J Theor Probab (2011) 24:821–848
DOI 10.1007/s10959-011-0341-z

On Free Stochastic Differential Equations

Vladislav Kargin

Received: 29 July 2009 / Revised: 6 December 2010 / Published online: 26 January 2011

Abstract The paper derives an equation for the Cauchy transform of the solution of a free stochastic differential equation (SDE). This new equation is used to solve several particular examples of free SDEs.

Keywords Free probability · Stochastic differential equations

Mathematics Subject Classification (2000) 60H10 · 46L54 · 15A52

1 Introduction

Free stochastic differential equations generalize classical stochastic differential equations to the setting of free probability. Here is an example of such an equation:

$$X_t = a(X_t)\,dt + b^*(X_t)(dZ_t)b(X_t).$$

In this equation, X_t is a self-adjoint operator, $a(X_t)$ and $b(X_t)$ are operator-valued functions of X_t, and the driving noise Z_t is an operator process with free increments. That is, the increments $Z_s - Z_t$, $s > t$, are assumed to be free from past realizations of Z_t. The process Z_t is usually the free Brownian process, in which case the increments have semicircle distributions; however, other choices are possible.

Informally, the reader may think about X_t as very large random matrices and Z_t as matrices with independent Gaussian random variables as entries. These entries follow independent Brownian motions and we are interested in the law of the eigenvalues of X_t. The free probability theory is a convenient abstraction which intends to model the situation when the size of the matrices is very large.

V. Kargin (✉)
Department of Mathematics, Stanford University, Palo Alto, CA 94305, USA
e-mail: kargin@stanford.edu

The study of free stochastic differential equations ("free stochastic calculus") is more difficult than in the classical case because of non-commutativity of coefficients and noise. This paper contributes by developing a new tool for the analysis of these equations.

The idea of free stochastic calculus was first suggested in [12]. It was later developed and formalized in [2, 3, 9], and [1], which introduced stochastic integration with respect to free Brownian motion as a rigorous basis for free stochastic calculus. They also derived an analog of the Itô formula, which allows us to obtain identities like the following:

$$\int_0^a \left[W_t^2(dW_t) + W_t(dW_t)W_t + (dW_t)W_t^2\right] = W_a^3 - 2\int_0^a W_t\,dt,$$

where W_t denotes the free Brownian motion (the Wigner process). An analogous formula in the classical situation is

$$\int_0^a 3B_t^2(dB_t) = B_a^3 - 3\int_0^a B_t\,dt,$$

where B_t is the standard Brownian motion. Note the different coefficient before the integral on the right-hand side.

The classical Itô formula is very helpful in the study of stochastic differential equations. Unfortunately, the range of applicability of the free Itô formula is smaller. This difficulty calls for a different method applicable to those free SDEs, which are not solvable with the Itô formula. One possibility is to seek an equation for the evolution of the spectral distribution of the solution. Such an equation was derived by Biane and Speicher in [4] for the equation

$$X_t = a(X_t)\,dt + dW_t. \tag{1}$$

They showed that the density of the spectral probability measure of X_t, which we denote p_t and which generalizes the eigenvalue distribution of a matrix, satisfies the free Fokker–Planck equation:

$$\frac{\partial p_t}{\partial t} = -\frac{\partial}{\partial x}\left[p_t(Hp_t + a)\right]. \tag{2}$$

Here H denotes a multiple of the Hilbert transform:

$$Hu(x) := \text{p.v.}\int \frac{u(y)}{x-y}\,dy. \tag{3}$$

Still, the approach through the free Fokker–Planck equation has its own disadvantages. First, it is applicable only to equations that have the special form (1), that is, only to equations with the constant diffusion coefficient. Second, the free Fokker–Planck equation (2) is not a bona fide partial differential equation since it includes the Hilbert transform operator. For this reason, it is somewhat difficult to solve this equation.

The purpose of this paper is to approach the free stochastic equations by deriving a differential equation for the Cauchy transform of the solution.

Recall that the *resolvent* of operator X_t is defined as the operator-valued function of a complex parameter $G_t(z) := (X_t - z)^{-1}$. The *Cauchy transform* of X_t is defined as the expectation of the resolvent: $g_t(z) := E[(X_t - z)^{-1}]$. It is useful because the knowledge of the Cauchy transform is sufficient to recover all properties of the spectral probability distribution of X_t. It turns out that if X_t solves

$$dX_t = a(X_t)\,dt + b(X_t)(dW_t)c(X_t),$$

then $g_t(z)$ satisfies the following equation:

$$\frac{dg_t}{dt} = -E\bigl(a_t G_t^2\bigr) + E(b_t c_t G_t)E\bigl(b_t c_t G_t^2\bigr), \tag{4}$$

where we use a_t, b_t, and c_t to denote $a(X_t)$, $b(X_t)$, and $c(X_t)$, respectively. This is the statement of Theorem 3.2 below.

