

# On fluctuations of Riemann's zeta zeros

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**Abstract** It is shown that the normalized fluctuations of Riemann's zeta zeros around their predicted locations follow the Gaussian law. It is also shown that fluctuations of two zeros,  $\gamma_k$  and  $\gamma_{k+x}$ , with  $x \sim (\log k)^\beta$ ,  $\beta > 0$ , for large  $k$  follow the two-variate Gaussian distribution with correlation  $(1 - \beta)_+$ .

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## 1 Introduction

This paper is concerned with the statistical properties of the Riemann zeta function zeros. This subject originated in 1944, when Selberg [26] showed that the number of zeros in a sufficiently long interval on the critical line can be described by the Gaussian law (see also [7, 10, 11, 24]). In the 1970s, Montgomery and Dyson discovered the remarkable fact that the spacings between the zeta zeros resemble the spacings between the eigenvalues of random Hermitian Gaussian matrices. This resemblance was substantiated analytically by Montgomery [19] and supported numerically by Odlyzko [21] (see also [3, 23]). The connection between zeta zeros and random matrix eigenvalues drew much attention, as can be seen for example from review papers in [18]. Recently, Bourgade [1] supported this connection by showing that at the mesoscopic level Riemann's zeros have correlations previously found by Diaconis and Evans [5] for eigenvalues of unitary random matrices (see also [2]).

The motivation for our study comes from a paper by Gustavsson [9], who showed that eigenvalues of random Hermitian matrices fluctuate according to the Gaussian

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law. Our goal is to investigate the statistical fluctuations of Riemann’s zeros around their predicted positions and to show that these fluctuations also follow the Gaussian law.

We denote the non-trivial zeros of Riemann’s zeta function by  $\beta_k + i\gamma_k$ . (we do not assume Riemann’s hypothesis in this paper). We consider only zeros with  $\beta_k \geq 1/2$  and positive imaginary part,  $\gamma_k > 0$ , and order them so that the imaginary part is non-decreasing,  $\gamma_1 \leq \gamma_2 \leq \dots$ .

Let  $\mathcal{N}(T)$  denote the number of zeros with the imaginary part strictly between 0 and  $T$ . If there is a zero with imaginary part equal to  $T$ , then we count this zero as  $1/2$ .

Define

$$S(T) := \frac{1}{\pi} \text{Im} \log \zeta \left( \frac{1}{2} + iT \right),$$

where the logarithm is calculated by continuous variation along the contour  $\sigma + iT$ , with  $\sigma$  changing from  $+\infty$  to  $1/2$ .

It is known (see Chapter 15 in [4]) that

$$\mathcal{N}(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8} + S(T) + O\left(\frac{1}{1+T}\right).$$

Let  $t_k$  be the solution of the equation

$$\frac{t}{2\pi} \log \frac{t}{2\pi e} + \frac{7}{8} = k - 1/2.$$

It is convenient to think about  $t_k$  as predicted imaginary parts of Riemann’s zeros,  $\gamma_k$ . Note that the distance between consecutive  $t_k$  are of order  $1/\log t_k$ . Let

$$\sigma_k := \frac{\sqrt{2 \log \log t_k}}{\log t_k},$$

and define

$$f_k = \frac{\gamma_k - t_k}{\sigma_k}. \tag{1}$$

The quantities  $f_k$  show normalized fluctuations of imaginary parts of Riemann’s zeros from their predicted locations  $t_k$ . In order to study the statistical properties of  $f_k$  we introduce a probability space  $\{\Omega, \mathcal{B}, \mathbb{P}\}$ , where  $\Omega = [0, 1]$ ,  $\mathcal{B}$  is the  $\sigma$ -algebra of Borel subsets of  $\Omega$ , and  $\mathbb{P}$  is the Lebesgue measure on  $\mathcal{B}$ .

Let us fix  $\theta \in (1/2, 1]$ . We define a sequence of random variables  $f^{(N)}$  by the following formula:

$$f^{(N)}(\omega) := f_{k(N,\omega)}, \tag{2}$$

where

$$k(N, \omega) := \lfloor N + \omega \lfloor N^\theta \rfloor \rfloor, \tag{3}$$

$\omega \in \Omega$ , and  $\lfloor x \rfloor$  denotes the largest integer which is less than or equal to  $x$ . Hence  $k(N, \omega)$  is a random variable uniformly distributed on  $I_N = \mathbb{Z} \cap [N, N + \lfloor N^\theta \rfloor - 1]$ . Note that

$$\mathbb{P} \left\{ f^{(N)} \in (a, b) \right\} = \frac{1}{\lfloor N^\theta \rfloor} |\{k : k \in I_N, f_k \in (a, b)\}|,$$

and

$$\mathbb{E} \left( f^{(N)} \right)^r = \frac{1}{\lfloor N^\theta \rfloor} \sum_{k \in I_N} (f_k)^r.$$

First, we will prove the following theorem.

**Theorem 1.1** *Suppose that random variables  $f^{(N)}$  are defined as in (2) with  $1/2 < \theta \leq 1$ . Then, as  $N \rightarrow \infty$ , we have:*

(i) *for every real  $\xi$ ,*

$$\mathbb{P} \left\{ f^{(N)} > \xi \right\} \rightarrow \frac{1}{\sqrt{2\pi}} \int_{\xi}^{\infty} e^{-x^2/2} dx, \text{ and}$$

(ii) *for every integer  $p \geq 0$ ,*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left( f^{(N)} \right)^p = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^p e^{-x^2/2} dx.$$

The requirement that  $\theta > 1/2$  comes from a density estimate for Riemann zeros. This estimate says that the number of zeros with the imaginary part in the interval  $[T, T + H]$  and the real part in  $(\sigma, \infty)$ ,  $\sigma \geq 1/2$ , is bounded by a multiple of  $HT^{-\alpha(\sigma-1/2)} \log T$ , where  $\alpha$  is a positive constant, provided that  $T$  is sufficiently large. The bound is uniform in  $\sigma$ . Selberg’s density theorem (Theorem 1 in [25]) establishes this result for  $H \geq T^\theta$ ,  $\theta > 1/2$ . Karatsuba [12, 13] established the density estimate for  $\theta > 27/82$ . Moreover, Korolev showed in [14] that the density estimate holds for “almost all”  $T$  if  $H > T^\varepsilon$ , where  $\varepsilon$  is an arbitrary positive constant. We expect that the results in our Theorem 1.1 can be improved to include the cases  $\theta > 27/82$  and perhaps even the case  $\theta > 0$  by using these density estimates.

It is also interesting to ask how  $f_k$  and  $f_{k'}$  are related when  $k$  and  $k'$  are sufficiently close to each other. More precisely, define random variables  $f_1^{(N)}$  and  $f_2^{(N)}$  by the formula:

$$f_i^{(N)}(\omega) = f_{k_i(N, \omega)}, \quad i = 1, 2, \tag{4}$$

where

$$k_1(N, \omega) = N + \lfloor \omega N \rfloor,$$

$$k_2(N, \omega) = N + \lfloor \omega N \rfloor + \lceil (\log N)^\beta \rceil,$$

and  $\beta > 0$  (we set here  $\theta = 1$  for simplicity. However, the result below is likely to hold for all  $\theta \in (27/82, 1)$ ).

**Theorem 1.2** *Suppose that random variables  $f_i^{(N)}$  are defined as in (4) with  $\beta > 0$ , and suppose that  $Y_1, Y_2$  are zero-mean Gaussian random variables with  $\mathbb{E}(Y_i^2) = 1$  and  $\mathbb{E}(Y_1 Y_2) = (1 - \beta)_+$ . Then as  $N \rightarrow \infty$ ,*

- (i) *the joint cumulative distribution function of  $(f_1^{(N)}, f_2^{(N)})$  converges pointwise to the joint cumulative distribution function of  $(Y_1, Y_2)$ , and*
- (ii) *the joint moments of  $(f_1^{(N)}, f_2^{(N)})$  converge to the corresponding joint moments of  $(Y_1, Y_2)$ .*

After the first version of this article was completed, the author learned from M. A. Korolev about his papers [15] and [16] (based on earlier results by Karatsuba and Korolev in [12] and [13]), that consider similar questions. See, for example, Theorem 10 in [16] which is similar to our Theorem 1.1. However, the joint distribution of the fluctuations of zeta zeros is not studied in these papers.

We have shown that the distribution of two zeta zero fluctuations approaches a two-variate Gaussian distribution. By a natural extension of the argument, with a more cumbersome notation, it is possible to show that the distribution of any finite number of fluctuations approaches a multivariate Gaussian distribution with the covariance matrix  $\mathbb{E}X_i X_j = (1 - \beta_{ij})_+$ , where

$$\beta_{ij} = \lim_{N \rightarrow \infty} \frac{\log |k_j(N, \omega) - k_i(N, \omega)|}{\log \log N},$$

and the limit is assumed to be positive and the same for all  $\omega$  (the fluctuations are around the predicted locations  $t_{k_i(N, \omega)}$ , and the functions  $k_i(N, \omega)$  are defined as in (4) with appropriate changes).

The positive numbers  $\beta_{ij}$  in the covariance matrix are not arbitrary but satisfy the ultrametric inequality:

$$\beta_{ik} \leq \max \{ \beta_{ij}, \beta_{jk} \}.$$

More about this covariance structure can be found in Section 4 of [1], where it is shown, in particular, how this structure can arise as a result of a branching process.

