Previous semester final w/ solns (2016 maybe?)

1. (10 points) Find all critical numbers of the function

$$f(x) = \frac{x^2}{3-x}$$

$$f'(x) = \frac{(3-x)2x - x^2(-1)}{(3-x)^2} = \frac{6x - 2x^2 + x^2}{(3-x)^2} = \frac{x(6-x)}{(3-x)^2}$$

$f' = 0$ when $x = 0$, $x = 6$

f' DNE when $x = 3$ (but 3 is not in Dom(f)).

Thus critical numbers are $x = 0$ and $x = 6$.

2. (10 points) Find the limits.

a) $\displaystyle\lim_{x\to-\infty} \frac{5x^3 - 3x + 1}{2x^3 + x^2}$

$$= \lim_{x\to-\infty} \frac{\frac{5x^3}{x^3} - \frac{3x}{x^3} + \frac{1}{x^3}}{\frac{2x^3}{x^3} + \frac{x^2}{x^3}} = \lim_{x\to-\infty} \frac{5 - \underbrace{\frac{3}{x^2}}_{\to 0} + \underbrace{\frac{1}{x^3}}_{\to 0}}{2 + \underbrace{\frac{1}{x}}_{\to 0}}$$

$$= \frac{5}{2}$$

b) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{3x^2-1}}{2x-3}$

$$= \lim_{x\to-\infty} \frac{\frac{\sqrt{3x^2-1}}{x}}{\frac{2x}{x} - \frac{3}{x}} = \lim_{x\to-\infty} \frac{\frac{\sqrt{3x^2-1}}{-\sqrt{x^2}}}{2 - \frac{3}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{3 - \underbrace{\frac{1}{x^2}}_{\to 0}}}{2 - \underbrace{\frac{3}{x}}_{\to 0}}$$

$$= -\frac{\sqrt{3}}{2}$$

Note that

$$\sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

and since $x \to -\infty$,

$$\sqrt{x^2} = -x$$

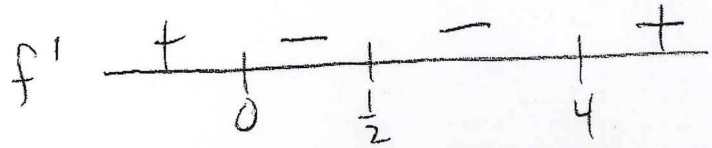
$$\Rightarrow -\sqrt{x^2} = x$$

3. (10 points) Given the derivative $f'(x)$ of the function $f(x)$, list all intervals on which $f(x)$ is increasing.

$$f'(x) = \frac{(2x-1)^2(x-4)}{x^3(x^2+3)}$$

f' can change signs at roots of numerator or denominator – in this case $x = \frac{1}{2}, 4, 0$

[(x^2+3) has no real roots]



f increases where $f' > 0$

So f is increasing on $(-\infty, 0) \cup (4, \infty)$

4. (15 points) Find the absolute maximum and the absolute minimum values on the closed interval $[0, \frac{3\pi}{2}]$ of the function

$$f(x) = \sin x + \cos^2 x$$

$$f'(x) = \cos(x) + 2\cos(x)(-\sin(x))$$
$$= \cos(x) - 2\cos(x)\sin(x)$$
$$= \cos(x)[1 - 2\sin(x)]$$

f' DNE? never.

$f' = 0 \Rightarrow \cos(x) = 0$ or $1 - 2\sin(x) = 0$

$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$ on $[0, \frac{3\pi}{2}]$

$\Rightarrow \sin(x) = \frac{1}{2}$

$\Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$ on $[0, \frac{3\pi}{2}]$

CANDIDATES TEST

x	$f(x)$
0	1
$\frac{\pi}{6}$	$\frac{5}{4}$
$\frac{\pi}{2}$	1
$\frac{5\pi}{6}$	$\frac{5}{4}$
$\frac{3\pi}{2}$	-1

absolute maximum: $5/4$

absolute minimum: -1

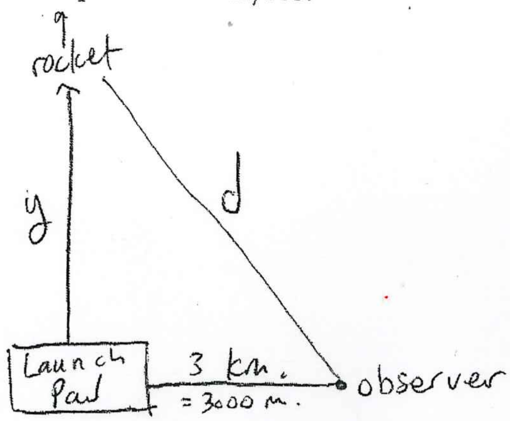
$f(0) = \sin(0) + [\cos(0)]^2 = 0 + 1 = 1$

$f(\frac{\pi}{6}) = \sin(\frac{\pi}{6}) + [\cos(\frac{\pi}{6})]^2 = \frac{1}{2} + (\frac{\sqrt{3}}{2})^2 = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$

$f(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) + [\cos(\frac{\pi}{2})]^2 = 1 + 0 = 1$

$f(\frac{5\pi}{6}) = \sin(\frac{5\pi}{6}) + [\cos(\frac{5\pi}{6})]^2 = \frac{1}{2} + (-\frac{\sqrt{3}}{2})^2 = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$

5. (15 points) An observer is positioned 3 km away from a rocket launch pad. How fast is the distance between the rocket and the observer increasing, when the rocket is 4 km above the ground and is moving straight up at the speed of 300 m/sec?



KNOWN: $\frac{dy}{dt} = 300$ m/s

NEED $\frac{dd}{dt}$ when $y = 4$ km
$= 4000$ m.