Equation (4) is not a usual differential equation since it involves expectations. In general, these expectations are difficult to compute because the coefficients a_t, b_t, and c_t, and the resolvent G_t are not free from each other. However, if the coefficients are polynomials, it is possible to perform further reduction to a differential equation as we will show in Proposition 3.4.

As we just said, the knowledge of the Cauchy transform can be used to recover the spectral probability distribution. In particular, the free Fokker–Planck equation (2) can be derived from (4) as will be shown in Corollary 3.6.

In certain cases it is not possible to compute the Cauchy transform explicitly, but it is possible to detect the behavior of its singularities. This knowledge can provide us with information about the support of the spectral distribution. In particular, it can show us how the norm of the solution grows.

For a simple example of this approach, let us consider the well-known case of the free Ornstein–Uhlenbeck equation:

$$dX_t = -\theta X_t\,dt + \sigma\,dW_t. \tag{5}$$

For this equation, it is easy to compute the Cauchy transform using (4) and recover the known result that for positive θ, the spectral probability distribution of X_t converges to a stationary solution, which is a semicircle distribution supported on the interval $[-\sigma\sqrt{2/|\theta|}, \sigma\sqrt{2/|\theta|}]$.

As a more difficult example, consider the equation

$$dX_t = \theta X_t\,dt + X_t^{1/2}(dW_t)X_t^{1/2},$$

which can be thought of as a free analog of the equation for the "geometric Brownian motion", $dx_t = \theta x_t\,dt + x_t\,dB_t$.

Let $X_0 = I$. Equation (4) leads to the following differential equation:

$$\frac{\partial g}{\partial t} + z(\theta - 1 - zg)\frac{\partial g}{\partial z} = -g(\theta - 1 - zg),$$

with the initial condition $g(0, z) = (1 - z)^{-1}$. The method of characteristics gives us a functional equation for the Cauchy transform:

$$z + g^{-1} = e^{(\theta - 1 - zg)t}. \tag{6}$$

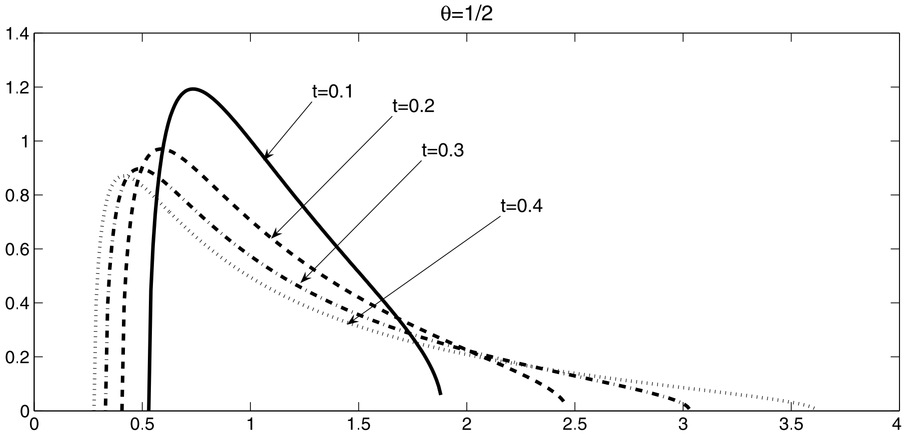

Fig. 1 $\theta = 1/2$

While it is difficult to extract an explicit analytical formula for the solution of this equation, we can investigate how the support of the distribution changes with time. It turns out that for $\theta < 0$, the support of the distribution shrinks to zero. If θ is between 0 and 1, then the lower boundary of the support decreases to zero and the upper boundary grows exponentially fast to infinity. If $\theta > 1$, then both the lower and upper boundary of the support grow exponentially fast to infinity.

We can solve (6) numerically and recover the density of the spectral distribution by using Stieltjes formula. Figures 1–4 show the evolution of the density for various values of parameters and illustrate the complexity of the behavior of the spectral distribution. For example, Fig. 2 shows that even if $\theta > 1$, the spectral distribution does not approach infinity immediately. There is a transition period in which a significant portion of the spectral distribution remains below $\lambda = 1$. Similarly, Fig. 3 shows that for $\theta < 0$, the distribution does not collapse to zero immediately. Only when time increases, the distribution begins the rapid approach to zero, as shown in Fig. 4.

