Absolute Value Summary

Refer to the examples below to solve all varieties of absolute value equations and inequalities.
(You should insert interval notation under the algebraic answer for the inequalities.)

Note that domain considerations for equation cases are not stated. For example, in the problem

$|x-4| = 9+x$, the two cases fully stated with domain consideration would be:

$x-4=9+x$ when $x-4>0,$ that is, when $x>4;$ while $x-4=-(9+x)$ when $x-4<0,$ that is, when $x<4.$

Although it's important to know when each case holds, as long as you check your answer into the original problem, you will be sure to discard anything that is a domain contradiction. The examples show this in a simple way.

Example A

$|x-4| = 9$

$x-4=9$ or $x-4=-9$

$x=13$ or $x=-5$

$|x-4| < 9$

$-9 < x-4 < 9$

$-5 < x < 13$

$|x-4| > 9$

$x-4<-9$ or $x-4>9$

$x<-5$ or $x>13$

Example B

$|x-4|+2=9$

$|x-4|=7$

$x-4=7$ or $x-4=-7$

$x=11$ or $x=-3$

$|x-4|+2<9$

$|x-4|<7$

$-7<x-4<7$

$-3<x<11$

$|x-4|+2>9$

$|x-4|>7$

$x-4<-7$ or $x-4>7$

$x<-3$ or $x>11$

Example C

$|x-4| = 9+x$

$x-4=9+x$ or $x-4=-(9+x)$

$-4=9$ (contradiction)

or $2x=-5,$ so $x=-\frac{5}{2}$

Check $x=-\frac{5}{2}$ into problem:

$\left|-\frac{5}{2}-4\right| = 9+-\frac{5}{2}$

$\left|-\frac{13}{2}\right| = \frac{13}{2}$ (true)

$|x-4| < 9+x$

$-(9+x) < x-4 < 9+x$

$-9-x < x-4 < 9+x$

Right side: $-4<9$ (not needed)

Left side: $-2x<5,$ so $x>-\frac{5}{2}$

Check $x=0$ into original problem:

$|0-4| < 9+0,$

$|-4| < 9$ (true)

$|x-4| > 9+x$

$x<-(9+x)$ or $x-4>9+x$

$2x<-9$ or $-4>9$ (contradiction)

$x<-\frac{9}{2}$ is the solution interval

Check $x=-5$ into original problem:

$|-5-4| > 9+-5$

$9>4$ (true)

Finally, if absolute value appears on both sides of the equation or inequality, it is often easiest to solve by using the property that $|a|^2 = a^2$ to avoid the cases completely. Squaring both sides eliminates the fences and the problem becomes a typical quadratic or linear inequality.

Caution: This only works if there is nothing outside the absolute value fences on either side. If there is, you have to go back to using the cases. See example at end of summary.

So far, we had $|f(x)| = |g(x)|$, which gives the possibilities:

$f(x) = g(x), \quad f(x) = -g(x), \quad -f(x) = g(x), \quad -f(x) = -g(x)$

But two of these cases are repeats, so we have only two cases to consider.

A final example:

Example F

$|5x+12| = |x| - 4$

Because of that constant outside the fences, there are four distinct cases.

$5x+12 = x-4$ or $5x+12 = -x-4$ or $-(5x-12) = x-4$ or $-(5x-12) = -x-4$

Solving each gives $x = -4, \quad x = -\frac{16}{6}, \quad x = \frac{16}{6}, \quad x = 4$

Checking each reveals in fact that there is no solution! Sometimes an absolute value problem is impossible.

Occasionally, it is obvious from the start; for example, $|f(x)| = -1$ (an abs value will never be negative), that is, $|f(x)| < 0$ has no solution.