Covariances that satisfy ultrametric inequalities are of interest in statistical physics. They are used, in particular, in the theory of frustrated disordered systems (“spin glasses”), where they are crucial in a proposed description of local equilibria by replica method (see [17, 22, 28]). A possible reason for the appearance of ultrametric structure

in this area of physics is the close relation of spin glasses with random matrices where ultrametric covariances describe the eigenvalue distribution.

In particular, Diaconis and Evans in [5] considered uniformly distributed  $N$ -by- $N$  random unitary matrices and determined the covariances for the eigenvalue counts in given intervals for large  $N$  (see Theorems 6.1 and 6.3 in their paper). They found that these covariances have an interesting and unusual structure. A similar structure was found for Gaussian Hermitian random matrices and other random matrix ensembles by Soshnikov in [27]. In fact, for random unitary matrices this structure can be seen as a consequence of the ultrametric covariances exhibited by characteristic polynomials of these matrices (Theorem 1.4 in Bourgade's paper [1]). Bourgade has also found a parallel result for counts of Riemann zeros in given intervals (Theorem 1.1 and Corollary 1.3 in [1]).

In another development, Gustavsson [9] studied eigenvalues of Gaussian Hermitian matrices and found the ultrametric structure in covariances defined by using the deviations of individual eigenvalues from their predicted locations (Theorems 1.3 and 1.4 in [9]). This setup is similar to what we do in this paper and the results are also remarkably similar.

However, while the results are similar, the methods are quite different. In random matrix papers, the method is based either on group representation theory which allows one to compute average traces of matrix powers (as in [1, 5]), or on explicit formulas for the distribution of eigenvalues (as in [9, 27]). In contrast, in number-theoretic papers, the method is based on the Selberg approximation formula for the number of Riemann zeros with ordinates between zero and  $T$ , and on a multitude of other facts from number theory, which allow one to estimate the powers of this approximate function. The fact that these distinct methods lead to very similar results is rather mysterious.

An interested reader can find more about relations of Riemann's zeros and random matrices in review papers mentioned in the beginning of this paper.

The rest of the paper is organized as follows. Section 2 outlines the scheme of the proof of Theorems 1.1 and 1.2. Section 3 introduces some technical tools that we will need in the proof of the main theorems. Section 4 proves a modification of the key approximation result by Selberg. Section 5 calculates the moments of the approximate function  $S_x$ . Section 6 calculates the moments of  $S$  and concludes the proof of Theorem 1.1. Section 7 proves Theorem 2.3. Section 8 proves Theorem 2.4 and concludes the proof of Theorem 1.2. And Sect. 9 concludes.

## 2 Outline of proofs

In the proof we use the strategy used by Gustavsson in his work on the fluctuations of eigenvalues in the Gaussian Unitary Ensemble. The first step in Gustavsson's proof is to relate fluctuations of an individual eigenvalue to fluctuations of eigenvalue counts in a fixed interval. This allows one to use existing methods for finding the distribution of eigenvalue counts.

In our setup, an analogous step requires connecting the random fluctuations  $f_{k(N, \omega)}$  to the number of Riemann zeros in the interval  $[0, t_{k(N, \omega)}]$ , which can be approximated by the function  $S(t_{k(N, \omega)})$ .

It is convenient to define

$$X_k := \frac{\sqrt{2\pi} S(t_k + \xi\sigma_k)}{\sqrt{\log \log t_k}},$$

and a corresponding sequence of random variables

$$X^{(N)}(\omega) := X_{k(N,\omega)}, \tag{5}$$

where  $k(N, \omega)$  is as in (3).

A connection between  $X^{(N)}$  and  $f^{(N)}$  can be seen as follows. For every real  $\xi$ ,

$$\begin{aligned} \mathbb{P}\{f^{(N)} > \xi\} &= \frac{1}{H_N} |\{k : k \in I_N, \gamma_k > t_k + \xi\sigma_k\}| \\ &= \frac{1}{H_N} |\{k : k \in I_N, \mathcal{N}(t_k + \xi\sigma_k) \leq k - 1/2\}| \\ &= \frac{1}{H_N} \left| \left\{ k : k \in I_N, \frac{t_k + \xi\sigma_k}{2\pi} \log \frac{t_k + \xi\sigma_k}{2\pi e} + \frac{7}{8} \right. \right. \\ &\quad \left. \left. + S(t_k + \xi\sigma_k) + O(1/t_k) \leq k - 1/2 \right\} \right| \\ &= \frac{1}{H_N} |\{k : k \in I_N, S(t_k + \xi\sigma_k) \\ &\leq -\xi \sqrt{\frac{\log \log t_k}{2\pi^2}} \left(1 + \frac{\log 2\pi}{\log t_k}\right) + o(1/t_k)\}| \\ &= \frac{1}{H_N} \left| \left\{ k : k \in I_N, \frac{\sqrt{2\pi} S(t_k + \xi\sigma_k)}{\sqrt{\log \log t_k}} \right. \right. \\ &\leq \left. \left. -\xi \left(1 + \frac{\log 2\pi}{\log t_k}\right) + o(1/t_k) \right\} \right|. \end{aligned}$$

Since  $t_k$  is asymptotically close to  $2\pi k / \log k$ , it follows that

$$\begin{aligned} \mathbb{P}\left\{X^{(N)} \leq -\xi - \frac{c_1 |\xi|}{\log N} + o\left(\frac{\log N}{N}\right)\right\} &\leq \mathbb{P}\{f^{(N)} > \xi\} \\ &\leq \mathbb{P}\left\{X^{(N)} \leq -\xi + \frac{c_2 |\xi|}{\log N} + o\left(\frac{\log N}{N}\right)\right\}, \end{aligned} \tag{6}$$

where  $c_1$  and  $c_2$  are two constants. Hence for large  $N$ , the distribution of the random variable  $f^{(N)}$  is essentially determined by the distribution of the random variable  $X^{(N)}$ .

In this connection, it is appropriate to recall the following theorem by Selberg (Theorem 3 in [26]). Let

$$X(t) := \frac{\sqrt{2\pi} S(t)}{\sqrt{\log \log t}}.$$

**Theorem 2.1** (Selberg) *Assume RH, and let  $T^a \leq H \leq T^2$ , where  $a > 0$ . Then for every  $k \geq 1$*

$$\frac{1}{H} \int_T^{T+H} |X(t)|^{2k} dt = \frac{2k!}{k!2^k} + O(1/\log \log T),$$

with the constant in the remainder term that depends only on  $k$  and  $a$ .

In other words, the even moments of the function  $X(t)$  behave as the moments of a standard Gaussian variable. This was refined in [7], where it was shown in particular that for every interval  $I$ ,

$$\frac{1}{H} \int_T^{T+H} \mathbf{1}_I [X(t)] dt = \frac{1}{\sqrt{2\pi}} \int_I e^{-x^2/2} dx + o(1),$$

where  $\mathbf{1}_I$  denotes the indicator function of interval  $I$ . We prove a modified version of this result.

**Theorem 2.2** *Suppose that random variables  $X^{(N)}$  are defined as in (5) with  $1/2 < \theta \leq 1$ . Then, for every real  $s$ , as  $N \rightarrow \infty$ ,*

$$\mathbb{E} \mathbf{1}_{(-\infty, s]} (X^{(N)}) \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^s e^{-x^2/2} dx.$$

We will prove Theorem 2.2 by the method of moments, which says that in order to establish the convergence of a sequence of r.v. in distribution to the Gaussian law it is enough to show the convergence of every moment (Example 2.23 on p. 18 in van der Vaart [29]). That is, it is enough to show that

$$\mathbb{E} \left( X^{(N)} \right)^r \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^r e^{-x^2/2} dx \tag{7}$$

for every integer  $r > 0$ . We will show this in Sect. 6 in Corollary 6.2.

The first claim in Theorem 1.1 follows immediately from Theorem 2.2 and inequalities (6). The second claim follows from the first one because (6) and (7) imply that  $(f^{(N)})^{2n}$  are asymptotically uniformly integrable for every  $n > 0$  and therefore the moments of  $f^{(N)}$  converge to the moments of the limiting Gaussian distribution (see Theorem 2.20 in van der Vaart [29]).