Let us compare this result with the classical analog. By using the Itô formula, it is easy to show that the solution of the classical equation for the geometric Brownian motion is

$$x_t = \exp\left\{\left(\theta - \frac{1}{2}\right)t + B_t\right\}.$$

Hence, with probability 1, the classical solution will decrease exponentially to zero if $\theta < 1/2$, and will grow exponentially to infinity if $\theta > 1/2$. However, the support of the solution distribution is $(0, \infty)$ for all t. This is quite unlike the behavior of the free SDE solution.

Note that the equation for the geometric Brownian motion can be generalized to the free probability setting in a different way:

$$dX_t = \theta X_t\, dt + X_t\, dW_t + (dW_t)X_t.$$

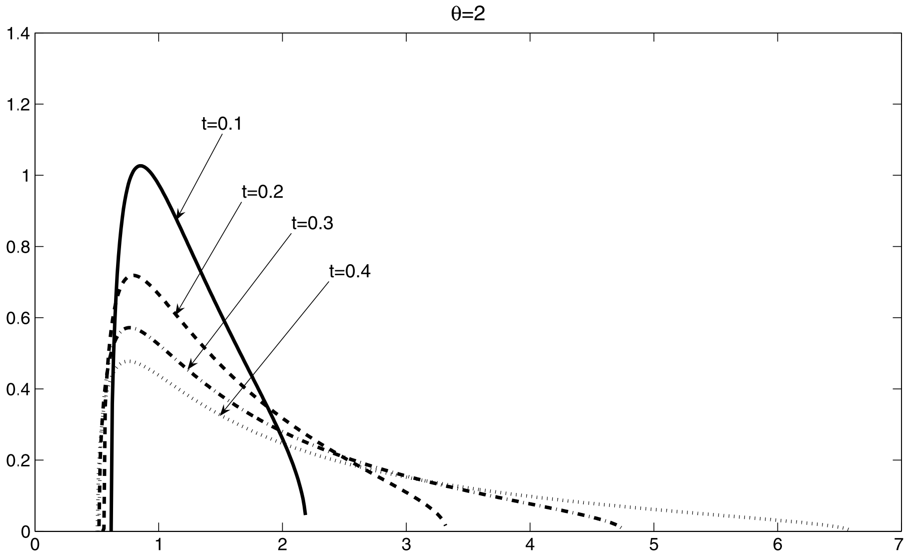

Fig. 2 $\theta = 2$

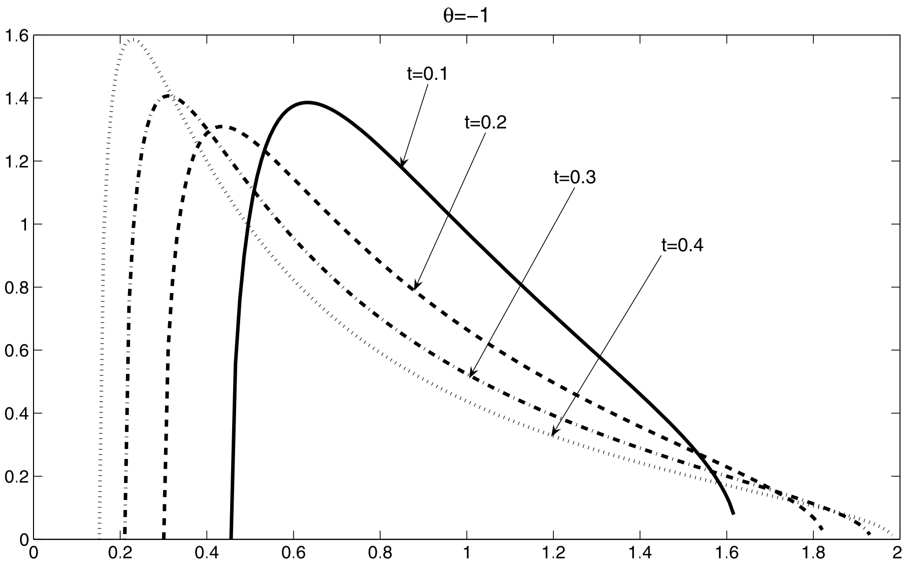

Fig. 3 $\theta = -1$

The behavior of the solution of this equations is quite different. In particular, the ratio of the standard deviation to the expectation is $\sqrt{2(e^{2t}-1)}$. This ratio grows exponentially fast with t, quite unlike the previous example, where this ratio equals $\sqrt{t}$. Unfortunately, the partial differential equation associated with equation is more dif-

