On properties of a family of orthogonal polynomials

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1. Scalar product

Let $\mathbb{H}$ denote the skew-field of quaternions, let $\Lambda \cong \mathbb{R}^3$ be the $\mathbb{R}$-vector space of imaginary quaternions, and let $\mathcal{M}$ be the right $\mathbb{H}$-module of polynomial functions on Λ. Elements of this module are defined as finite sums of $z^k a_k$, where a_k are arbitrary quaternions and $z \in \Lambda$,

$$P(z) = \sum_{k=0}^{n} z^k a_k.$$

Let

$$d\mu(z) = f(z)\, dm(z),$$

where $dm(z)$ is the Lebesgue measure on $\mathbb{R}^3$, and

$$f(z) = (2\pi)^{-3/2} e^{-|z|^2/2}. \tag{1}$$

Define the *scalar product* on $\mathcal{M}$ as

$$\langle u(z), w(z)\rangle = \int_\Lambda \overline{u}(z) w(z)\, d\mu(z). \tag{2}$$

This function is linear in the second argument

$$\langle u(z), w(z)\alpha\rangle = \langle u(z), w(z)\rangle\, \alpha$$

for every $\alpha \in \mathbb{H}$, and it also has the property that $\langle w(z), u(z)\rangle = \overline{\langle u(z), w(z)\rangle}$.

In addition, the scalar product is positive definite:

$$\langle u(z), u(z)\rangle \geq 0 \text{ for all } u(z) \in \mathcal{M},$$

and equality holds only if $u(z) = 0$.

As usual, the norm of a function is defined as the square root of the scalar product of the function with itself,

$$\|u(z)\| = \sqrt{\langle u(z), u(z)\rangle}.$$

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m/n	0	1	2	3	4	5	6
0	1	0	-3	0	$5 \cdot 3$	0	$-7!!$
1	0	3	0	$-5 \cdot 3$	0	$7!!$	0
2	-3	0	$5 \cdot 3$	0	$-7!!$	0	$9!!$
3	0	$-5 \cdot 3$	0	$7!!$	0	$-9!!$	0
4	$5 \cdot 3$	0	$-7!!$	0	$9!!$	0	$-11!!$
5	0	$7!!$	0	$-9!!$	0	$11!!$	0
6	$-7!!$	0	$9!!$	0	$-11!!$	0	$13!!$

Table 1. Scalar products of monomials, $\langle z^m, z^n \rangle$.

We say that a family of quaternionic functions on Λ, $\{u_k\}$ is *orthonormal*, if

$$\langle u_k, u_l \rangle = \begin{cases} 1, & \text{if } k = l, \\ 0, & \text{if } k \neq l. \end{cases} \tag{3}$$

Let $P_k(z)$ denote the system of monic orthogonal polynomials. We can show that this system exists and unique by the Gram-Schmidt orthogonalization process.

In order to understand properties of this system of functions, we calculate the scalar products of monomials explicitly.

Theorem 1.1 —Scalar products of monomials —.
For all non-negative integers m and n

$$\langle z^m, z^n \rangle = \begin{cases} (-1)^{\frac{n-m}{2}} (m+n+1)!!, & \textit{if } n-m \textit{ is even,} \\ 0 & \textit{if } n-m \textit{ is odd.} \end{cases}$$

Proof of Theorem 1.1. We start with a useful lemma.

Lemma 1.2.
For every non-negative integers l and k,
the integral $\int_\Lambda |z|^{2l} z^k d\mu(z)$ is real.

Proof of Lemma. We write $z = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ and expand the expression $|z|^{2l} z^k$. We claim that every monomial coefficient before $\mathbf{i}$, $\mathbf{j}$, or $\mathbf{k}$ in this expansion has one of its variables x_i in the odd power. If this claim holds, then the integral of these monomials with respect to measure μ is 0, by the symmetry of μ, and the lemma is proved.

It is sufficient to prove the claim for $l = 0$, since $|z|^{2l} = (x_1^2+x_2^2+x_3^2)^l$ is real and all monomials in its expansion have variables in the even power.

