = 1 - x^3$
15. $G(x) = x^3 - 6x^2 + 10$
16. $f(x) = 12 + 9x - 3x^2 - x^3$
17. $g(x) = x^3 - x^4$
18. $f(x) = x^4 - 2x^3$
19. $f(x) = \frac{1}{3}x^3 - 2x^2 + 4x - 1$
20. $F(x) = -\frac{1}{3}x^3 + 3x^2 - 9x + 2$
21. $g(x) = 2x^4 - 20x^2 + 18$
22. $f(x) = 3x^4 - 15x^2 + 12$
23. $F(x) = \sqrt[3]{x - 1}$
24. $G(x) = \sqrt[3]{x + 2}$
25. $f(x) = 1 - x^{2/3}$
26. $f(x) = (x + 3)^{2/3} - 5$
27. $G(x) = \dfrac{-8}{x^2 + 1}$
28. $F(x) = \dfrac{5}{x^2 + 1}$
29. $g(x) = \dfrac{4x}{x^2 + 1}$
30. $g(x) = \dfrac{x^2}{x^2 + 1}$

31. $f(x) = \sqrt[3]{x}$

32. $f(x) = (x + 1)^{1/3}$

33. $g(x) = \sqrt{x^2 + 2x + 5}$

34. $F(x) = \dfrac{1}{\sqrt{x^2 + 1}}$

35–68. Check the results of Exercises 1–34 using a calculator.

For Exercises 69–84, draw a graph to match the description given. Answers will vary.

69. $f(x)$ is increasing over $(-\infty, 2)$ and decreasing over $(2, \infty)$.

70. $g(x)$ is decreasing over $(-\infty, -3)$ and increasing over $(-3, \infty)$.

71. $G(x)$ is decreasing over $(-\infty, 4)$ and $(9, \infty)$ and increasing over $(4, 9)$.

72. $F(x)$ is increasing over $(-\infty, 5)$ and $(12, \infty)$ and decreasing over $(5, 12)$.

73. $g(x)$ has a positive derivative over $(-\infty, -3)$ and a negative derivative over $(-3, \infty)$.

74. $f(x)$ has a negative derivative over $(-\infty, 1)$ and a positive derivative over $(1, \infty)$.

75. $F(x)$ has a negative derivative over $(-\infty, 2)$ and $(5, 9)$ and a positive derivative over $(2, 5)$ and $(9, \infty)$.

76. $G(x)$ has a positive derivative over $(-\infty, -2)$ and $(4, 7)$ and a negative derivative over $(-2, 4)$ and $(7, \infty)$.

77. $f(x)$ has a positive derivative over $(-\infty, 3)$ and $(3, 9)$, a negative derivative over $(9, \infty)$, and a derivative equal to 0 at $x = 3$.

78. $g(x)$ has a negative derivative over $(-\infty, 5)$ and $(5, 8)$, a positive derivative over $(8, \infty)$, and a derivative equal to 0 at $x = 5$.

79. $F(x)$ has a negative derivative over $(-\infty, -1)$ and a positive derivative over $(-1, \infty)$, and $F'(-1)$ does not exist.

80. $G(x)$ has a positive derivative over $(-\infty, 0)$ and $(3, \infty)$ and a negative derivative over $(0, 3)$, but neither $G'(0)$ nor $G'(3)$ exists.

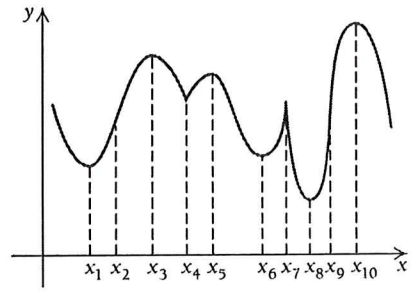
81. $f(x)$ has a negative derivative over $(-\infty, -2)$ and $(1, \infty)$ and a positive derivative over $(-2, 1)$, and $f'(-2) = 0$, but $f'(1)$ does not exist.

82. $g(x)$ has a positive derivative over $(-\infty, -3)$ and $(0, 3)$, a negative derivative over $(-3, 0)$ and $(3, \infty)$, and a derivative equal to 0 at $x = -3$ and $x = 3$, but $g'(0)$ does not exist.