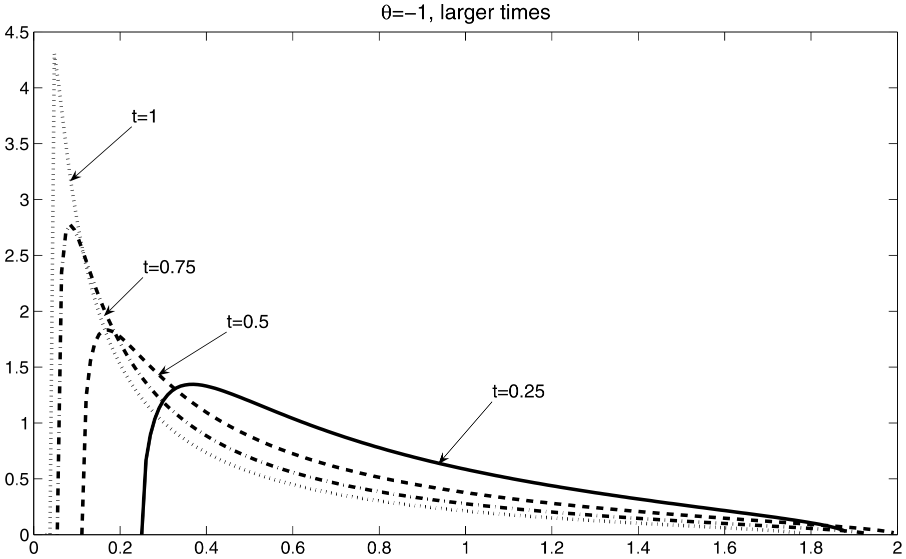

Fig. 4 $\theta = -1$, large times

ficult to solve and it is not clear whether the solution becomes unbounded in finite time.

Finally, let us consider the following equation:

$$dX_t = kX_t(dW_t)X_t,$$

and let the initial condition be $X_0 = aI$. For this equation it is possible to write an explicit formula for the spectral distribution of the solution. An interesting feature of this equation is that the solution blows up in finite time $\tau = (ak)^{-2}$, by which we means that the operator norm of the solution becomes infinite as t approaches τ.

Another interesting feature is that as time t approaches τ, the spectral distribution converges to a fixed distribution. If $k = a^{-1}$, then the density of this distribution is

$$f(\xi) = \frac{\sqrt{4\xi - 1}}{2\pi\xi^3},$$

which is supported on the interval $[1/4, \infty)$. Otherwise, it is a scaled version of this distribution. The behavior of the solution density for various times is illustrated in Figs. 5 and 6.

Several specific classes of free SDE have already been investigated in the literature. Biane and Speicher in [4] and Gao in [7] studied the free Ornstein–Uhlenbeck equation (5). Biane and Speicher proved that its solution converges to a stationary process with a semicircle distribution. Gao considered free Ornstein–Uhlenbeck processes with a free Levy driving noise, and showed that every self-decomposable probability measure on the real line can be realized as a distribution of such a process.

