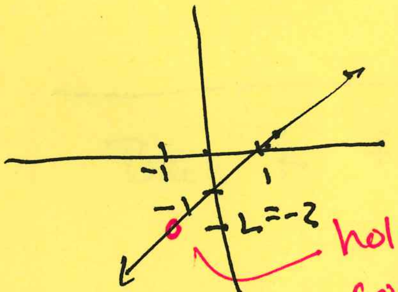
Week 7 / Limits - Interpreted by graphs; also see algebraic computation

$f(x) = \frac{x^2-1}{x+1}$ $\quad D_f: x \neq -1$ $\quad$ simplified $\frac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}}$

$f(x) = 1x - 1$, where $x \neq -1$



hole's coordinates
$(-1, -2)$
(x, y)

$$\lim_{x \to -1} \frac{x^2-1}{x+1} = \lim_{x \to -1} (x-1)$$

$L = -2$

Conclude: The limit exists as $x \to -1$ though the fcn does not at $x = -1$

Warning
$\lim_{x \to a} f(x)$ does not have to be $f(a)$.
In fact, $f(a)$ might fail to exist.

by next Monday

* $f(x)$ is "continuous" at $x = a$ if $\lim_{x \to a} f(x) = f(a)$

Holes in graphs of $f(x)$ do not rule out limits there. In fact, holes are sites of limits.
We don't care what happens at $x = a$.

"$\equiv$" identically equal

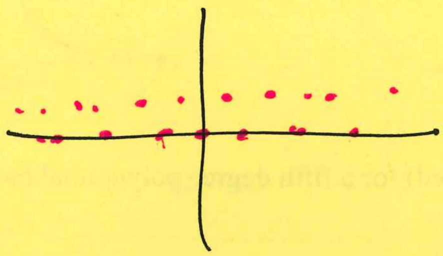
Noah $\quad f(a) \not\equiv \lim_{x \to a} f(x)$

unless f is continuous at $x = a$

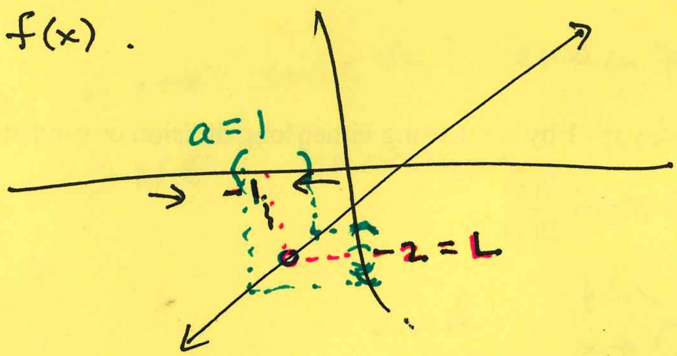
$\underline{\text{Ex}}$ Weird fcn. $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q}, \; x \in \mathbb{Q}' \end{cases}$

$\downarrow$ y

$\lim_{x \to a} f(x)$ DNE does not exist for any $x \in \mathbb{R}$.



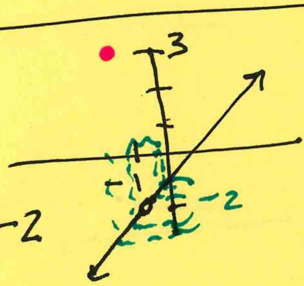
Back to $f(x)$.



Big mistake I see: $\lim_{x \to -1} f(x) = -1.99$ not true

Precise def $\quad 0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$\underline{\text{Ex}}$



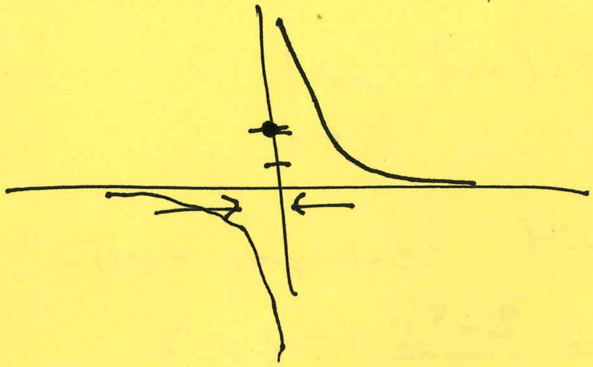
$f(x) = \begin{cases} \frac{x^2-1}{x+1}, & x \neq -1 \\ 3, & x = -1 \end{cases}$