Notice: when $y = 4000$ m,
$d = 5000$ m.

Equation relating y and d:

$$3000^2 + y^2 = d^2$$

$$\Rightarrow \quad 2y\frac{dy}{dt} = 2d\frac{dd}{dt}$$

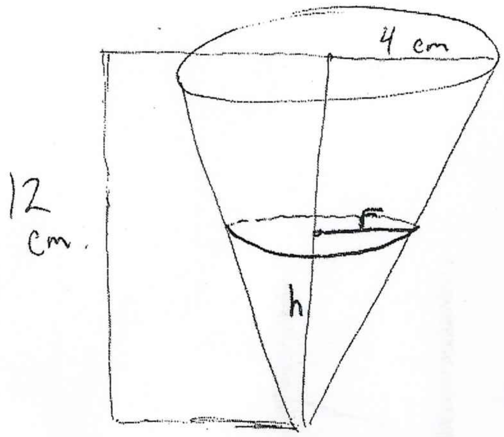
$$\Rightarrow \quad 2(4000)(300) = 2(5000)\frac{dd}{dt}$$

$$\Rightarrow \quad \frac{dd}{dt} = \frac{2(4000)(300)}{2(5000)}$$

$$= \frac{4}{5}(300)$$

$$= 240 \text{ m/s}$$

6. (15 points) Water is leaking from a conical cup at the constant rate of 2 cm^3/min. The height of the cup is 12 cm and the radius of the top is 4 cm. How fast is the level of the water in the cup decreasing when the water is 3 cm deep? (The volume of a right circular cone is given by the formula $V = \frac{1}{3}\pi r^2 h$.)



KNOWN: $\frac{dV}{dt} = -2 \text{ cm}^3/\text{min}$

NEED $\frac{dh}{dt}$ when $h = 3$

Volume $= \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h$

$\Rightarrow V = \frac{1}{27}\pi h^3$

So $\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt}$

$-2 = \frac{1}{9}\pi (3)^2 \frac{dh}{dt}$

$-2 = \frac{1}{9}\pi \cdot 9 \frac{dh}{dt}$

$\Rightarrow -\frac{2}{\pi} = \frac{dh}{dt}$.

Notice that $\frac{r}{h} = \frac{4}{12}$

$\Rightarrow r = \frac{4h}{12}$

$\Rightarrow r = \frac{h}{3}$

The water level in the cup is decreasing by $\frac{2}{\pi}$ cm/min when the water is 3 cm. deep.

7. (35 points) Let $g(x) = \frac{x^2}{3x-2}$. To save you time, I'm giving you the derivatives of g: $g'(x) = \frac{3x^2-4x}{(3x-2)^2}$ and $g''(x) = \frac{8}{(3x-2)^3}$.

a. Give the vertical asymptotes. (If there are none, say so.) Remember to justify your answer.

$$\lim_{x \to \frac{2}{3}^-} \frac{x^2}{3x-2} = -\infty \quad \text{and} \quad \lim_{x \to \frac{2}{3}^+} \frac{x^2}{3x-2} = \infty,$$

so $x = \frac{2}{3}$ is a vertical asymptote.

b. Give the horizontal asymptotes. (If there are none, say so.) Remember to justify your answer.

$$\lim_{x \to \infty} \frac{x^2}{3x-2} = \lim_{x \to \infty} \frac{\frac{x^2}{x}}{\frac{3x}{x} - \frac{2}{x}} = \lim_{x \to \infty} \frac{x}{3 - \frac{2}{x}} = \infty$$

$$\lim_{x \to -\infty} \frac{x^2}{3x-2} = \lim_{x \to -\infty} \frac{x}{3 - \frac{2}{x}} = -\infty$$

Thus there are no Horizontal Asymptotes.

Optional observation $\left\{ \right.$ But: $3x - 2 \,\overline{)\, x^2 + 0x + 0}$ with quotient $\frac{1}{3}x + \frac{2}{9}$; $-(x^2 - \frac{2}{3}x)$; $\frac{2}{3}x + 0$ $\quad$ ($y = \frac{1}{3}x + \frac{2}{9}$ is a Slant Asymptote)$\left. \right\}$

c. Give the intervals of increasing and decreasing, and give all local maxima and local minima.

$$g\left(\tfrac{4}{3}\right) = \frac{\left(\frac{4}{3}\right)^2}{3\left(\frac{4}{3}\right) - 2} = \frac{\frac{16}{9}}{4-2} = \frac{\left(\frac{16}{9}\right)}{2} = \frac{16}{18} = \frac{8}{9}$$

$$g'(x) = \frac{3x^2 - 4x}{(3x-2)^2} = \frac{x(3x-4)}{(3x-2)^2}$$

g': $+$ | 0 | $-$ | $\frac{2}{3}$ | $-$ | $\frac{4}{3}$ | $+$

g is increasing on $(-\infty, 0) \cup (\frac{4}{3}, \infty)$

g is decreasing on $(0, \frac{2}{3}) \cup (\frac{2}{3}, \frac{4}{3})$

local max: $(0, 0)$

local min: $(\frac{4}{3}, \frac{8}{9})$

(continued on next page)

d. Give the intervals of concavity and the inflection points.

$$g''(x) = \frac{8}{(3x-2)^3}$$

g'' — (−) at $\frac{2}{3}$ (+)

g is concave downward on $(-\infty, \frac{2}{3})$
g is concave upward on $(\frac{2}{3}, \infty)$

g changes concavity at $x = \frac{2}{3}$
but $\frac{2}{3}$ is not in $Dom(g)$,
thus not an inflection point.

e. Sketch the graph of g.