In order to prove Theorem 1.2, define random variables

$$X_i^{(N)}(\omega) := X_{k_i(N, \omega)}, \quad i = 1, 2, \tag{8}$$

where  $k_i(N, \omega)$  are as in (4). Let us use notations

$$I_1^{(N)} = [N, 2N - 1], I_2^{(N)} := [N + [(\log N)^\beta], 2N - 1 + [(\log N)^\beta]], \text{ and } \mathbf{s} := (s_1, s_2).$$

Then,

$$\begin{aligned} \mathbb{P} \left\{ f_1^{(N)} > \xi_1, f_2^{(N)} > \xi_2 \right\} &= N^{-2} \left| \left\{ \mathbf{s} \in \mathbb{Z}^2 : s_i \in I_i^{(N)}, \gamma_{s_i} > t_{s_i} + \xi_i \sigma_{s_i}, i = 1, 2 \right\} \right| \\ &= N^{-2} \left| \left\{ \mathbf{s} : s_i \in I_i^{(N)}, \mathcal{N}(t_{s_i} + \xi_i \sigma_{s_i}) \leq k_i - 1/2 \right\} \right| \\ &= N^{-2} \left| \left\{ \mathbf{s} : s_i \in I_i^{(N)}, \frac{\sqrt{2\pi} S(t_{s_i} + \xi_i \sigma_{s_i})}{\sqrt{\log \log t_{s_i}}} \right. \right. \\ &\quad \left. \left. \leq -\xi_i \left( 1 + \frac{\log 2\pi}{\log t_{s_i}} \right) + o(1/t_{s_i}) \right\} \right|. \end{aligned}$$

That is, with some positive  $c_1$  and  $c_2$ , we have

$$\mathbb{P} \left\{ f_1^{(N)} > \xi_1, f_2^{(N)} > \xi_2 \right\} \leq \mathbb{P} \left\{ X_i^{(N)} \leq -\xi_i + \frac{c_1}{\log N} |\xi_i| + o\left(\frac{\log N}{N}\right), i = 1, 2 \right\}, \tag{9}$$

and

$$\mathbb{P} \left\{ f_1^{(N)} > \xi_1, f_2^{(N)} > \xi_2 \right\} \geq \mathbb{P} \left\{ X_i^{(N)} \leq -\xi_i - \frac{c_2}{\log N} |\xi_i| + o\left(\frac{\log N}{N}\right), i = 1, 2 \right\}. \tag{10}$$

In words, the joint cumulative distribution function of  $f_1^{(N)}$  and  $f_2^{(N)}$  approaches that of  $X_1^{(N)}$  and  $X_2^{(N)}$ .

First of all, we have the following result for the random variables  $X_1^{(N)}$  and  $X_2^{(N)}$ .

**Theorem 2.3** *Let  $X_i^{(N)}$  be defined as in (8). Then,*

$$\lim_{N \rightarrow \infty} \mathbb{E} X_1^{(N)} X_2^{(N)} = (1 - \beta)_+ := \begin{cases} 1 - \beta, & \text{if } \beta \in (0, 1), \\ 0, & \text{if } \beta \geq 1. \end{cases}$$

More generally, the following result holds.

**Theorem 2.4** *Let  $X_i^{(N)}$  be defined as in (8). Then for every  $l, m \geq 0$ ,  $\mathbb{E} \left( X_1^{(N)} \right)^l \left( X_2^{(N)} \right)^m$  converges to  $\mathbb{E} (Y_1)^l (Y_2)^m$  where  $(Y_1, Y_2)$  is a zero-mean Gaussian random variable with  $\mathbb{E} (Y_i^2) = 1$  and  $\mathbb{E} (Y_1 Y_2) = (1 - \beta)_+$ .*



Theorem 2.3 is a particular case of Theorem 2.4. However, we will prove it separately, since its proof is more transparent and shows how the proof of the more general Theorem 2.4 proceeds.

Given Theorem 2.4, we can prove Theorem 1.2.

*Proof of Theorem 1.2* Theorem 2.4 implies that the cumulative distribution function of  $(X_1^{(N)}, X_2^{(N)})$  converges pointwise to the cumulative distribution function of the Gaussian variable  $(X_1, X_2)$ . The first claim of the theorem follows immediately from this fact and inequalities (9) and (10). In addition, Theorem 2.4 and inequalities (9) and (10) imply that for all integer  $a, b \geq 0$ , the random variables  $(f_1^{(N)})^a (f_2^{(N)})^b$  are asymptotically uniformly integrable. Hence, their expectations converge to the corresponding expectation of the limit,  $\mathbb{E}(Y_1)^a (Y_2)^b$  (by Theorem 2.20 in van der Vaart [29]). This completes the proof of the second claim of the theorem.  $\square$

The proof of the convergence of moments of  $X^{(N)}$  follows the plan of the argument in Selberg [26].

Recall that  $X^{(N)}$  is a rescaled version of  $S(t_k + \xi\sigma_k)$  where  $k$  is random. The first step in Selberg’s proof is to show that  $S(t)$  can be approximated by  $S_x(t)$ , where

$$S_x(t) := -\frac{1}{\pi} \sum_{p \leq x^3} \frac{\sin(t \log p)}{\sqrt{p}}.$$

That is, Selberg shows that

$$\frac{1}{H} \int_K^{K+H} (S(t) - S_x(t))^{2n} dt$$

is small provided that  $K$  and  $H$  are sufficiently large and that  $x \sim K^\varepsilon$  with a sufficiently small  $\varepsilon > 0$ . In our case we will need to modify this result in order to show that the integral can be replaced by a sum over a discrete set of points.

The next step in Selberg’s proof is to calculate the moments

$$\frac{1}{H} \int_K^{K+H} |S_x(t)|^{2n} dt.$$

Again it will be necessary to prove a corresponding result for a sum over a discrete set of points.

Given the results in these two steps, it is relatively easy to calculate the moments of the random variable  $S(t_k + \xi\sigma_k)$ . This will be done essentially as in Selberg’s paper. However, we will need to extend the calculation to the multivariate case with two random variables  $S(t_k + \xi_1\sigma_k)$  and  $S(t_k + \xi_2\sigma_k)$ .

### 3 Exponential sums

The changes in Selberg's proof make it necessary to estimate certain exponential sums. The main additional tool that we use to handle these sums is the following theorem by van der Corput (Theorem 2.2 in [8]). Let  $e(f(n))$  denote  $\exp[2\pi i f(n)]$ .

**Theorem 3.1** (van der Corput) *Suppose that  $f$  is a real valued function with two continuous derivatives on interval  $I$ . Suppose also that there is some  $\lambda > 0$  and some  $\kappa \geq 1$  such that*

$$\lambda \leq |f''(x)| \leq \kappa\lambda$$

on  $I$ . Then,

$$\sum_{n \in I} e(f(n)) = O\left(\kappa |I| \lambda^{1/2} + \lambda^{-1/2}\right).$$

In order to apply this theorem in our situation, we need to estimate derivatives of a function  $g(x)$  that we are about to define. Let  $t(x)$  be the functional inverse of the function

$$x(t) = \frac{t}{2\pi} \log \frac{t}{2\pi e} + \frac{11}{8} \quad (11)$$

on interval  $[t_0, \infty)$  where  $t_0$  is sufficiently large (the coefficients  $11/8$  is not necessary is not necessary for the argument. It is included only because its presence makes  $t(k)$  an unbiased estimator of  $\gamma_k$ ). Note that  $t(x)$  is an increasing concave function. Let

$$g(x) := t(x) + \xi \frac{\sqrt{2 \log \log t(x)}}{\log t(x)}, \quad (12)$$

where  $\xi$  is a real constant. This function is well defined for  $x$  greater than some numeric constant  $x_0$ . For  $x$  between 0 and  $x_0$ , we define  $g(x)$  in an arbitrary fashion such that  $g(x)$  has a continuous 3-rd derivative for all  $x \geq 0$ .

**Lemma 3.2** *When  $x \rightarrow \infty$ ,*

$$g''(x) \sim -\frac{2\pi}{x (\log x)^2}$$

and

$$g'''(x) \sim \frac{2\pi}{x^2 (\log x)^2}.$$

*Proof* The identity

$$x = \frac{t(x)}{2\pi} \log \frac{t(x)}{2\pi e} + \frac{11}{8}$$

implies that

$$t \sim 2\pi \frac{x}{\log x}, t' \sim 2\pi \frac{1}{\log x}, t'' \sim -\frac{2\pi}{x (\log x)^2}, \text{ and } t''' \sim \frac{2\pi}{x^2 (\log x)^2}.$$

If

$$h := \frac{\sqrt{\log \log t(x)}}{\log t(x)},$$

then a calculation shows that  $h'' = o(t'')$ ,  $h''' = o(t''')$  and therefore

$$g'' \sim -\frac{2\pi}{x (\log x)^2} \text{ and } g''' \sim \frac{2\pi}{x^2 (\log x)^2}.$$

□

In the following we will use the notation  $g_k$  for  $g(k) \equiv t_k + \xi \sigma_k$ .

**Lemma 3.3** *Let*

$$\theta = \log \left( \frac{p_{l+1} \cdots p_{2n}}{p_1 \cdots p_l} \right),$$

where  $1 \leq l \leq 2n$ ,  $\{p_1, \dots, p_l\} \neq \{p_{l+1}, \dots, p_{2n}\}$  and primes  $p_i < y$  for all  $i$ . Assume  $1 \leq H \leq cK$ . Then,

$$\sum_{k=K}^{K+H-1} e^{i\theta g_k} = O \left( Hn \frac{y^{n/2} \log y}{K^{1/2} \log K} + y^{n/2} K^{1/2} \log K \right).$$

*Proof* From the assumption, we obtain

$$c/y^n \leq |\theta| \leq 2n \log y.$$

(In order to see the first inequality, let  $l \leq n$ . Then

$$\left| 1 - \frac{p_{l+1} \cdots p_{2n}}{p_1 \cdots p_l} \right| = \left| \frac{p_1 \cdots p_l - p_{l+1} \cdots p_{2n}}{p_1 \cdots p_l} \right| \geq \frac{1}{y^n}$$

by the uniqueness of integer factorization, and the desired inequality follows. The case  $l \geq n$  is similar).