Consider a single term in the expansion of z^k, for example, $x_1\mathbf{i}x_3\mathbf{k}x_2\mathbf{j}x_2\mathbf{j}\ldots$, It can be either imaginary or real, and it is clear that it is imaginary if and only if the term contain at least one of the variables in the odd power. Indeed, we can do transpositions of imaginary units in the expansion and this will only introduce real factors. Hence, if all powers are even then all imaginary units in the product can be paired off and cancelled out, so that the product is real.

Therefore the claim and the lemma are proved. □

Now, let us calculate the real part of the expression $|z|^{2l}\, z^k$. Since $|z|^{2l}$ is real, we only need to calculate the real part of z^k.

Lemma 1.3.
Let $z = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$. Then,

$$\operatorname{Re} z^k = \operatorname{Re} \overline{z}^k = \begin{cases} (-1)^r \left(x_1^2 + x_2^2 + x_3^2\right)^r, & \text{if } k = 2r, \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

Proof of Lemma. We write the quaternion z in its matrix form:

$$\varphi(z) = \begin{pmatrix} x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & -x_1 i \end{pmatrix},$$

and note that for every quaternion w its real part can be computed as $\frac{1}{2}\operatorname{Tr}\varphi(w)$. The eigenvalues of $\varphi(z)$ are $\pm i\sqrt{x_1^2 + x_2^2 + x_3^2}$. Hence, we compute:

$$\operatorname{Re} z^k = \frac{1}{2}\operatorname{Tr}\varphi(z^k) = \begin{cases} (-1)^r \left(x_1^2 + x_2^2 + x_3^2\right)^l, & \text{if } k = 2l, \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

The case of $\operatorname{Re}\overline{z}^k$ is similar. □

Now we can finish the proof of Theorem 1.1.

Let $m \leq n$ and note that

$$\mathrm{Re}\,(\overline{z}^m z^n) = |z|^{2m}\mathrm{Re}\,z^{n-m} = \begin{cases} (-1)^{\frac{n-m}{2}}(x_1^2+x_2^2+x_3^2)^{\frac{m+n}{2}}, & \text{if } n-m \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Next we calculate:

$$\begin{aligned} \int_{\mathbb{R}^3} \left(x_1^2+x_2^2+x_3^2\right)^l d\mu(z) &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{S}^2}\int_{\mathbb{R}} r^{2l+2} e^{-r^2/2} dr dS \\ &= \frac{2^{l+1}}{\sqrt{\pi}}\Gamma\left(l+\frac{3}{2}\right). \end{aligned}$$

Hence,

$$\int \overline{z}^m z^n d\mu(z) = (-1)^{\frac{n-m}{2}} \frac{2^{\frac{m+n}{2}+1}}{\sqrt{\pi}}\Gamma\left(\frac{m+n+3}{2}\right) = (-1)^{\frac{n-m}{2}}(m{+}n{+}1)!!$$

□

Since all entries in the matrix of scalar products are real, we derive an important consequence that the coefficients of monic orthogonal polynomials are real.

Note that $\langle z, z\rangle \neq \langle 1, z^2\rangle$, which means that the scalar product cannot be written as $\langle P(z), Q(z)\rangle = \int_{\mathbb{R}} P(x)Q(x)\mu(dx)$ for a measure μ on the real line.

2. Three-term recurrence relation

By usual means, we can derive the three-term recurrence relation for the orthogonal polynomials.

Theorem 2.1 —Three-term recurrence for P - polynomials —.