y

$\lim_{x \to -1} f(x) = -2$

$f(-1) = 3$

Ex $f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$



$\boxed{\lim_{x \to 0} f(x) \text{ DNE}}$

$\lim_{x \to 0} f(x) = 2$

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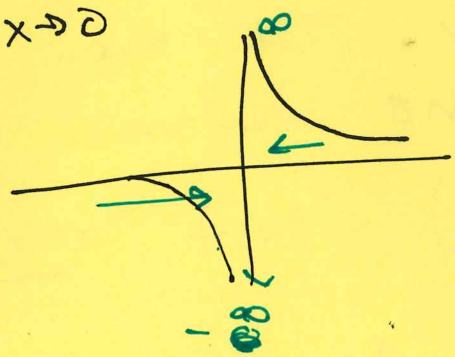
* Famous, more useful example:

Ex $\lim_{x \to 0} \frac{1}{x}$ DNE because $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

and $\lim_{x \to 0^+} \frac{1}{x} = \infty$

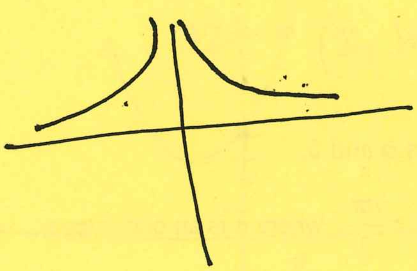


We agree to call infinities "limits" even tho' they ain't #s"

The limit as $x \to 0$ DNE because

LHL ≠ RHL

Ex



$y = \left| \frac{1}{x} \right|$ $\lim_{x \to 0^-} f(x) = \infty$

$x \to 0^+$ $= \infty$

so L is ∞

__Ex__ $$f(x) = \frac{x^3 - 8}{x - 2} =$$

$$\lim_{x \to 2} f(x) = ? = 12$$

First impulse: Plug in $x=2$ into $f(x)$.

$$f(2) = \frac{2^3-8}{2-2} = \frac{0}{0}$$ "Indeterminate form" I.F.

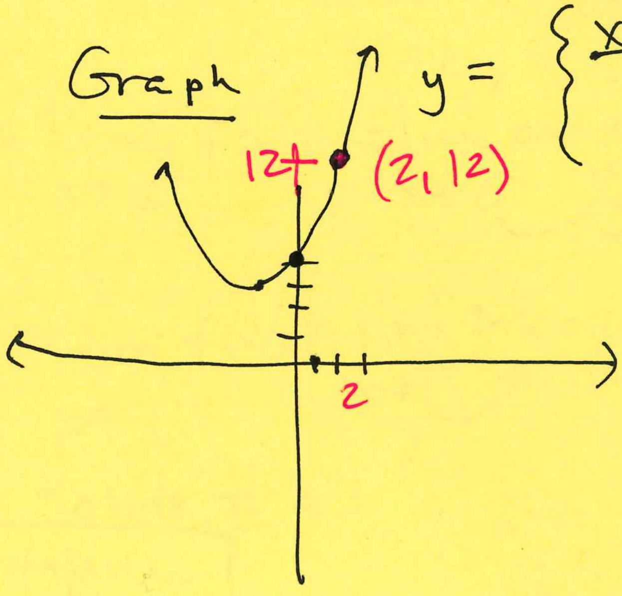
"I.F." does not indicate "DNE". In fact,

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, and several other forms are I.F. When you get an I.F. do not conclude DNE. Instead, do __algebra__!

__Factor__ $$f(x) = \frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}} = x^2+2x+4$$

$$\lim_{x \to 2} x^2+2x+4 = 2^2+2\cdot2+4 = 12$$

__Graph__ $$y = \begin{cases} x^2+2x+4, & x \neq 2 \\ 12 & x=2 \end{cases}$$



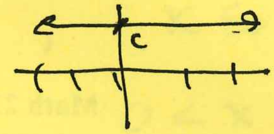
$$x^2+2x+4=0$$

$$x = \frac{-2 \pm \sqrt{2^2-4(1)(4)}}{2(1)}$$

4 — **Fact**

Ex $f(x) = c$

$$\lim_{x\to a} c = c$$



Ex $f(x) = -9$, $\lim_{x\to 2} -9 = -9$

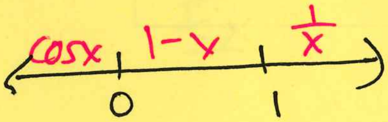
$x \to 0$

$x \to -\pi$

Ex $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

$\lim_{x\to d} f(x) = f(d)$. Polynomials have limits at all $x \in \mathbb{R}$.