Hence, by using Lemma 3.2, we find that

$$\lambda \leq |\theta g''(x)| \leq \kappa \lambda$$

with

$$\lambda = \frac{c}{y^n K (\log K)^2},$$

and

$$\kappa = O(ny^n \log y)$$

By applying van der Corput’s theorem, we obtain

$$\sum_{k=K}^{K+H-1} e^{i\theta g_k} = O\left(Hn \frac{y^{n/2} \log y}{K^{1/2} \log K} + y^{n/2} K^{1/2} \log K\right).$$

□

**Lemma 3.4** *Suppose  $1 \leq c_1 K^\theta \leq H \leq c_2 K$ , where  $\theta > 1/2$  and  $c_1, c_2 > 0$ . Let  $r$  be a positive integer,  $y \leq K^{\frac{2\theta-1}{3r}-\varepsilon}$ , and assume that*

$$|\alpha_p| < A \frac{\log p}{\log y} \text{ for } p < y.$$

Then, we have

$$\sum_{k=K}^{K+H-1} \left| \sum_{p < y} \frac{\alpha_p}{p^{1/2+ig_k}} \right|^{2r} = O(H).$$

*Proof* We can write

$$\left( \sum_{p < y} \frac{\alpha_p}{p^{1/2+ig_k}} \right)^r = \sum_{n < y^r} \frac{\beta_n}{n^{1/2+ig_k}},$$

where  $\beta_n \leq A^r$ . Hence,

$$\begin{aligned} \sum_{k=K}^{K+H-1} \left| \sum_{p < y} \frac{\alpha_p}{p^{1/2+ig_k}} \right|^{2r} &= \sum_{m, n < y^r} \frac{\beta_m \bar{\beta}_n}{\sqrt{mn}} \sum_{k=K}^{K+H-1} \left(\frac{m}{n}\right)^{ig_k} \\ &\leq H \sum_{n < y^r} \frac{|\beta_n|^2}{n} + 2 \sum_{m < n < y^r} \frac{|\beta_m \beta_n|}{\sqrt{mn}} \left| \sum_{k=K}^{K+H-1} \left(\frac{m}{n}\right)^{ig_k} \right|. \end{aligned}$$

The first sum can be estimated as follows:

$$\sum_{n < y^r} \frac{|\beta_n|^2}{n} \leq A^r \sum_{n < y^r} \frac{|\beta_n|}{n} \leq A^r \left( \sum_{p < y} \frac{|\alpha_p|}{p} \right)^r = O(1),$$

where we used Mertens’ result  $\sum_{p < y} \frac{\log p}{p} = O(\log y)$  in the last step.

In order to estimate the second sum we note that

$$1/y^r < \log |n/m| < r \log y;$$

hence we can apply van der Corput’s theorem and estimate

$$\left| \sum_{k=K}^{K+H-1} \left(\frac{m}{n}\right)^{ig_k} \right| \leq O \left( Hr \frac{y^{r/2} \log y}{K^{1/2} \log K} + y^{r/2} K^{1/2} \log K \right).$$

Besides,

$$\sum_{m < n < y^r} \frac{|\beta_m \beta_n|}{\sqrt{mn}} \leq \left( \sum_{p < y} \frac{\alpha_p}{\sqrt{p}} \right)^{2r} = O(y^r).$$

By assumptions about  $H$  and  $y$ , it follows that

$$\begin{aligned} \sum_{m < n < y^r} \frac{|\beta_m \beta_n|}{\sqrt{mn}} \left| \sum_{k=K}^{K+H-1} \left(\frac{m}{n}\right)^{ig_k} \right| &= O(y^r) O \left( Hr \frac{y^{r/2} \log y}{K^{1/2} \log K} + y^{r/2} K^{1/2} \log K \right) \\ &= O(H). \end{aligned}$$

□

**Lemma 3.5** *Suppose  $1 \leq c_1 K^\theta \leq H \leq c_2 K$ , where  $\theta > 1/2$  and  $c_1, c_2 > 0$ . Let  $r$  be a positive integer,  $y \leq K^{\frac{2\theta-1}{3r}-\varepsilon}$ , and assume that*

$$|\alpha_p| < A \text{ for } p < y.$$

Then,

$$\sum_{k=K}^{K+H-1} \left| \sum_{p < y} \frac{\alpha_p}{p^{1+ig_k}} \right|^{2r} = O(H).$$

The proof of this lemma is similar to the proof of the previous one.

#### 4 A consequence of Selberg’s approximation formula

Recall that

$$S_x(t) := -\frac{1}{\pi} \sum_{p \leq x^3} \frac{\sin(t \log p)}{\sqrt{p}}.$$

Our goal in this section is to prove the following result.

**Proposition 4.1** *Suppose  $1 \leq c_1 K^\theta \leq H \leq c_2 K$ , where  $1/2 < \theta \leq 1$  and  $c_1, c_2 > 0$ . Let  $x = K^{\frac{\theta-1/2}{20n}}$ . Then, we have*

$$\sum_{k=K}^{K+H-1} |S(g_k) - S_x(g_k)|^{2n} = O(H).$$

*Proof* Let  $\Lambda(n) = \log p$ , if  $n$  is a power of the prime number  $p$ , and  $\Lambda(n) = 0$ , otherwise. Also, define

$$\Lambda_x(n) = \begin{cases} \Lambda(n), & \text{for } 1 \leq n \leq x, \\ \Lambda(n) \left( \frac{\log^2 \frac{x^3}{n} - 2 \log^2 \frac{x^2}{n}}{2 \log^2 x} \right), & \text{for } x \leq n \leq x^2, \\ \Lambda(n) \frac{\log^2 \frac{x^3}{n}}{2 \log^2 x}, & \text{for } x^2 \leq n \leq x^3. \end{cases}$$

□

Let  $a \in (1/2, 1]$ ,  $x = T^{\frac{a-1/2}{60k}}$ ,  $T^a \leq H \leq T$ ,  $T \leq t \leq T + H$ .

The first formula on p. 37 in [25] [immediately before formula (5.2)] states that

$$\begin{aligned} S(t) - S_x(t) &= O \left( \left| \sum_{p < x^3} \frac{\Lambda(p) - \Lambda_x(p)}{\sqrt{p} \log p} p^{-it} \right| \right) \\ &+ O \left( \left| \sum_{p < x^{3/2}} \frac{\Lambda_x(p^2)}{p \log p} p^{-2it} \right| \right) + O \left( \left( \sigma_{x,t} - \frac{1}{2} \right) \log T \right) \\ &+ O \left( \left( \sigma_{x,t} - \frac{1}{2} \right) x^{\sigma_{x,t} - \frac{1}{2}} \int_{1/2}^{\infty} x^{1/2 - \sigma} \left| \sum_{p < x^3} \frac{\Lambda_x(p) \log(xp)}{p^{\sigma + it}} \right| d\sigma \right), \end{aligned}$$

where

$$\sigma_{x,t} = \frac{1}{2} + \max_{\rho} \left( \beta - \frac{1}{2}, \frac{2}{\log x} \right)$$

and the maximum is taken over all zeros  $\beta + i\gamma$  for which

$$|t - \gamma| \leq \frac{x^{3|\beta-1/2|}}{\log x}.$$

It follows that

$$\begin{aligned}
 & \sum_{k=K}^{K+H-1} |S(g_k) - S_x(g_k)|^{2n} \\
 &= O \left( \sum_{k=K}^{K+H-1} \left| \sum_{p < x^3} \frac{\Lambda(p) - \Lambda_x(p)}{\sqrt{p} \log p} p^{-ig_k} \right|^{2n} \right) \\
 &+ O \left( \sum_{k=K}^{K+H-1} \left| \sum_{p < x^{3/2}} \frac{\Lambda_x(p^2)}{p \log p} p^{-2ig_k} \right|^{2n} \right) \\
 &+ O \left( (\log K)^{2n} \sum_{k=K}^{K+H-1} \left( \sigma_{x, g_k} - \frac{1}{2} \right)^{2n} \right) \\
 &+ O \left( \sum_{k=K}^{K+H-1} \left( \sigma_{x, g_k} - \frac{1}{2} \right)^{2n} x^{2n(\sigma_{x, g_k} - \frac{1}{2})} \right. \\
 &\quad \left. \left\{ \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p < x^3} \frac{\Lambda_x(p) \log(xp)}{p^{\sigma+ig_k}} \right| d\sigma \right\}^{2n} \right). \tag{13}
 \end{aligned}$$

By applying Lemmas 3.4 and 3.5, we find that the first two sums on the right-hand side are  $O(H)$ . For example, for the first term we can apply Lemma 3.4 with

$$\alpha_p = \frac{\Lambda(p) - \Lambda_x(p)}{\log p}.$$

The third term can be estimated by using the following lemma.

