Dec 7 - Sec 3.4 Limits at infinity - Several examples

Definition $\lim_{x\to\infty} f(x) = L$ if that for every $\varepsilon > 0$

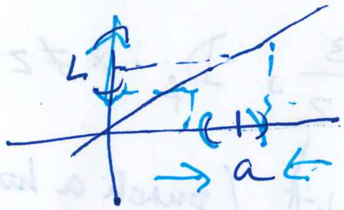
there is a corresponding number N such that

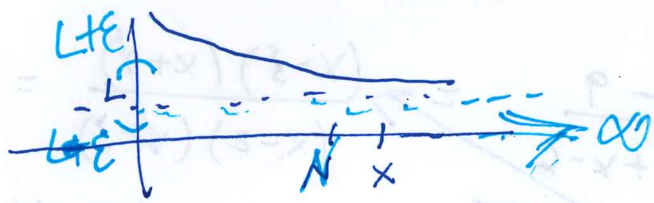
if $x > N$, $|f(x) - L| < \varepsilon$

Notice the difference btwn this limit def. and $\lim_{x\to a} f(x) = L$

which we recall is that for any $\varepsilon > 0$ there is a $\delta > 0$

such that if $0 < |x-a| < \delta$, $\underbrace{\quad}_{\text{then}}$ $|f(x) - L| < \varepsilon$.





Ex $\boxed{f(x) = x^3}$, $\lim_{x\to\infty} x^3 = \infty^3 = \infty$, $\lim_{x\to-\infty} x^3 = (-\infty)^3 = -\infty$ $\quad (-1)^3(\infty)^3$

$f(x) = x^3 - x^2 - x$, $\lim_{x\to\infty} f(x) = \infty^3 - \infty^2 - \infty$ IF

$$\lim_{x\to\infty} x^3\left(1 - \frac{1}{x} - \frac{1}{x^2}\right) = \lim_{x\to\infty} x^3 \cdot \lim_{x\to\infty} 1 - \frac{1}{x} - \frac{1}{x^2} = \infty^3 \cdot 1 = \infty$$

$$\lim_{x\to-\infty} x^3\left(1 - \frac{1}{x} - \frac{1}{x^2}\right) = \lim_{x\to-\infty} x^3 \cdot \lim_{x\to\infty}\left(1 - \frac{1}{x} - \frac{1}{x^2}\right) = -\infty \cdot 1 = -\infty$$

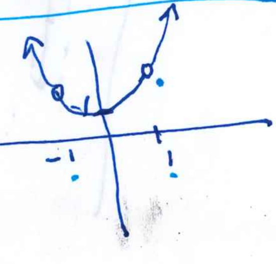
Ex $f(x) = \dfrac{x^4 - 1}{x^2 - 1}$ $\quad x^2 - 1 = 0$ at $x = \pm 1$ $\quad = \dfrac{\cancel{(x^2-1)}(x^2+1)}{\cancel{(x^2-1)}} = x^2 + 1$, $\quad$ Domain $x \neq \pm 1$

(No VA)

$$\boxed{\lim_{x\to 1} f(x) = 1^2 + 1 = 2, \quad \lim_{x\to -1} f(x) = 2}$$

There are holes at $(1, 2), (-1, 2)$

$\lim_{x\to\infty} x^2 + 1 = \infty$
$x\to-\infty$ No HA



$$\lim_{x\to 1} x^2 + 1 = 1^2 + 1 = 2$$

$$\lim_{x\to -1} x^2 + 1 = (-1)^2 + 1 = 2$$

$\underline{\text{Ex}}$ $f(x) = \dfrac{x^2-9}{x^2-4x-3} = \dfrac{(x-3)(x+3)}{(x-3)(x-1)} = \dfrac{x+3}{x-1}$

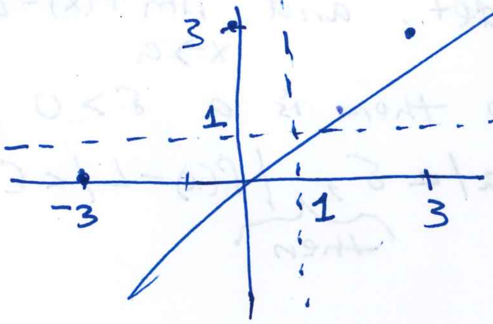
$D_f: x \neq 3, 1$; $\lim_{x\to 3} f(x) = \frac{6}{2} = 3$, $\lim_{x\to 1^+} f(x) = \infty$, $\lim_{x\to 1^-} f(x) = -\infty$