83. $H(x)$ is increasing over $(-\infty, \infty)$, but the derivative does not exist at $x = 1$.

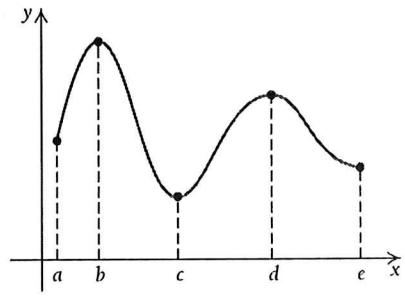
84. $K(x)$ is decreasing over $(-\infty, \infty)$, but the derivative does not exist at $x = 0$ and $x = 2$.

85. Consider this graph.



Explain the idea of a critical value. Then determine which x-values are critical values, and state why.

86. Consider this graph.



Using the graph and the intervals noted, explain how to relate the concept of the function being increasing or decreasing to the first derivative.

APPLICATIONS

Business and Economics

87. Employment. According to the U.S. Bureau of Labor Statistics, the number of professional services employees fluctuated during the period 2000–2009, as modeled by

$$E(t) = -28.31t^3 + 381.86t^2 - 1162.07t + 16{,}905.87,$$

where t is the number of years since 2000 ($t = 0$ corresponds to 2000) and E is thousands of employees. (*Source*: www.data.bls.gov.) Find the relative extrema of this function, and sketch the graph. Interpret the meaning of the relative extrema.

88. Advertising. Brody Electronics estimates that it will sell N units of a new toy after spending a thousands of dollars on advertising, where

$$N(a) = -a^2 + 300a + 6, \quad 0 \le a \le 300.$$

Find the relative extrema and sketch a graph of the function.

Life and Physical Sciences

89. Temperature during an illness. The temperature of a person during an intestinal illness is given by

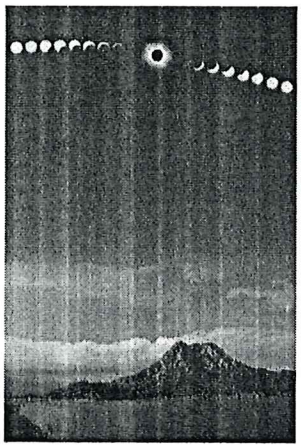
$$T(t) = -0.1t^2 + 1.2t + 98.6, \quad 0 \le t \le 12,$$

where T is the temperature (°F) at time t, in days. Find the relative extrema and sketch a graph of the function.

90. Solar eclipse. On January 15, 2010, the longest annular solar eclipse until 3040 occurred over Africa and the Indian Ocean (in an annular eclipse, the sun is partially obscured by the moon and looks like a ring). The path of the full eclipse on the earth's surface is modeled by

$$f(x) = 0.0125x^2 - 1.157x + 22.864, \quad 15 < x < 90,$$

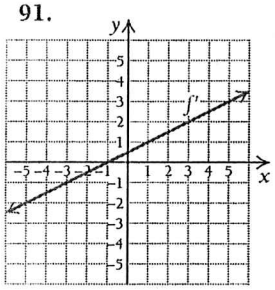
where x is the number of degrees of longitude east of the prime meridian and $f(x)$ is the number of degrees of latitude north (positive) or south (negative) of the equator. (*Source*: NASA.) Find the longitude and latitude of the southernmost point at which the full eclipse could be viewed.



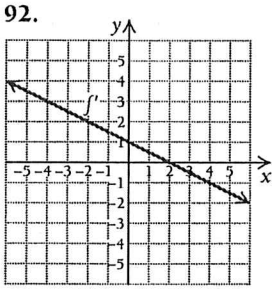
SYNTHESIS

In Exercises 91–96, the graph of a derivative f' is shown. Use the information in each graph to determine where f is increasing or decreasing and the x-values of any extrema. Then sketch a possible graph of f.