**Lemma 4.2** *Suppose  $1 \leq c_1 K^\theta \leq H \leq c_2 K$ , where  $1/2 < \theta \leq 1$  and  $c_1, c_2 > 0$ . Next, suppose that  $x \geq 2, 1 \leq \xi \leq x^{8k}, x^3 \xi^2 \leq \left(\frac{H}{\sqrt{K}}\right)^{1/4}$ . Then we have for  $0 \leq \nu \leq 8k$ ,*

$$\sum_{k=K}^{K+H-1} \left( \sigma_{x, g_k} - \frac{1}{2} \right)^\nu \xi^{\sigma_{x, g_k} - \frac{1}{2}} = O \left( \frac{H}{(\log x)^\nu} \right).$$

This lemma is an analog of Lemma 12 on p.33 in [25] and its proof is the same as the proof of Lemma 12 with minor changes (at this step, Selberg’s density estimate is used).

By applying Lemma 4.2 with  $\xi = 1$  and  $\nu = 2n$  we find that the third term is

$$O \left( H \left( \frac{\log K}{\log x} \right)^{2n} \right) = O(H),$$

provided that, for example,

$$x = K^{\frac{\theta-1/2}{20n}}.$$

It remains to bound the fourth term.

$$\begin{aligned} & \frac{1}{H} \sum_{k=K}^{K+H-1} \left(\sigma_{x, g_k} - \frac{1}{2}\right)^{2n} x^{2n(\sigma_{x, g_k} - \frac{1}{2})} \left\{ \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p < x^3} \frac{\Lambda_x(p) \log(xp)}{p^{\sigma+ig_k}} \right| d\sigma \right\}^{2n} \\ & \leq \left\{ \frac{1}{H} \sum_{k=K}^{K+H-1} \left(\sigma_{x, g_k} - \frac{1}{2}\right)^{4n} x^{4n(\sigma_{x, g_k} - \frac{1}{2})} \right\}^{1/2} \\ & \quad \left\{ \frac{1}{H} \sum_{k=K}^{K+H-1} \left[ \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p < x^3} \frac{\Lambda_x(p) \log(xp)}{p^{\sigma+ig_k}} \right| d\sigma \right]^{4n} \right\}^{1/2} \end{aligned}$$

by the Schwarz inequality. The first term in the product can be estimated as

$$O\left(\frac{1}{(\log x)^{2n}}\right)$$

by Lemma 4.2 with  $\xi = x^{4n}$ . For the second term, we have

$$\begin{aligned} & \left[ \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p < x^3} \frac{\Lambda_x(p) \log(xp)}{p^{\sigma+ig_k}} \right| d\sigma \right]^{4n} \\ & \leq \left[ \int_{1/2}^{\infty} x^{1/2-\sigma} d\sigma \right]^{4n-1} \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p < x^3} \frac{\Lambda_x(p) \log(xp)}{p^{\sigma+ig_k}} \right|^{4n} d\sigma \\ & = \frac{1}{(\log x)^{4n-1}} \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p < x^3} \frac{\Lambda_x(p) \log(xp)}{p^{\sigma+ig_k}} \right|^{4n} d\sigma, \end{aligned}$$

where the second line follows by the Hölder inequality.

Hence, by Lemma 3.4, we obtain

$$\frac{1}{H} \sum_{k=K}^{K+H-1} \left(\sigma_{x, g_k} - \frac{1}{2}\right)^{2n} x^{2n(\sigma_{x, g_k} - \frac{1}{2})} \left\{ \int_{1/2}^{\infty} x^{1/2-\sigma} \left| \sum_{p < x^3} \frac{\Lambda_x(p) \log(xp)}{p^{\sigma+ig_k}} \right| d\sigma \right\}^{2n}$$



$$\begin{aligned}
 &= O\left(\sqrt{\log x} \int_{1/2}^{\infty} x^{1/2-\sigma} \frac{1}{H} \sum_{k=K}^{K+H-1} \left| \sum_{p < x^3} \frac{\Lambda_x(p) \log(xp)}{p^{\sigma+ig_k} \log^2 x} \right|^{4n} d\sigma\right) \\
 &= O\left(\sqrt{\log x} \int_{1/2}^{\infty} x^{1/2-\sigma} d\sigma\right) = O(1)
 \end{aligned}$$

provided that  $x = K^{\frac{\theta-1/2}{20n}}$ . By using this in (13), we find that

$$\sum_{k=K}^{K+H-1} |S(g_k) - S_x(g_k)|^{2n} = O(H).$$

□

### 5 Moments of the approximation to S(t)

In the next step we show that the moments of  $S_x(t)$  are approximately Gaussian.

**Lemma 5.1** *Suppose  $1 \leq c_1 K^\theta \leq H \leq c_2 K$ , where  $\theta > 1/2$  and  $c_1, c_2 > 0$ . Let  $x \leq K^{\frac{2\theta-1}{6n}-\varepsilon}$ . Then, for every integer  $n \geq 1$ ,*

(i)

$$\sum_{k=K}^{K+H-1} |S_x(g_k)|^{2n} = \frac{(2n)!}{(2\pi)^{2n} n!} \left( H (\log \log K)^n + O\left( H (\log \log K)^{n-1} \right) \right),$$

and

(ii)

$$\sum_{k=K}^{K+H-1} S_x(g_k)^{2n-1} = O(H).$$

*Proof* First, we can write

$$S_x(t) = \frac{1}{2\pi i} (\eta - \bar{\eta}),$$

where

$$\eta = \eta(t) = \sum_{p < x^3} p^{-1/2-it}. \tag{14}$$

Hence,

$$\sum_{k=K}^{K+H-1} |S_x(g_k)|^{2n} = \frac{1}{(2\pi)^{2n}} \sum_{l=0}^{2n} (-1)^{n-l} \binom{2n}{l} \sum_{k=K}^{K+H-1} \eta(g_k)^l \bar{\eta}(g_k)^{2n-l}.$$

Here

$$\sum_{k=K}^{K+H-1} \eta(g_k)^l \bar{\eta}(g_k)^{2n-l} = \sum_{p_i < x^3} \frac{1}{\sqrt{p_1 \dots p_{2n}}} \sum_{k=K}^{K+H-1} \left( \frac{p_{l+1} \dots p_{2n}}{p_1 \dots p_l} \right)^{ig_k}.$$

If  $\{p_1, \dots, p_l\} \neq \{p_{l+1}, \dots, p_{2n}\}$ , then by using Lemma 3.3, we obtain

$$\sum_{k=K}^{K+H-1} \left( \frac{p_{l+1} \dots p_{2n}}{p_1 \dots p_l} \right)^{ig_k} \leq O\left( \left( Hn \frac{x^n \log x}{K^{1/2} \log K} + x^n K^{1/2} \log K \right) \right).$$

Since

$$\sum_{p_i < x^3} \frac{1}{\sqrt{p_1 \dots p_{2n}}} = O\left( \left( \sum_{p \leq x^3} \frac{1}{\sqrt{p}} \right)^{2n} \right) = O(x^{2n}),$$

hence in the case  $l \neq n$ , we have

$$\begin{aligned} \sum_{k=K}^{K+H-1} \eta(g_k)^l \bar{\eta}(g_k)^{2n-l} &= O\left( x^{2n} \left( Hn \frac{x^n \log x}{K^{1/2} \log K} + x^n K^{1/2} \log K \right) \right) \\ &= O(H). \end{aligned}$$

If  $l = n$ , then

$$\begin{aligned} \sum_{k=K}^{K+H-1} \eta(g_k)^n \bar{\eta}(g_k)^n &= H \sum_{\substack{p < x^3 \\ p_1 \dots p_n = p_{n+1} \dots p_{2n}}} \frac{1}{p_1 \dots p_n} + O(H) \\ &= n! H \left( \sum_{p < x^3} \frac{1}{p} \right)^n + O\left( n! H \sum_{p < x^3} \frac{1}{p_1 \dots p_{n-2} p_n^2} \right) + O(H), \end{aligned}$$

where the second equality follows from the fact that the number of ways in which a number of the form  $p_1 \dots p_n$  can be written as a product of  $n$  primes is equal to  $n!$  if the primes are all different and less than  $n!$  if two or more of the primes are equal.

Hence,

$$\begin{aligned} \sum_{k=K}^{K+H-1} \eta(g_k)^n \bar{\eta}(g_k)^n &= n!H (\log \log x)^n + O\left(n!H (\log \log x)^{n-1}\right) \\ &= n!H (\log \log K)^n + O\left(n!H (\log \log K)^{n-1}\right). \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{k=K}^{K+H-1} |S_x(g_k)|^{2n} &= \frac{1}{(2\pi)^{2n}} \sum_{l=0}^{2n} (-1)^{n-l} \binom{2n}{l} \sum_{k=K}^{K+H-1} \eta(g_k)^l \bar{\eta}(g_k)^{2n-l} \\ &= \frac{(2n)!}{(2\pi)^{2n} n!} \left( H (\log \log K)^n + O\left( H (\log \log K)^{n-1} \right) \right). \end{aligned}$$

The proof of (ii) is similar, except that in this case it is always true that  $\{p_1, \dots, p_l\} \neq \{p_{l+1}, \dots, p_{2n-1}\}$ . □

### 6 Moments of S(t)

**Theorem 6.1** *Suppose  $1 \leq c_1 K^\theta \leq H \leq c_2 K$ , where  $\theta > 1/2$  and  $c_1, c_2 > 0$ . Then, for every integer  $n \geq 1$ ,*

