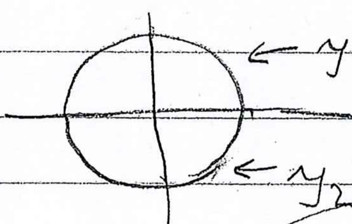
Sec 13 HW – Implicit Differentiation

1. $x^2 + y^2 = 1$ is not a function, but it can be written as two fcns:

$$y^2 = 1 - x^2 \longrightarrow y_1 = \sqrt{1-x^2}, \quad y_2 = -\sqrt{1-x^2}$$



$\leftarrow y_1 = \sqrt{1-x^2} \rightarrow y_1' = \frac{1}{2}(1-x^2)^{-1/2}(-2x)$

$$y_1' = \frac{-x}{\sqrt{1-x^2}}$$

$\leftarrow y_2 = -\sqrt{1-x^2}$

$$y_2' = \frac{+x}{\sqrt{1-x^2}}$$

Are these the same as what implicit differentiation give?

$$x^2 + y^2 = 1$$

$$2x + 2y\frac{dy}{dx} = 0 \longrightarrow \frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}y} =$$

$$\boxed{\frac{dy}{dx} = -\frac{x}{y}} = \frac{-x}{\pm\sqrt{1-x^2}} = \frac{\mp x}{\sqrt{1-x^2}} = y_1' \ \& \ y_2'$$

2a) $x^2 + 3y^2 = 6$

$$2x + 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{6y} = \frac{-x}{3y}$$

$$\boxed{\frac{dy}{dx} = \frac{-x}{3y}}$$

b) $9x - x^2y^2 = 2xy$

Requires product rule:

$$9 - 2xy^2 - x^2 \cdot 2y\frac{dy}{dx} = 2y + 2x\frac{dy}{dx}$$

$\longrightarrow$

2b) $9-2xy^2-2y = 2x^2y\frac{dy}{dx} + 2x\frac{dy}{dx}$

$9-2xy^2-2y = (2x^2y+2x)\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{9-2xy^2-2y}{2x^2y+2x}}$$

c) $3xy - \frac{y}{3} = 2x^{-1}$

$3y + 3x\frac{dy}{dx} - \frac{1}{3}\frac{dy}{dx} = -2x^{-2}$

$\left(3x - \frac{1}{3}\right)\frac{dy}{dx} = -2x^{-2} - 3y$

$\frac{dy}{dx} = \frac{-2x^{-2}-3y}{3x-1/3} = \frac{\frac{-2}{x^2} - 3y}{3x - \frac{1}{3}}$ (LCD $= 3x^2$)

$\frac{dy}{dx} = \frac{-3x^2}{-3x^2} \cdot \frac{\frac{-2}{x^2} - 3y}{3x - \frac{1}{3}} = \frac{6+9x^2y}{-9x^3+x^2}$ (book has negative factored out)

f. $3x^2 - 4y^3 + 3 = \sqrt{5x+y}$

$6x - 12y^2\frac{dy}{dx} = \frac{1}{2}(5x+y)^{-1/2} \cdot \left(5 + \frac{dy}{dx}\right)$

$6x - 12y^2\frac{dy}{dx} = \frac{5}{2}\left(\frac{1}{\sqrt{5x+y}}\right) + \frac{1}{\sqrt{5x+2y}}\frac{dy}{dx}$

$6x - \frac{5}{2\sqrt{5x+y}} = \left(12y^2 + \frac{1}{\sqrt{5x+2y}}\right)\frac{dy}{dx}$

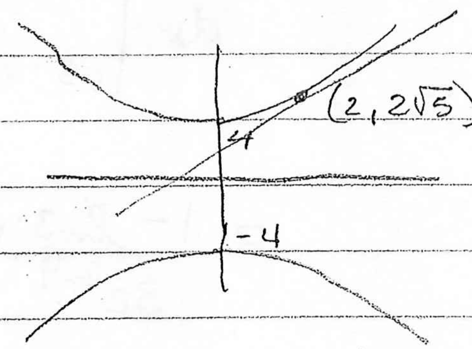
Sec 13 con'd

#2f

$$\frac{dy}{dx} = \frac{6x - \dfrac{5}{2\sqrt{5x+y}}}{12y^2 + \dfrac{1}{\sqrt{5x+y}}}$$

(same as book, bit different form)

#3. $y^2 - x^2 = 16$ hyperbola



Eqn. of tangent line at $(2, 2\sqrt{5})$ is found:

$$2y\frac{dy}{dx} - 2x = 0$$

$$\left.\frac{dy}{dx}\right|_{(2,2\sqrt{5})} = \left.\frac{x}{y}\right|_{(2,2\sqrt{5})} = \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}} = \text{slope}$$

$$\boxed{y - 2\sqrt{5} = \frac{1}{\sqrt{5}}(x-2)}$$

6. $\boxed{e^{xy} = x}$ Find slope at $x=3$, that is,

$\left.\frac{dy}{dx}\right|_{x=3}$ (you'll need y value, so plug $x=3$ into $e^{xy}=x$ first)

$$e^{3y} = 3 \longrightarrow \ln e^{3y} = \ln 3 \to 3y = \ln 3$$

$$\longrightarrow \boxed{y = \frac{\ln 3}{3} \text{ when } x = 3}$$

Now ind dy/dx: $\underbrace{e^{xy}}_{e^u}\underbrace{\left(1y + x\frac{dy}{dx}\right)}_{du} = 1 \longrightarrow e^{xy}y + xe^{xy}\frac{dy}{dx} = 1$

$$xe^{xy}\frac{dy}{dx} = 1 - ye^{xy} \longrightarrow \boxed{\frac{dy}{dx} = \frac{1-ye^{xy}}{xe^{xy}}}$$

$$\text{So } \left.\frac{dy}{dx}\right|_{(3,\ln 3)} = \frac{1 - \ln 3\, e^{3\ln 3}}{3e^{3\ln 3}}$$

book is wrong

$$= \frac{1 - \ln 3 \cdot e^{\ln 9}}{3e^{\ln 9}} = \frac{1 - \ln 3 \cdot 9}{3 \cdot 9} = \boxed{\frac{1 - 9\ln 3}{27}}$$