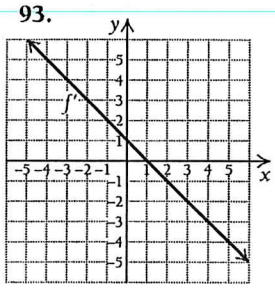
91.



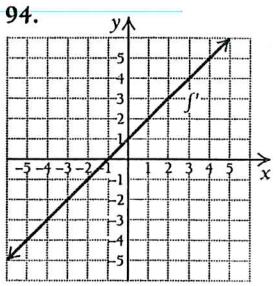
92.



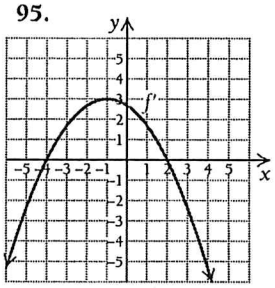
93.



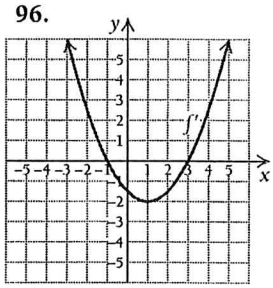
94.



95.



96.



TECHNOLOGY CONNECTION

Graph each function. Then estimate any relative extrema.

97. $f(x) = -x^6 - 4x^5 + 54x^4 + 160x^3 - 641x^2 - 828x + 1200$

98. $f(x) = x^4 + 4x^3 - 36x^2 - 160x + 400$

99. $f(x) = \sqrt[3]{|4 - x^2|} + 1$ **100.** $f(x) = x\sqrt{9 - x^2}$

Use your calculator's absolute-value feature to graph the following functions and determine relative extrema and intervals over which the function is increasing or decreasing. State the x-values at which the derivative does not exist.

101. $f(x) = |x - 2|$

102. $f(x) = |2x - 5|$

103. $f(x) = |x^2 - 1|$

104. $f(x) = |x^2 - 3x + 2|$

105. $f(x) = |9 - x^2|$

106. $f(x) = |-x^2 + 4x - 4|$

107. $f(x) = |x^3 - 1|$

108. $f(x) = |x^4 - 2x^2|$

Life science: caloric intake and life expectancy. *The data in the following table give, for various countries, daily caloric intake, projected life expectancy, and infant mortality. Use the data for Exercises 109 and 110.*

Country	Daily Caloric Intake	Life Expectancy at Birth (in years)	Infant Mortality (number of deaths before age 1 per 1000 births)
Argentina	3004	77	13
Australia	3057	82	5
Bolivia	2175	67	46
Canada	3557	81	5
Dominican Republic	2298	74	30
Germany	3491	79	4
Haiti	1835	61	62
Mexico	3265	76	17
United States	3826	78	6
Venezuela	2453	74	17

(Source: *U.N. FAO Statistical Yearbook, 2009.*)

109. Life expectancy and daily caloric intake.

a) Use the regression procedures of Section R.6 to fit a cubic function $y = f(x)$ to the data in the table, where x is daily caloric intake and y is life expectancy. Then fit a quartic function and decide which fits best. Explain.

b) What is the domain of the function?

c) Does the function have any relative extrema? Explain.

110. Infant mortality and daily caloric intake.

a) Use the regression procedures of Section R.6 to fit a cubic function $y = f(x)$ to the data in the table, where x is daily caloric intake and y is infant mortality. Then fit a quartic function and decide which fits best. Explain.

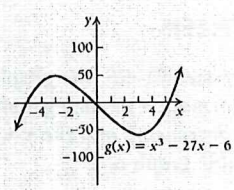
b) What is the domain of the function?

c) Does the function have any relative extrema? Explain.

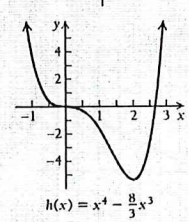
111. Describe a procedure that can be used to select an appropriate viewing window for the functions given in **(a)** Exercises 1–16 and **(b)** Exercises 97–100.

Answers to Quick Checks

1. Relative maximum at $(-3, 48)$, relative minimum at $(3, -60)$



2. Relative minimum at $\left(2, -\frac{16}{3}\right)$



3. There are no extrema.