(i)

$$\sum_{k=K}^{K+H-1} |S(g_k)|^{2n} = \frac{(2n)!}{(2\pi)^{2n} n!} \left( H (\log \log K)^n + O\left( H (\log \log K)^{n-1/2} \right) \right),$$

and

(ii)

$$\sum_{k=K}^{K+H-1} (S(g_k))^{2n-1} = O\left( H (\log \log K)^{n-1} \right).$$

*Proof* Take  $x = K^{\frac{2\theta-1}{6n}-\varepsilon}$ . Then, the triangle inequality for  $L^p$  norms implies that

$$\begin{aligned} &\left| \left( \frac{1}{H} \sum_{k=K}^{K+H-1} |S(g_k)|^{2n} \right)^{1/2n} - \left( \frac{1}{H} \sum_{k=K}^{K+H-1} |S_x(g_k)|^{2n} \right)^{1/2n} \right| \\ &\leq \left( \frac{1}{H} \sum_{k=K}^{K+H-1} |S(g_k) - S_x(g_k)|^{2n} \right)^{1/2n} = O(1). \end{aligned}$$

Hence,

$$\left(\frac{1}{H} \sum_{k=K}^{K+H-1} |S(g_k)|^{2n}\right)^{1/2n} = \left(\frac{(2n)!}{(2\pi)^{2n} n!} \left[ (\log \log K)^n + O\left((\log \log K)^{n-1}\right) \right]\right)^{1/2n} + O(1),$$

and

$$\frac{1}{H} \sum_{k=K}^{K+H-1} |S(g_k)|^{2n} = \frac{(2n)!}{(2\pi)^{2n} n!} (\log \log K)^n + O\left((\log \log K)^{n-1/2}\right).$$

For the proof of (ii), we estimate

$$S(g_k)^{2n-1} - S_x(g_k)^{2n-1} = O\left(\sum_{v=1}^{2n-1} |S_x(g_k)|^{2n-1-v} |S(g_k) - S_x(g_k)|^v\right),$$

and note that

$$\sum_{k=K}^{K+H-1} |S_x(g_k)|^{2n-1-v} |S(g_k) - S_x(g_k)|^v \leq \left(\sum_{k=K}^{K+H-1} |S_x(g_k)|^{2n}\right)^{\frac{2n-1-v}{2n}} \left(\sum_{k=K}^{K+H-1} |S(g_k) - S_x(g_k)|^{\frac{v}{1+v} 2n}\right)^{\frac{1+v}{2n}},$$

where we used the Hölder inequality with  $p = 2n / (2n - 1 - v)$  and  $q = 2n / (1 + v)$ .

Next, we use the inequality

$$\left|\frac{1}{H} \sum_{k=K}^{K+H-1} |S(g_k) - S_x(g_k)|^{\frac{v}{1+v} 2n}\right|^{\frac{1+v}{v} \frac{1}{2n}} \leq \left|\frac{1}{H} \sum_{k=K}^{K+H-1} |S(g_k) - S_x(g_k)|^{2n}\right|^{\frac{1}{2n}} = O(1)$$

in order to conclude that

$$\left|\sum_{k=K}^{K+H-1} |S(g_k) - S_x(g_k)|^{\frac{v}{1+v} 2n}\right|^{\frac{1+v}{2n}} \leq O\left(H^{\frac{1+v}{2n}}\right).$$

Therefore,

$$\begin{aligned} \sum_{k=K}^{K+H-1} |S_x(g_k)|^{2n-1-v} |S(g_k) - S_x(g_k)|^v &= O\left(H^{\frac{2n-1-v}{2n}} (\log \log K)^{n-\frac{1+v}{2}} H^{\frac{1+v}{2n}}\right) \\ &= O\left(H (\log \log K)^{n-\frac{1+v}{2}}\right) \\ &= O\left(H (\log \log K)^{n-1}\right) \end{aligned}$$

for  $1 \leq \nu \leq 2n - 1$ . Hence,

$$\begin{aligned} \sum_{k=K}^{K+H-1} S(g_k)^{2n-1} &= \sum_{k=K}^{K+H-1} S_x(g_k)^{2n-1} + O\left(H(\log \log K)^{n-1}\right) \\ &= O\left(H(\log \log K)^{n-1}\right). \end{aligned}$$

□

**Corollary 6.2** *Suppose  $1 \leq c_1 K^\theta \leq H \leq c_2 K$ , where  $\theta > 1/2$  and  $c_1, c_2 > 0$ . Then,*

(i)

$$\frac{1}{H} \sum_{k=K}^{K+H-1} \left| \frac{\sqrt{2\pi} S(t_k + \xi \sigma_k)}{\sqrt{\log \log t_k}} \right|^{2n} = \frac{(2n)!}{(2\pi)^{2n} n!} \left( 1 + O\left((\log \log K)^{-1/2}\right) \right),$$

and

(ii)

$$\frac{1}{H} \sum_{k=K}^{K+H-1} \left( \frac{\sqrt{2\pi} S(t_k + \xi \sigma_k)}{\sqrt{\log \log t_k}} \right)^{2n-1} = O\left((\log \log K)^{-1/2}\right),$$

This Corollary implies Theorem 2.2 (and Theorem 1.1 as a consequence).

### 7 Covariance

**Lemma 7.1** *Let two non-equal primes  $p_1, p_2$  be both less than  $y \leq cK$ . Assume  $1 \leq H \leq cK$ , and let  $k' = k + x$ , where  $0 < x < K^\epsilon$ , with  $\epsilon \in [0, 1)$ . Then,*

$$\sum_{k=K}^{K+H-1} \exp(-i(g_k \log p_1 - g_{k'} \log p_2)) = O\left(H \frac{y^{1/2} \log y}{K^{1/2} \log K} + y^{1/2} K^{1/2} \log K\right).$$

*Proof* By using Lemma 3.2, we can estimate:

$$\begin{aligned} g''(t) \log p_1 - g''(t+x) \log p_2 &= \frac{-2\pi}{t(\log t)^2} (\log p_1 - \log p_2) + o\left(\frac{\log y}{t(\log t)^2}\right) \\ &\quad + o\left(\frac{x \log y}{t^2}\right). \end{aligned}$$

It follows that

$$\left| g''(t) \log p_1 - g''(t+x) \log p_2 \right| \geq c \frac{1}{yK(\log K)^2},$$

and

$$|g''(t) \log p_1 - g''(t+x) \log p_2| \leq cy \log y \frac{1}{yK (\log K)^2}.$$

The conclusion of the lemma follows by applying Theorem 3.1. □

**Lemma 7.2** *Suppose that  $s(x) = c(\log x)^{\beta-1} + O((\log x)^{\beta-2})$ , where  $\beta > 0$ . Then,*

$$\sum_{p \leq x} \frac{1}{p} p^{is(x)} = (1 - \beta)_+ \log \log x + O(1).$$

*Proof* This is a direct consequence of Lemma 3.4 in [1]. □

*Proof of Theorem 2.3* In order to compute  $\mathbb{E}X_1^{(N)}X_2^{(N)}$ , we proceed as above in the calculation of  $\mathbb{E}\left(X_1^{(N)}\right)^2$ .

Since by Proposition 4.1,

$$\sum_{k=K}^{K+H-1} |S(g_k) - S_x(g_k)|^{2n} = O(H),$$

therefore, it is essential to compute

$$\frac{1}{N} \sum_{k=N}^{2N-1} S_x(g_k) S_x(g_{k'}),$$

where  $k' = k + (\log N)^\beta$ . By using function  $\eta$ , defined in (14), we obtain:

$$\begin{aligned} \sum_{k=N}^{2N-1} S_x(g_k) S_x(g_{k'}) &= -\frac{1}{(2\pi)^2} \sum_{k=N}^{2N-1} (\eta(g_k) \eta(g_{k'}) - \eta(g_k) \bar{\eta}(g_{k'}) \\ &\quad - \bar{\eta}(g_k) \eta(g_{k'}) + \bar{\eta}(g_k) \bar{\eta}(g_{k'})). \end{aligned}$$

For the first term in this sum, we write

$$\sum_{k=N}^{2N-1} \eta(g_k) \eta(g_{k'}) = \sum_{p_1, p_2 \leq x^2} \frac{1}{\sqrt{p_1 p_2}} \sum_{k=N}^{2N-1} p_1^{-ig_k} p_2^{-ig_{k'}}.$$

Note that the sum

$$\sum_{k=N}^{2N-1} p_1^{-ig_k} p_2^{-ig_{k'}} = \sum_{k=N}^{2N-1} \exp[-i(g_k \log p_1 + g_{k+(\log N)^\beta} \log p_2)]$$

is an exponential sum, and it can be estimated by using van der Corput’s theorem by noticing that the second derivative of the function

$$g(s) \log p_1 + g(s + \alpha (\log N)^\beta) \log p_2$$

is bounded by  $O(N^{-1} (\log N)^{-2})$  from below and by  $O(N^{-1} (\log N)^{-2} \log x)$  from above. This implies that with an appropriate choice of  $x$ ,

$$\frac{1}{N} \sum_{k=N}^{2N-1} \eta(g_k) \eta(g_{k'}) = O(1),$$

and similarly for  $N^{-1} \sum_{k=N}^{2N-1} \bar{\eta}(g_k) \bar{\eta}(g_{k'})$ .