Suppose that $P_n(z)$ are monic polynomials orthogonal with respect to the scalar product in (2), and that $P_0(z) = 1$, and $P_1(z) = z$. Then these polynomials satisfy the following recurrence relation:

$$P_{n+1}(z) = zP_n(z) + \beta_n P_{n-1}(z), \tag{4}$$

where β_n are some real positive coefficients, and

$$\beta_n = \frac{\langle P_n, P_n\rangle}{\langle P_{n-1}, P_{n-1}\rangle}.$$

n	P_n	h_n	β_n
0	1	1	*
1	z	3	3
2	z^2+3	6	2
3	z^3+5z	30	5
4	z^4+10z^2+15	120	4
5	z^5+14z^3+35z	840	7
6	$z^6+21z^4+105z^2+105$	5,040	6
7	$z^7+27z^5+189z^3+315z$	45,360	9
8	$z^8+36z^6+378z^4+1260z^2+945$	362,880	8
9	$z^9+44z^7+594z^5+2772z^3+3465z$	3,991,680	11

Table 2. Monic orthogonal polynomials, their squared norms and β_n.

The proof is standard and omitted.

Table 2 is the table of the first orthogonal monic polynomials together with recursion coefficients β_n and the squared norms of polynomials $h_n := \langle P_n, P_n \rangle$.

In the next step, we are going to derive more explicit formulas for the orthogonal polynomials P_n, their squared norms h_n, and coefficients β_n.

3. Determinantal formulas

Let $s_{ij} := \langle z^i, z^j \rangle$. (These are elements of the infinite matrix in Table 1.) And let D_n denote the principal submatrices of the matrix of scalar products:

$$D_n = \begin{pmatrix} s_{00} & s_{01} & \cdots & s_{0n} \\ s_{10} & s_{11} & \cdots & s_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n0} & s_{n1} & \cdots & s_{nn} \end{pmatrix}.$$

Finally let $|D_n|$ denotes $\det(D_n)$.

Theorem 3.1 —Determinantal formula for for P - polynomials —.
The monic orthogonal polynomials are given by the formula

$$P_n(z) = \frac{1}{|D_{n-1}|} \det \begin{pmatrix} s_{00} & s_{01} & \cdots & s_{0,n-1} & s_{0n} \\ s_{10} & s_{11} & \cdots & s_{1,n-1} & s_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{n-1,0} & s_{n-1,1} & \cdots & s_{n-1,n-1} & s_{n-1,n} \\ 1 & z & \cdots & z^{n-1} & z^n \end{pmatrix}.$$

Their squared norms are $h_n := \langle P_n(z), P_n(z) \rangle = |D_n| / |D_{n-1}|$.

Proof: The polynomials are clearly monic. In order to prove orthogonality, we write

$$\langle z^m, P_n(z) \rangle = \frac{1}{|D_{n-1}|} \det \begin{pmatrix} s_{00} & s_{01} & \cdots & s_{0,n-1} & s_{0n} \\ s_{10} & s_{11} & \cdots & s_{1,n-1} & s_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{n-1,0} & s_{n-1,1} & \cdots & s_{n-1,n-1} & s_{n-1,n} \\ \langle z^m, 1 \rangle & \langle z^m, z \rangle & \cdots & \langle z^m, z^{n-1} \rangle & \langle z^m, z^n \rangle \end{pmatrix}.$$

This equals 0 for $m \leq n-1$ because there are two coinciding rows.

For $m = n$, we have $\langle z^n, P_n(z) \rangle = |D_n| / |D_{n-1}|$. Since the polynomials are monic, $\langle P_n(z), P_n(z) \rangle = \langle z^n, P_n(z) \rangle = |D_n| / |D_{n-1}|$. □

3.1. **Norm of polynomials.**

Theorem 3.2 —Norms and recurrence coefficients for P - polynomials —.

$$h_n = \begin{cases} n!(n+2), & \text{if } n \text{ is odd,} \\ (n+1)!, & \text{if } n \text{ is even.} \end{cases}$$

$$\beta_n = \begin{cases} n+2, & \text{if } n \text{ is odd,} \\ n, & \text{if } n \text{ is even.} \end{cases}$$

As a corollary, we find that $|D_n| > 0$ for all n and, therefore, all eigenvalues of matrix D_n are positive. This implies that the scalar product $\langle u, v \rangle$ is positive definite on $\mathcal{M}$.