10. $y^2 = x \longrightarrow 2y\frac{dy}{dx} = 1 \longrightarrow \boxed{\frac{dy}{dx} = \frac{1}{2y}}$

a) So $\frac{d^2y}{dx^2} = -1(2y)^{-2} \cdot 2\frac{dy}{dx}$ ← Chain rule + implicit

thus $\frac{d^2y}{dx^2} = \frac{-\cancel{2}}{\cancel{2}4y^2}\frac{dy}{dx}$ ← Now substitute

$$\frac{d^2y}{dx^2} = \frac{-1}{2y^2} \cdot \frac{1}{2y} = \boxed{\frac{-1}{4y^3}}$$

b) Find d^3y/dx^3 by quotient or power rule:

Quotient rule: $\frac{d^3y}{dx^3} = \frac{0(4y^3) - (-1)(12y^2\, dy/dx)}{(4y^3)^2} = \frac{12y^2\, dy/dx}{16y^6}$

$$= \frac{3\, dy/dx}{4y^4} \quad \underset{dy/dx}{\text{substitute}} = \frac{3\left(\frac{1}{2y}\right)}{4y^4} = \boxed{\frac{3}{8y^5}}$$

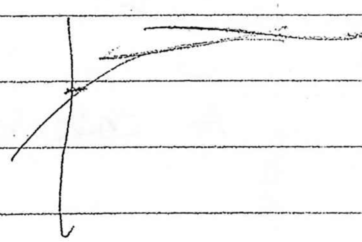
Power rule

$$\frac{d^2y}{dx^2} = -(4y^3)^{-1} \rightarrow \frac{d^3y}{dx^3} = +(4y^3)^{-2}\left(12y^2\frac{dy}{dx}\right)$$

$$= \frac{12y^2}{(4y^3)^2}\frac{dy}{dx} = \frac{12y^2}{16y^6}\frac{dy}{dx} = \frac{3}{4y^4}\frac{dy}{dx}$$

Substituting dy/dx $\quad \frac{d^3y}{dx^2} = \frac{3}{4y^4}\cdot\frac{1}{2y} = \boxed{\frac{3}{8y^5}}$

#8 For a certain product we have the following fcns implicit in q, the first because it is not explicitly solved for C, the second, not explicitly solved for R.



Cost $\quad C^2 = q^2 + 100\sqrt{q} + 100$

Revenue $\quad 900(q-4)^2 + R^2 = 25{,}500$

Marg. Cost $C'(q)$ $\quad 2C\frac{dC}{dq} = 2q + \frac{1}{2}\cdot 100q^{-1/2} + 0$

$$2C\frac{dC}{dq} = 2q + \frac{50}{\sqrt{q}} \rightarrow \frac{dC}{dq} = \frac{2q}{2C} + \frac{50}{2C\sqrt{q}} = \boxed{\frac{q}{C} + \frac{25}{C\sqrt{q}}}$$

Marginal cost $C'(q)$

Marg. Rev $R'(q)$ $\quad 1800(q-4)^1 + 2R\frac{dR}{dq} = 0$

$$\frac{dR}{dq} = \frac{-1800(q-4)}{2R} = \boxed{\frac{-900(q-4)}{R}}$$

marginal revenue fcn $R'(q)$.

To find $C'(5)$ + $R'(5)$ we need to find what C + R would be at $q=5$, since the marg fns are not explicit.

(1) Cost at $q=5$: $C^2 = 5^2 + 100\sqrt{5} + 100$

$$C = \sqrt{125+100\sqrt{5}}$$
$$= \sqrt{25(5+4\sqrt{5})} = 5\sqrt{5+4\sqrt{5}}$$

$$C'(5) = \frac{q\sqrt{q}+25}{C\sqrt{q}} = \frac{5\sqrt{5}+25}{5\sqrt{5+4\sqrt{5}}\sqrt{5}} = \frac{\sqrt{5}+5}{\sqrt{5+4\sqrt{5}}\sqrt{5}}$$

Multiply top + bottom by $\sqrt{5}$:

$$C'(q) = \frac{\sqrt{5}+5 \cdot \sqrt{5}}{\sqrt{5+4\sqrt{5}}\sqrt{5} \cdot \sqrt{5}} = \frac{5+5\sqrt{5}}{\sqrt{5+4\sqrt{5}} \cdot 5} = \frac{1+\sqrt{5}}{\sqrt{5+4\sqrt{5}}}$$

A calculator tells $\approx .87$, or a rate of $\frac{.87}{100 \text{ items}}$

(2) Revenue at $q=5$: $900(5-4)^{\frac{1}{2}} + R^2 = 25{,}500$

$$R^2 = 25{,}500 - 900 = 24{,}600$$
$$R = \sqrt{24{,}600} = 10\sqrt{246}$$

Then $R'(5) = \dfrac{-900(5-4)}{10\sqrt{246}} = \dfrac{-90}{\sqrt{246}}$ or $\dfrac{-5.74}{100 \text{ items}}$ rate

Meaning: When production is 500 units, costs are increasing at a rate of 87¢/next 100 items

When 500 units are sold, revenue for next 100 decreases by \$5.74