Therefore

$$\frac{1}{N} \sum_{k=N}^{2N-1} S_x(g_k) S_x(g_{k'}) = \frac{1}{2\pi^2} \operatorname{Re} \left[ \frac{1}{N} \sum_{k=N}^{2N-1} \eta(g_k) \bar{\eta}(g_{k'}) \right] + O(1),$$

where

$$\sum_{k=N}^{2N-1} \eta(g_k) \bar{\eta}(g_{k'}) = \sum_{p_1, p_2 \leq x^3} \frac{1}{\sqrt{p_1 p_2}} \sum_{k=N}^{2N-1} p_1^{-ig_k} p_2^{ig_{k'}}.$$

If  $p_1 \neq p_2$ , then by using Lemma 7.1, we can estimate

$$\sum_{p_1, p_2 \leq x^2} \frac{1}{\sqrt{p_1 p_2}} \sum_{k=N}^{2N-1} p_1^{-ig_k} p_2^{ig_{k'}} = O(x^3 N^{1/2} \log N) = O(N),$$

provided that  $x = N^\kappa$  and  $\kappa \leq 1/6$ .

If  $p_1 = p_2$ , then we have

$$\frac{1}{N} \sum_{k=N}^{2N-1} \sum_{p \leq x^3} \frac{1}{p} p^{-i(g_k - g_{k'})}$$

If one sets  $x = N^\kappa$ , then by using the definition of function  $g$ , it is easy to see that for every  $k \in [N, 2N - 1]$ , and  $k' = k + (\log N)^\beta$ , we have

$$\begin{aligned} g_{k'} - g_k &= 2\pi (\log N)^{\beta-1} + O((\log N)^{\beta-2}) \\ &= (2\pi/\kappa^{\beta-1}) (\log x)^{\beta-1} + O((\log x)^{\beta-2}), \end{aligned}$$

where the implicit constant in the  $O$ -term does not depend on  $k$ .

Hence, by Lemma 7.2,

$$\frac{1}{N} \sum_{k=N}^{2N-1} \sum_{p \leq x^3} \frac{1}{p} p^{-i(g_k - g_{k'})} = (1 - \beta)_+ \log \log x + O(1).$$

It follows that

$$\frac{1}{N} \sum_{k=N}^{2N-1} S_x(g_k) S_x(g_{k'}) = \frac{1}{2\pi^2} (1 - \beta)_+ \log \log x + O(1).$$

Next we note that

$$\begin{aligned} \frac{1}{N} \sum_{k=N}^{2N-1} S(g_k) S(g_{k'}) &= \frac{1}{N} \sum_{k=N}^{2N-1} S_x(g_k) S_x(g_{k'}) \\ &\quad + \frac{1}{N} \sum_{k=N}^{2N-1} S_x(g_k) (S(g_{k'}) - S_x(g_{k'})) \\ &\quad + \frac{1}{N} \sum_{k=N}^{2N-1} (S(g_k) - S_x(g_k)) S_x(g_{k'}) \\ &\quad + \frac{1}{N} \sum_{k=N}^{2N-1} (S(g_k) - S_x(g_k)) (S(g_{k'}) - S_x(g_{k'})). \end{aligned}$$

By the Schwarz inequality, the last three terms can be estimated as  $(\log \log N)^{1/2}$ , and therefore we have

$$\frac{1}{N} \sum_{k=N}^{2N-1} S(g_k) S(g_{k'}) = \frac{1}{2\pi^2} (1 - \beta)_+ \log \log x + O\left((\log \log x)^{1/2}\right),$$

This implies that

$$\mathbb{E}X_1^{(N)} X_2^{(N)} = (1 - \beta)_+.$$

□

### 8 Joint moments

*Proof of Theorem 2.4* It is clearly enough to prove the corresponding result for random variables  $S(g_{k_1})$  and  $S(g_{k_2})$  since  $X_i^{(N)}$  are the rescaled versions of these random variables. In fact, as a consequence of the Selberg approximation result, it is enough to show that  $S_x(g_{k_1})$  and  $S_x(g_{k_2})$  have the required moments.



Indeed,

$$\begin{aligned}
 & S(g_{k_1})^a S(g_{k_2})^b \\
 &= (S_x(g_{k_1}) + S(g_{k_1}) - S_x(g_{k_1}))^a (S_x(g_{k_2}) + S(g_{k_2}) - S_x(g_{k_2}))^b \\
 &= S_x(g_{k_1})^a S_x(g_{k_2})^b \\
 &\quad + O\left(\sum_{s,t} S_x(g_{k_1})^s (S(g_{k_1}) - S_x(g_{k_1}))^{a-s} S_x(g_{k_2})^t (S(g_{k_2}) - S_x(g_{k_2}))^{b-t}\right),
 \end{aligned}$$

where the sum is over  $s$  and  $t$  such that  $0 \leq s \leq a$ ,  $0 \leq t \leq b$ , and  $s + t < a + b$ . After we sum over  $k_1$  and apply the Schwarz inequality twice, we find that

$$\begin{aligned}
 & \frac{1}{N} \sum_{k_1=N}^{2N-1} (S(g_{k_1})^a S(g_{k_2})^b - S_x(g_{k_1})^a S_x(g_{k_2})^b) \\
 &= O\left(\sum_{s,t} \left(\frac{1}{N} \sum_{k_1=N}^{2N-1} S_x(g_{k_1})^{4s}\right)^{1/4} \left(\frac{1}{N} \sum_{k_1=N}^{2N-1} S_x(g_{k_2})^{4t}\right)^{1/4}\right. \\
 &\quad \times \left.\left(\frac{1}{N} \sum_{k_1=N}^{2N-1} (S(g_{k_1}) - S_x(g_{k_1}))^{4(a-s)}\right)^{1/4} \left(\frac{1}{N} \sum_{k_1=N}^{2N-1} (S(g_{k_1}) - S_x(g_{k_1}))^{4(a-s)}\right)^{1/4}\right) \\
 &= O\left((\log \log N)^{(a+b-1)/2}\right).
 \end{aligned}$$

Hence, if variables  $S$  and  $S_x$  are scaled by  $(\log \log N)^{-1}$ , the difference in their moments is of order  $(\log \log N)^{-1/2}$ .

The result about moments of the scaled versions of  $S_x(g_{k_1})$  and  $S_x(g_{k_2})$  follows from the result for random variables

$$\eta_i^{(N)} := \frac{1}{\sqrt{\log \log N}} \eta(g_{k_i(N,\omega)}),$$

where  $i = 1, 2$ , and  $\eta(t)$  is as defined in (14).

**Theorem 8.1** *Let  $a_1, a_2, b_1, b_2 \geq 0$ . The joint moments of random variables  $\eta_1^{(N)}$  and  $\eta_2^{(N)}$ ,*

$$m_N(a_1, a_2, b_1, b_2) := \mathbb{E} \left( \eta_1^{(N)} \right)^{a_1} \left( \overline{\eta_1^{(N)}} \right)^{a_2} \left( \eta_2^{(N)} \right)^{b_1} \left( \overline{\eta_2^{(N)}} \right)^{b_2},$$

converge to the corresponding joint moments of complex Gaussian random variables  $\eta_1$  and  $\eta_2$ , which have the following covariance structure:  $\mathbb{E}\eta_i^2 = \mathbb{E}\overline{\eta_i^2} = \mathbb{E}\eta_1\eta_2 = \mathbb{E}\overline{\eta_1}\overline{\eta_2} = 0$ ,  $\mathbb{E}\eta_i\overline{\eta_i} = 1$ ,  $\mathbb{E}\eta_1\overline{\eta_2} = \mathbb{E}\overline{\eta_1}\eta_2 = (1 - \beta)_+$ .

Indeed, if this result holds, then the joint moments of (real) random variables

$$S_x(g_{k_j}) = \frac{1}{2\pi i} \left( \eta_j^{(N)} - \bar{\eta}_j^{(N)} \right)$$

converge to the corresponding joint moments of Gaussian random variables  $S_1$  and  $S_2$ , where

$$E(S_1^2) = E(S_2^2) = \frac{1}{2\pi^2},$$

and

$$E(S_1 S_2) = \frac{1}{2\pi^2} (1 - \beta)_+.$$

This implies the statement of Theorem 2.4.

Before attacking Theorem 8.1, let us recall the Wick Rule for the joint moments of Gaussian random variables, namely,

$$\mathbb{E}[x_{i_1} \dots x_{i_k}] = \sum_{\pi \in P_2(\{1, \dots, k\})} \prod_{(r,s) \in \pi} \mathbb{E}[x_{i_r} x_{i_s}],$$

where the sum is over all pairings of indices  $1, \dots, k$ . (In particular, if  $k$  is odd, then the sum is empty.) (See, for example, Theorem 22.3 in [20] or Appendix 1 on p. 13 in [30]).

If we apply this rule to random variables  $\eta_i, \bar{\eta}_i$ , then we find that

$$m(a_1, a_2, b_1, b_2) := \mathbb{E}(\eta_1)^{a_1} (\bar{\eta}_1)^{a_2} (\eta_2)^{b_1} (\bar{\eta}_2)^{b_2}$$

is zero unless  $a_1 + b_1 = a_2 + b_2$ . If  $a_1 + b_1 = a_2 + b_2$ , then

$$m(a_1, a_2, b_1, b_2) = n(k, a_1, a_2, b_1, b_2) (1 - \beta)_+^k, \tag{15}$$

where  $n(k, a_1, a_2, b_1, b_2)$  is a number of ways to pair  $a_1$  elements  $\eta_1$  and  $b_1$  elements  $\eta_2$  with  $a_2$  elements  $\bar{\eta}_1$  and  $b_2$  elements  $\bar{\eta}_2$  so that exactly  $k$  elements are connected with an element that has a different index.