VA : $x = 1$

HA : $y = 1$

$\lim_{x\to\infty} \dfrac{x+3}{x-1} = \lim_{x\to\infty} \dfrac{1+3/x}{1-1/x} = \dfrac{1+0}{1-0} = 1$

$\lim_{x\to-\infty} \dfrac{x+3}{x-1} = \lim_{x\to-\infty} \dfrac{1+3/x}{1-1/x} = 1$



$f(0) = -9/-3 = 3$, $f = 0$ at $x = -3$

$\underline{\text{*Ex}}$ $f(x) = \dfrac{x^2-9}{x^2+x-6} = \dfrac{(x-3)(x+3)}{(x-2)(x+3)} = \dfrac{x-3}{x-2}$, $D_f : x \neq 2, -3$

$\lim_{x\to -3} f(x) = \dfrac{-6}{-5} = \boxed{\dfrac{6}{5}}$ hole at $(-3, 6/5)$ point (punch a hole at this pt)

VA: $x = 2$ remaining factor

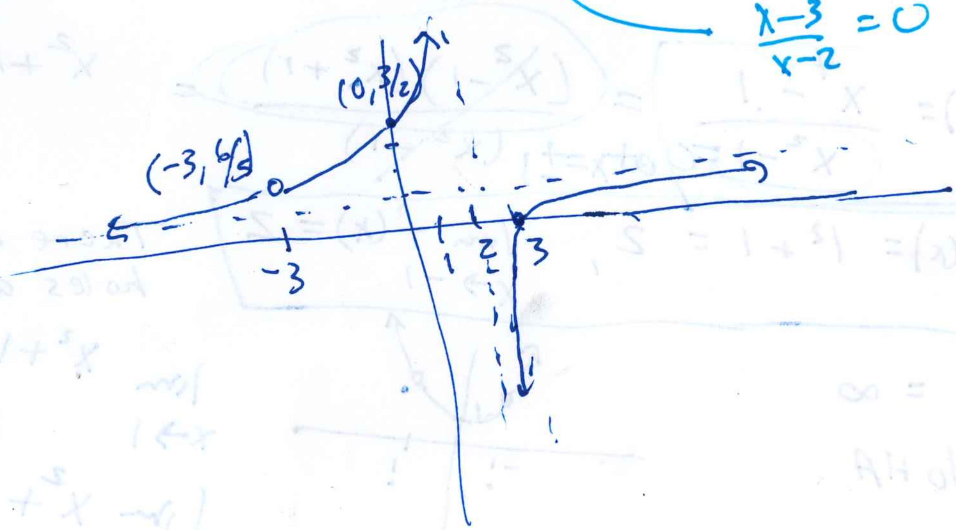
$\lim_{x\to 2^+} f(x) = -\infty$ from $-1/+0$, $\lim_{x\to 2^-} f(x) = \infty$ from $-1/-0$,

$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\infty} \dfrac{x-3}{x-2} = \lim_{x\to\infty} \dfrac{1-3/x}{1-2/x} = \dfrac{1-3/\infty}{1-2/\infty} = \dfrac{1-0}{1-0} = 1$

$\boxed{\text{HA: } y = 1}$

Intercepts $f(0) = \frac{3}{2}$, $f = 0$ at $x = 3$

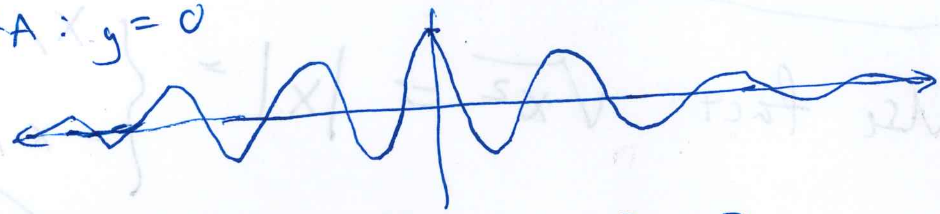
$\dfrac{x-3}{x-2} = 0$ + $f(0)$



Ex A fcn can cross its HA.