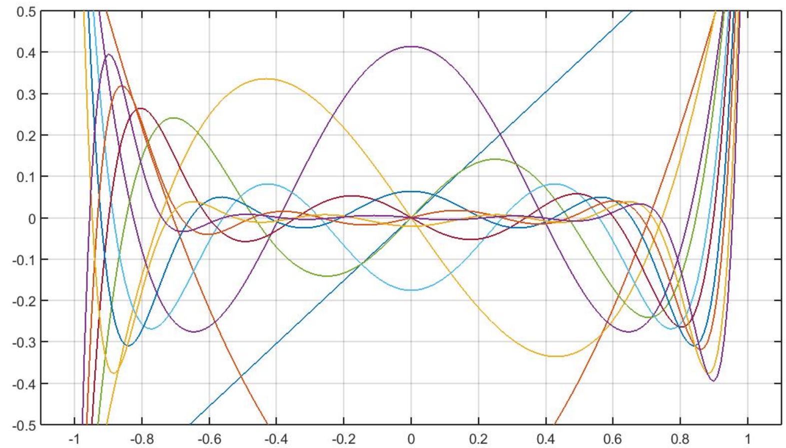

Figure 1. Plot of $h_n^{-3/4}P_n(i\sqrt{3n}x)$ for $n = 1, \ldots, 9$. The scaling exponent $-3/4$ was chosen ad hoc to fit the plots on the Figure.

Proof of explicit formulas for h_n and β_n. The proof proceeds by exhibiting an explicit formula for the determinant $|D_n|$.

Theorem 3.3.

$$|D_n| = \begin{cases} \left(\prod_{m=0}^{\lfloor \frac{n}{2} \rfloor}(2m+1)!\right)^2 (n+2)!!, & \textit{if } n \textit{ is odd,} \\ \left(\prod_{m=0}^{\lfloor \frac{n}{2} \rfloor}(2m+1)!\right)^2 \frac{(n+1)!!}{(n+1)!}, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof: First consider the matrix D' that consists of odd columns and odd rows of matrix D_n. (Note that the enumeration of entries in columns and rows starts from 0. However, we call the column with entries $s_{k,0}$ the first column and consider it as the odd column.)

$$D' = \begin{pmatrix} 1 & 3!! & \cdots & (2m+1)!! \\ 3!! & 5!! & \cdots & (2m+3)!! \\ \vdots & \vdots & \ddots & \vdots \\ (2m+1)!! & (2m+3)!! & \cdots & (4m+1)!! \end{pmatrix},$$

where $m = \left\lfloor \frac{n}{2} \right\rfloor$. We claim that

$$AD' = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{1,0} & 1 & 0 & \cdots & 0 \\ a_{2,0} & a_{2,1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,0} & a_{m,1} & a_{m,2} & \cdots & 1 \end{pmatrix} D'$$

$$= \begin{pmatrix} 1 & * & * & \cdots & * \\ 0 & 3! & * & \cdots & * \\ 0 & 0 & 5! & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (2m+1)! \end{pmatrix},$$

where matrix entries a_{st} are given by formula

$$a_{st} = \begin{cases} (-2)^{t-s} \frac{(2s+1)!}{(2t+1)!(s-t)!}, & \text{if } 0 \le t \le s \le m, \\ 0, & \text{otherwise.} \end{cases}$$

Indeed,

$$\begin{aligned} \left(AD'\right)_{s,p} &= \sum_{t=0}^{s} a_{st} D'_{tp} \\ &= \sum_{t=0}^{s} (-2)^{t-s} \frac{(2s+1)!}{(2t+1)!(s-t)!} \left(2(t+p)+1\right)!! \\ &= (-1)^s \left(2s+1\right)!! \sum_{t=0}^{s} \left(-1\right)^t \binom{s}{t} M_p\left(t\right), \qquad (5) \end{aligned}$$

where

$$M_p\left(t\right) = \frac{(2(t+p)+1)!!}{(2t+1)!!}$$

is a polynomial of degree p in t.

Lemma 3.4.
For every polynomial $f_p(t)$ of degree $p < s$,

$$\sum_{t=0}^{s} (-1)^t \binom{s}{t} f_p(t) = 0.$$

If $p = s$, then

$$\sum_{t=0}^{s} (-1)^t \binom{s}{t} f_p(t) = (-1)^s s! a_0,$$

where a_0 is the coefficient before the highest term t^p in $f_p(t)$.