Also, we need a generalization of Lemma 3.3.

**Lemma 8.2** *Let*

$$h(x) = g(x) \left( \sum_{k=1}^{a_2} \log q_k - \sum_{k=1}^{a_1} \log p_k \right) + g(x+u) \left( \sum_{k=a_2+1}^{a_2+b_2} \log q_k - \sum_{k=a_1+1}^{a_1+b_1} \log p_k \right),$$

where  $g(x)$  is as defined in (12),  $\{p_1, \dots, p_{a_1+b_1}\} \neq \{q_1, \dots, q_{a_2+b_2}\}$  and primes  $p_i$  and  $q_i$  are less than  $y \leq K^\varepsilon$  for all  $i$  and  $\varepsilon < 1/n$ . Assume  $1 \leq H \leq cK$  and  $u \leq \alpha(\log K)^\beta$ . Let  $n = \lfloor (a_1 + a_2 + b_1 + b_2) / 2 \rfloor$ . Then,

$$\sum_{k=K}^{K+H-1} e^{ih(k)} = O\left(H \frac{y^{n/2} \log y}{K^{1/2} \log K} + y^{n/2} K^{1/2} \log K\right).$$

*Proof* We can re-write the definition of  $h(x)$  as follows:

$$h(x) = h_1(x) + h_2(x) = g(x) \left( \sum_{k=1}^{a_2+b_2} \log q_k - \sum_{k=1}^{a_1+b_1} \log p_k \right) + (g(x+u) - g(x)) \left( \sum_{k=a_2+1}^{a_2+b_2} \log q_k - \sum_{k=a_1+1}^{a_1+b_1} \log p_k \right).$$

The second derivative of the first term can be estimated as in Lemma 3.3: If  $x \in [K, K + H]$ , then

$$h_1''(x) \in [\lambda, \kappa\lambda],$$

where

$$\lambda = \frac{c_1}{y^n K (\log K)^2} \text{ and } \kappa = c_2 y^n \log y.$$

For the second term, we note that

$$(g(x+u) - g(x))'' = g'''(\theta)u,$$

where  $\theta \in [x, x + u]$ , and by using Lemma 3.2 we find that

$$h_2''(x) = O\left(\frac{(\log K)^{\beta-2}}{K^2} \log y\right) = o(h_1''(x)),$$

provided that  $y \leq K^\varepsilon$  with  $\varepsilon < 1/n$ . It follows that  $h''(x) \sim h_1''(x)$ , and the conclusion of the lemma follows by an application of Theorem 3.1 as in Lemma 3.3.  $\square$

*Proof of Theorem 8.1* By definition, we write

$$\begin{aligned} \mathbb{E} \left( \eta_1^{(N)} \right)^{a_1} \left( \overline{\eta_1^{(N)}} \right)^{a_2} \left( \eta_2^{(N)} \right)^{b_1} \left( \overline{\eta_2^{(N)}} \right)^{b_2} &= \frac{1}{N (\log \log N)^{(a_1+a_2+b_1+b_2)/2}} \\ &\times \sum_{k_1=N}^{2N-1} \left( \sum_{p \leq x^2} \frac{p^{-ig_{k_1}}}{\sqrt{p}} \right)^{a_1} \left( \sum_{q \leq x^2} \frac{q^{ig_{k_1}}}{\sqrt{q}} \right)^{a_2} \\ &\times \left( \sum_{p \leq x^2} \frac{p^{-ig_{k_2}}}{\sqrt{p}} \right)^{b_1} \left( \sum_{q \leq x^2} \frac{q^{ig_{k_2}}}{\sqrt{q}} \right)^{b_2}, \end{aligned}$$

where  $k_2 = k_1 + [\alpha (\log N)^\beta]$ . If we expand the product of sums, we get for a general term

$$t(p, p', q, q') := \frac{1}{\sqrt{p_1 \dots p_{a_1+b_1} q_1 \dots q_{a_2+b_2}}} \frac{(q_1 \dots q_{a_2})^{ig_{k_1}} (q_{a_2+1} \dots q_{a_2+b_2})^{ig_{k_2}}}{(p_1 \dots p_{a_1})^{ig_{k_1}} (p_{a_1+1} \dots p_{a_1+b_1})^{ig_{k_2}}},$$

where  $p := (p_1, \dots, p_{a_1})$ ,  $p' := (p_{a_1 + 1}, \dots, p_{a_1+a_2})$ ,  $q := (q_1, \dots, q_{b_1})$ , and  $q' := (q_{b_1 + 1}, \dots, q_{b_1+b_2})$ .

By using Lemma 8.2, we find that after we sum this term over  $k_1$  and divide it by  $N (\log \log N)^{(a_1+a_2+b_1+b_2)/2}$ , we get a non-negligible contribution if and only if there is a pairing that puts every  $q_i$  in a correspondence with a  $p_j$ , so that  $q_i = p_j$ . In particular, it must be true that  $a_1 + b_1 = a_2 + b_2 = n$ . Hence, the moment is asymptotically equivalent to

$$\frac{1}{N (\log \log N)^n} \sum_{k_1=N}^{2N-1} \sum_{p \cdot p' = q \cdot q'} t(p, p', q, q'), \tag{16}$$

where  $p \cdot p'$  denotes the product of primes in  $p$  and  $p'$ , and similar for  $q \cdot q'$ . If we consider the sum over all  $(p, p')$ , in which at least one  $p_i$  appears twice, then we can see that this sum can be estimated as

$$O \left( \sum_{k_1=N}^{2N-1} \left( \sum_{p \leq x^2} \frac{1}{p^2} \right) \left( \sum_{p \leq x^2} \frac{1}{p} \right)^{n-2} \right) = O \left( N (\log \log N)^{n-2} \right),$$

which gives a negligible contribution to the moment.

Otherwise, if every prime appears only once in  $(p, p')$  and if  $p \cdot p' = q \cdot q'$ , then there is a unique pairing between elements of  $(p, p')$  and  $(q, q')$ . Let this pairing be called  $\pi$ . That is,  $\pi(i) = j$  means that  $p_i = q_j$ .

The terms that satisfy pairing  $\pi$  give the following contribution to the sum in (16):

$$\sum_{k_1=N}^{2N-1} \left( \sum_{p \leq x^2} \frac{1}{p} \right)^{n_{11}+n_{22}} \left( \sum_{p \leq x^2} \frac{p^{i(g_{k_1}-g_{k_2})}}{p} \right)^{n_{12}} \left( \sum_{p \leq x^2} \frac{p^{-i(g_{k_1}-g_{k_2})}}{p} \right)^{n_{21}} + O(N (\log \log N)^{n-2}),$$

where

$$n_{11} = |\{i, j : \pi(i) = j, 1 \leq i \leq a_2, 1 \leq j \leq a_1\}|,$$

$$n_{12} = |\{i, j : \pi(i) = j, 1 \leq i \leq a_2, a_1 \leq j \leq a_1 + b_1\}|,$$

and so on.

By Lemma 7.2, this can be computed as

$$\left((1 - \beta)_+\right)^{n_{12}+n_{21}} N (\log \log N)^n + O(N (\log \log N)^{n-1}).$$

After summing over all pairings we find that the moment equals

$$\sum_{\pi} \left((1 - \beta)_+\right)^{n_{12}+n_{21}} + O\left((\log \log N)^{-1}\right).$$

Recall  $n(k, a_1, a_2, b_1, b_2)$  is a number of ways to pair  $a_1$  elements  $\eta_1$  and  $b_1$  elements  $\eta_2$  with  $a_2$  elements  $\bar{\eta}_1$  and  $b_2$  elements  $\bar{\eta}_2$  so that exactly  $k$  elements are connected with an element that has a different index. That is,  $n(k, a_1, a_2, b_1, b_2)$  is the number of pairings  $\pi$  for which  $n_{12} + n_{21} = k$ .

It follows that asymptotically, the moment tends to  $n(k, a_1, a_2, b_1, b_2) (1 - \beta)_+^k$ , which is exactly the corresponding joint moment of the Gaussian variables that we obtained in formula (15).  $\square$

## 9 Conclusion

We have shown that the distribution of two zeta zero fluctuations  $f_k$  and  $f_{k+x}$  approaches a two-variate Gaussian distribution with covariance  $(1 - \beta)_+$ , provided that  $x \sim (\log k)^\beta$ . This gives an analogue of Gustavsson's results for fluctuations of eigenvalues from Gaussian Unitary Ensemble. It is of obvious interest to study the correlation of zeros at shorter distances. However, methods of this paper are not easy to generalize to this case.

Some of the methods in this paper could perhaps be useful to extend Gustavsson's results to other ensembles of random matrices, in particular to the ensemble of uniformly distributed unitary random matrices. The proof would proceed along the similar lines by using the additional tool by Diaconis and Shashahani [6] about expected values of traces of moments of  $U$ .

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