Ex $f(x) = \dfrac{\sin x}{x}$ $\qquad \lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1, \quad \lim\limits_{x\to\infty} \dfrac{\sin x}{x} = 0$

No VA; HA: $y=0$



Ex A rational fcn may have no HA. $f(x) = \dfrac{x^3 - x^2 + 2}{x-3} \begin{matrix}\div x^3 \\ \div x^3\end{matrix}$

$$\lim_{x\to\infty} \frac{1 - 1/x + 2/x^3}{1/x^2 - 3/x^3} = \frac{+1 - 0 + 0}{0} = \infty \qquad \frac{K}{0} \sim \infty$$

$$\lim_{x\to -\infty} \frac{1 - 1/x + 2/x^3}{1/x^2 - 3/x^3} = \frac{1 - 0 + 0}{0 - 0} = \infty$$

When is $\lim\limits_{x\to\infty} f(x) \neq \lim\limits_{x\to -\infty} f(x)$? Relevant fcns will contain $\sqrt{\ }$

Ex $f(x) = \dfrac{\sqrt{2x^2+1}}{3x-5}$, $\quad \lim\limits_{x\to\infty} f(x) = \dfrac{\sqrt{\infty^2+1}}{3\cdot\infty - 5} = \dfrac{\infty}{\infty}$ IF

Write $f(x) = \dfrac{\sqrt{x^2(2 + 1/x^2)}}{3x-5} = \dfrac{x\sqrt{2 + 1/x^2}}{3x-5}$ $\quad 1/x^2 \to 0$ $\quad \Rightarrow \lim\limits_{x\to\infty} f(x) = \dfrac{\infty\cdot\sqrt{2}}{3\infty - 5}$ still IF

2

Write $\lim\limits_{x\to\infty} \dfrac{\sqrt{2x^2+1}}{3x-5} = \lim\limits_{x\to\infty} \dfrac{\sqrt{\dfrac{2x^2}{x^2} + \dfrac{1}{x^2}}}{\dfrac{3x}{x} - \dfrac{5}{x}} = \boxed{\lim\limits_{x\to\infty} \dfrac{\frac{1}{x}\sqrt{2x^2+1}}{\frac{1}{x}(3x-5)}}$

1

3 $\quad \lim\limits_{x\to +\infty} \dfrac{\sqrt{2 + \frac{1}{x^2}}}{3 - 5/x} = \dfrac{\sqrt{2}}{3}$, HA: $y = \sqrt{2}/3$
$+\infty$

con'd

$$\lim_{x\to-\infty} \frac{\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{\frac{3x}{x}-\frac{5}{x}} = ? \quad -\frac{\sqrt{2}}{3}, \quad \lim_{x\to\infty} \frac{\sqrt{\frac{2x^2}{x^2}+\frac{1}{x^2}}}{\frac{3x}{x}-\frac{5}{x}} = \sqrt{2}/3$$

Use fact $\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Then $\lim_{x\to-\infty} \sqrt{\frac{2x^2+1}{x^2}} = \dfrac{\lim_{x\to-\infty} \sqrt{2+1/x^2}}{\lim_{x\to-\infty} \sqrt{x^2}} = -\sqrt{2}$

$= \cancel{\frac{\sqrt{2+0}}{-x}}$ Hence, $\lim_{x\to-\infty} f(x) = -\sqrt{2}/3$

HA at $-\infty$ $y = -\frac{\sqrt{2}}{3}$

Ex $f(x) = \underset{a}{\sqrt{x^2+1}} \underset{-b}{-x}$, $\lim_{x\to\infty} f(x) = \infty - \infty$ IF

Use conjugate of $\sqrt{x^2+1} - x$:

$$\lim_{x\to\infty} \underset{a}{\sqrt{x^2+1}} \underset{-b}{-x} \cdot \frac{\overset{a}{\sqrt{x^2+1}} \overset{+b}{+x}}{\sqrt{x^2+1}+x} = \lim_{x\to\infty} \frac{\cancel{x^2}+1-\cancel{x^2}}{\sqrt{x^2+1}+x} = 0$$