Proof of Lemma 1.8. For $p = 0$, this is clear,

$$\sum_{t=0}^{s} (-1)^t \binom{s}{t} = (1-1)^s = 0.$$

Consider $g(x) = (1-x)^s$. Then

$$g'(x) = -s(1-x)^{s-1} = \sum_{t=1}^{s} (-1)^t \binom{s}{t} t x^{t-1},$$

and if $s > 1$, then by substituting $x = 1$ we obtain

$$\sum_{t=0}^{s} (-1)^t \binom{s}{t} t = 0.$$

Similarly, by differentiating $g(x)$ twice, we get for $s > 2$,

$$\sum_{t=0}^{s} (-1)^t \binom{s}{t} t(t-1) = 0.$$

Together with the previous result, this implies that

$$\sum_{t=0}^{s} (-1)^t \binom{s}{t} t^2 = 0.$$

We can then proceed by induction, and prove that for every $p < s$,

$$\sum_{t=0}^{s} (-1)^t \binom{s}{t} t^p = 0.$$

Finally, for $p = s$, we find that

$$\sum_{t=0}^{s} (-1)^t \binom{s}{t} t^p = (-1)^s s!$$

□

An application of this lemma to the expression (5) shows that $(AD')_{s,p} = 0$ for $p < s$ and $(AD')_{s,p} = (2s+1)!!s!2^s = (2s+1)!$ This completes the proof of the claim.

The next step is to consider matrix D'' that consists of even rows and columns of matrix D_n. This submatrix is defined only for $n \geq 1$,

$$D'' = \begin{pmatrix} 3!! & 5!! & \cdots & (2m+3)!! \\ 5!! & 7!! & \cdots & (2m+5)!! \\ \vdots & \vdots & \ddots & \vdots \\ (2m+3)!! & (2m+5)!! & \cdots & (4m+3)!! \end{pmatrix},$$

where $m = \left\lfloor \frac{n-1}{2} \right\rfloor$.

Now, we find a lower-triangular matrix B such that BD'' is upper triangular. Namely, we claim that

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ b_{1,0} & 1 & 0 & \cdots & 0 \\ b_{2,0} & b_{2,1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{m,0} & b_{m,1} & b_{m,2} & \cdots & 1 \end{pmatrix} D''$$

$$= \begin{pmatrix} 1! \times 3 & * & * & \cdots & * \\ 0 & 3! \times 5 & * & \cdots & * \\ 0 & 0 & 5! \times 7 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (2m+1)! \times (2m+3) \end{pmatrix},$$

where b_{st} are given by the formula

$$b_{st} = \frac{2s+3}{2t+3} a_{st} = \begin{cases} (-\frac{1}{2})^{s-t} \frac{2s+3}{2t+3} \frac{(2s+1)!}{(2t+1)!(s-t)!}, & \text{if } 0 \leq t \leq s \leq m, \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, similar to the calculation in (5) we find that

$$(BD'')_{s,p} == (-1)^s (2s+3)(2s+1)!! \sum_{t=0}^{s} (-1)^t \binom{s}{t} N_p(t),$$

where

$$N_p(t) = \frac{(2(t+p)+3)!!}{(2t+3)!!}$$

is a polynomial of degree p in t. By applying Lemma 1.8, we find that $(BD'')_{s,p} = 0$ for $p < s$ and

$$(BD'')_{s,s} = 2^s s!\,(2s+3)\,(2s+1)!! = (2s+3)\,(2s+1)!,$$

which proves the claim.