#11.



$$f(x) = \begin{cases} \cos x, & \text{if } x \le 0 \\ 1-x, & \text{if } 0 < x < 1 \\ \frac{1}{x}, & \text{if } x \ge 1 \end{cases}$$

Find all $x = a$ such that $\lim_{x\to a} f(x)$ exists!

<u>Soln</u> ~~you~~ Check all LHL, RHL at each "important" x value.

$x = 0, 1$

$\boxed{\lim_{x\to 0} f(x) = 1} \Leftarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \cos x = 1$ sub. $x=0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1-x = 1$

$=$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1-x = 1-1 = 0$

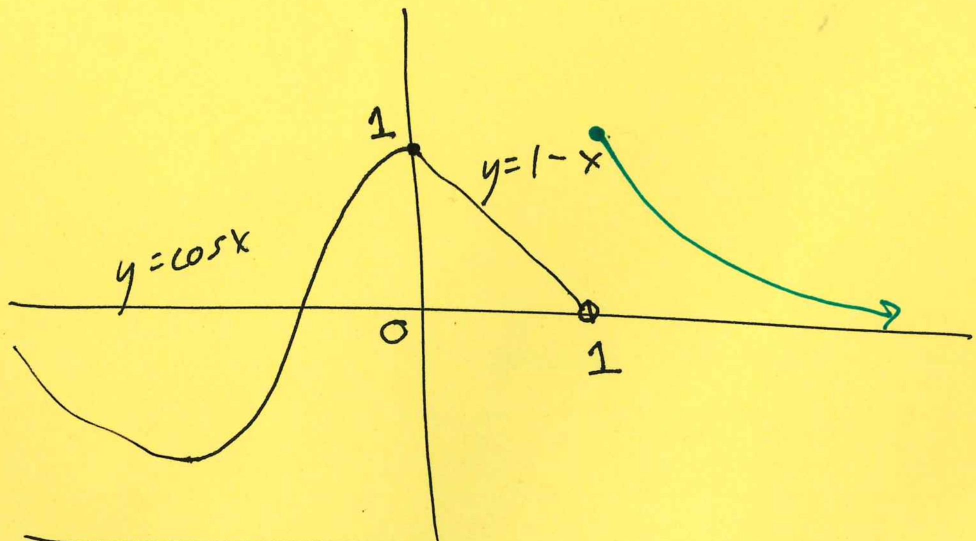
$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{1}{x} = \frac{1}{1} = 1$

$\neq$

$\lim_{x\to 1} f(x)$ DNE b/c LHL $\neq$ RHL

"close to a"

Graph $f(x) = \begin{cases} \cos x, & x \le 0 \\ 1-x & 0 < x < 1 \\ \frac{1}{x} & x \geq 1 \end{cases}$



Infinite limits — Vertical asymptotes

Ex $f(x) = \dfrac{x+1}{x-5}$, $\lim\limits_{x \to 5^+} f(x) = ? = \infty$

get really close

Process

x	$x-5$	$\frac{x+1}{x-5}$
5.1	.1	$\sim 6/.1 = 60$
5.01	.01	$\sim 6/.01 = 600$
5.001	.001	$\sim 6/.001 = 6000$

$6 \approx x+1$ $\quad \to \infty$

$\lim\limits_{x \to 5^-} f(x) = ? = -\infty$

x	$x-5$	$\frac{x+1}{x-5}$
4.9	-.1	$\sim 6/-.1 = -60$
4.99	-.01	$\sim 6/-.01 = -600$
4.999	-.001	$\sim 6/-.001 = -6000$

$x+1 \approx 6$ $\quad \to -\infty$

LHL $= -\infty$
RHL $= \infty$
SO L DNE