$\frac{k}{\infty} = 0$

Ex $\lim_{x\to\infty} \sin\frac{1}{x} = \sin\frac{1}{\infty} = \sin 0 \neq 0$

I.F. $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $\infty \cdot 0$, $(\infty)\cdot(-\infty)$, ∞^0

won't be an issue

Otherwise $\infty + \infty = \infty$, $\infty \cdot \infty = \infty$

$K/0 = \pm\infty$

Sec 3.4 #7, 8, 11, 13, 17/18 $\frac{1}{t^{3/2}}$

#13 $\lim\limits_{t\to\infty} \frac{\sqrt{t} + t^2}{2t - t^2} = \lim\limits_{t\to\infty} \frac{\frac{\sqrt{t}}{t^2} + 1}{\frac{2}{t} - 1} = \frac{0+1}{0-1} = \boxed{-1}$

#17 $\cancel{\lim\limits_{x\to\infty} \sqrt{\frac{1+4x^6}{2-x^3}} = \lim\limits_{x\to\infty} \sqrt{\frac{\frac{1}{x^6} + \frac{4x^6}{x^6}}{\frac{2}{x^3} - \frac{x^3}{x^3}}} = \lim\limits_{x\to\infty} \sqrt{\frac{1+}{x^3}}}$

#17/18 Use $\sqrt{x^{2n}} = x^n$, $x \geq 0$

& $\sqrt{x^{2n}} = -x^n$, $x < 0$

$\lim\limits_{x\to\infty} \frac{\sqrt{1+4x^6}}{2-x^3} = \lim\limits_{x\to\infty} \frac{\sqrt{1+4x^6}/x^3}{(2-x^3)/x^3}$ since $\sqrt{x^6} = x^3$, $x > 0$

$= \lim\limits_{x\to\infty} \frac{\sqrt{1/x^6 + 4x^6/x^6}}{2/x^3 - x^3/x^3} = \lim\limits_{x\to\infty} (+)\frac{\sqrt{1/x^6 + 4}}{2/x^3 - 1} = \frac{\sqrt{4}}{-1} = \boxed{-2}$

$\lim\limits_{x\to-\infty} \frac{\sqrt{1+4x^6}/x^3}{(2-x^3)/x^3} = \lim\limits_{x\to-\infty} (-)\frac{\sqrt{1/x^6 + 4x^6/x^6}}{2/x^3 - x^3/x^3}$

Since $\sqrt{x^6} = -x^3$

$= -\frac{\sqrt{0+4}}{0-1} = \frac{\sqrt{4}}{1} = \boxed{2}$

Sec 3.4

#13 $\lim\limits_{t\to\infty} \dfrac{\sqrt{t}+t^2}{2t-t^2} = \dfrac{\infty}{\infty-\infty}$ IF

Treat top + bottom as poly's;

deg top = 2, deg bottom = 2

$\div t^2 = \lim\limits_{t\to\infty} \dfrac{\sqrt{t}/t^2 + t^2/t^2}{2t/t^2 - t^2/t^2}$

$= \lim\limits_{t\to\infty} \dfrac{\frac{1}{t^{3/2}} + 1}{\frac{2}{t} - 1} = \dfrac{\frac{1}{\infty^{3/2}} + 1}{\frac{2}{\infty} - 1} = -1$

$\sqrt{t} = t^{1/2}$

#17/18 $\lim\limits_{\substack{x\to\infty \\ x\to-\infty}} \dfrac{\sqrt{1+4x^6}}{2-x^3} = \dfrac{\infty}{\infty}$ Essentially equal degrees

HA: at ∞ $y=-2$

at $-\infty$ $y=2$

$\div x^3 \quad = \lim\limits_{\substack{x\to\infty \\ x>0 \\ x\to-\infty \\ x<0}} \dfrac{\frac{1}{x^3}\sqrt{1+4x^6}}{\frac{1}{x^3}\left(2-x^3\right)}$

$= \lim\limits_{\substack{x\to\infty \\ x\to-\infty}} \dfrac{\sqrt{\frac{1}{x^6}+\frac{4x^6}{x^6}}}{\frac{2}{x^3}-\frac{x^3}{x^3}} = \dfrac{\pm\sqrt{0+4}}{0-1} = \mp 2$