Combining these two pieces, we find that the determinant of D_n is given by the formula

$$\det(D_n) = \begin{cases} \left(\prod_{m=0}^{\lfloor \frac{n}{2} \rfloor} (2m+1)!\right)^2 (n+2)!!, & \text{if } n \text{ is odd,} \\ \left(\prod_{m=0}^{\lfloor \frac{n}{2} \rfloor} (2m+1)!\right)^2 \frac{(n+1)!!}{(n+1)!}, & \text{if } n \text{ is even.} \end{cases}$$

□

Corollary 3.5.

$$h_n = \begin{cases} n!(n+2), & \textit{if } n \textit{ is odd,} \\ (n+1)!, & \textit{if } n \textit{ is even.} \end{cases}$$

$$\beta_n = \begin{cases} (n+2), & \textit{if } n \textit{ is odd,} \\ n, & \textit{if } n \textit{ is even.} \end{cases}$$

Proof: This follows by the direct calculation from the formula for $\det(D_n)$. □

4. Q POLYNOMIALS

Let s be real and define the polynomials $Q(s)$ by the formula:

$$Q(s) = \mathbf{i}^{-n} P_n(\mathbf{i}s). \tag{6}$$

The first ten Q_n are shown in Table 3.

n	$Q_n(x)$
0	1
1	x
2	$x^2 - 3$
3	$x^3 - 5x$
4	$x^4 - 10x^2 + 15$
5	$x^5 - 14x^3 + 35x$
6	$x^6 - 21x^4 + 105x^2 - 105$
7	$x^7 - 27x^5 + 189x^3 - 315x$
8	$x^8 - 36x^6 + 378x^4 - 1260x^2 + 945$
9	$x^9 - 44x^7 + 594x^5 - 2772x^3 + 3465x$

Table 3. $Q_n(x)$ polynomials

Theorem 4.1 —Properties of Q - polynomials —.
(i) Polynomials $Q_n(x)$ satisfy the following recursion:

(7) $$Q_{n+1}(x) = xQ_n(x) - \beta_n Q_{n-1}(x),$$

(ii) Polynomials $Q_n(x)$ are orthogonal with respect to a non-negative measure ν on $\mathbb{R}$.
(iii) The coefficients of every polynomial $Q_n(x)$ are real.
(iv) All the zeros of a polynomial $Q_n(x)$ are simple and real.
(v) Any two zeros of a polynomial $Q_n(x)$ are separated by a zero of polynomial $Q_{n-1}(x)$ and vice versa.

Proof. Formula (7) follows from the properties of $P_n(x)$. Claim (ii) follows by Favard's theorem, because β_n are positive. Claim (iii) is implied by (i) becase β_n are real. Claims (iv) and (v) are implied by (ii), see Theorem 1.2.2 in Akhieser [1]. □

Theorem 4.2 —Orthogonality of Q - polynomials —. *The polynomials $Q_n(x)$ are monic orthogonal polynomials with respect to measure ν with density*

$$f(t) = \frac{1}{\sqrt{2\pi}}\, t^2\, e^{-t^2/2},$$

defined on all real line.

Proof. The moments of this measure are $m_{2k+1} = 0$ and $m_{2k} = (2k+1)!!$. By using these moments, we can calculate the coefficients in the 3-term recurrence relation for orthogonal polynomials related to this measure. It turns out that these coefficients are the same as for the polynomials Q_n. Since the initial conditions are also satisfied, Q_n are the monic orthogonal polynomials for the measure ν. □

Theorem 4.3 —Relation of the Q- and Laguerre polynomials —.

$$Q_{2n}(x) = (-2)^n n!\, L_n^{(1/2)}\left(\frac{x^2}{2}\right),$$

$$Q_{2n+1}(x) = (-2)^n n!\, x\, L_n^{(3/2)}\left(\frac{x^2}{2}\right),$$

where $L_n^{(\alpha)}(x)$ denote the Laguerre polynomials.

Proof. The orthogonality relations for the Laguerre polynomials are

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)\, dx = \delta_{n\,m}\Gamma(\alpha+1)\binom{n+\alpha}{n}.$$

After a change of variable in this relations, the claim of the theorem directly follows from Theorem 1.11. □

References

[1] N. I. Akhieser. *The Classical Problem of Moments*. State Publishing House of Physical and Mathematical Literature, Moscow, 1961. In Russian. An English translation is available.