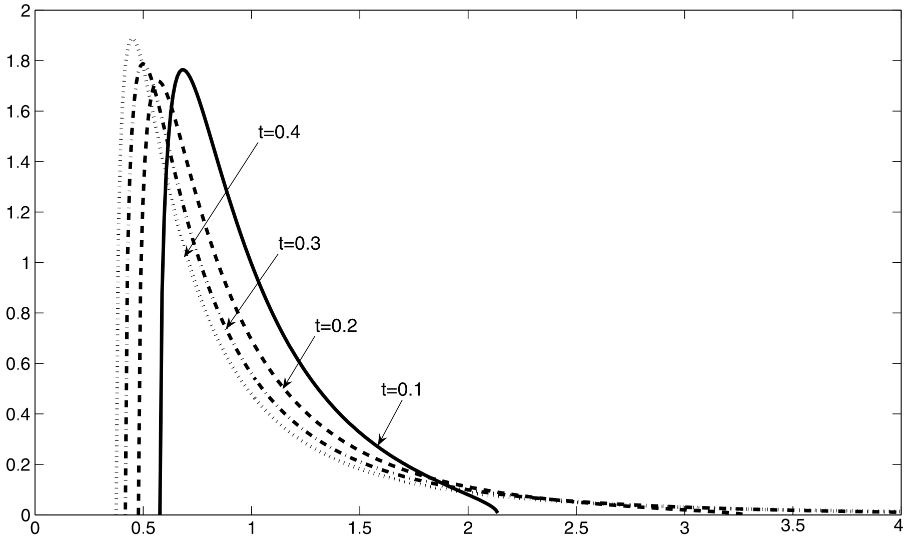

Fig. 5 Density functions for $t = 0.1, 0.2, 0.3, 0.4$

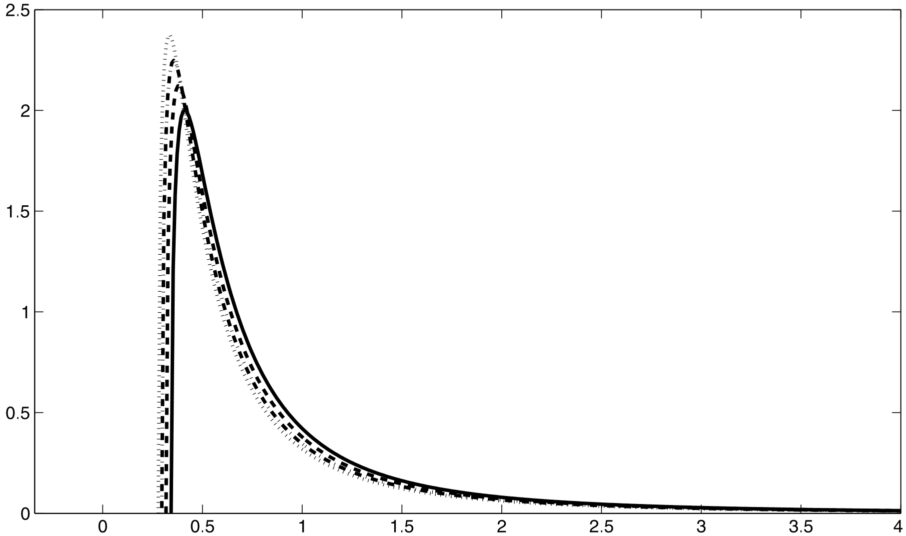

Fig. 6 Density functions for $t = 0.5, 0.6, 0.7, 0.8$

Capitaine and Donati-Martin in [5] defined the free Wishart process and found that it satisfies the free SDE of the form:

$$dX_t = \lambda\, dt + \sqrt{X_t}\, dZ_t + dZ_t^* \sqrt{X_t},$$

where Z_t is the complex Wigner process. Demni in [6] studied the so-called free Jacobi processes which satisfy equations similar to the following:

$$dX_t = (\theta I - X)\,dt + \sqrt{I - X_t}\,dZ_t\sqrt{X_t} + \sqrt{X_t}\,dZ_t^*\sqrt{I - X_t}.$$

With exception of the free Ornstein–Uhlenbeck process, we study a different set of free SDEs, and we approach these equations with a different point of view based on the differential equations for the Cauchy transform.

For the free Ornstein–Uhlenbeck process, our results agree with results in [4].

The rest of the paper is organized as follows. Section 2 provides preliminary information about free stochastic integration and Itô formulas. Section 3 describes main results. In particular, Sect. 3.1 is devoted to a local existence and uniqueness result. Section 3.2 presents general results about the Cauchy transform of the solution and Sect. 3.3 provides examples.

2 Free Stochastic Integration

2.1 The Free Brownian Motion

For the basics of free probability theory we refer to [13] and [10]. All operators that we consider belong to a non-commutative W^*-probability space $(\mathcal{A}, E)$, that is, to a von Neumann operator algebra $\mathcal{A}$ with a faithful normal trace E. We denote the usual operator norm by $\|X\|$, and the L^2-norm by $\|X\|_2 := \sqrt{E(X^*X)}$.

The *spectral probability distribution* of a self-adjoint operator $X \in \mathcal{A}$ is a probability measure μ on $\mathbb{R}$ such that

$$E(X^k) = \int_{\mathbb{R}} x^k \mu(dx).$$

Its Cauchy transform is the function

$$g_X(z) = \int_{\mathbb{R}} \frac{\mu(dx)}{x - z}.$$

It can be defined directly in terms of operator X as the expectation of the resolvent: $g_X(z) = E[G_X(z)]$, where $G_X(z) := (X - z)^{-1}$. The probability measure μ can be recovered from its Cauchy transform by the Stieltjes inversion formula:

$$\mu(B) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \int_B \operatorname{Im} g(x + i\varepsilon)\,dx, \tag{7}$$

provided that B is Borel and $\mu(\partial B) = 0$.

This fact is the starting point of our approach, since we will study the evolution of the Cauchy transform as a tool to investigate the evolution of the corresponding probability measure.

The most important concept in free probability theory is that of free independence. Let $\overline{A_i}$ denote an arbitrary element of algebra $\mathcal{A}_i$. The sub-algebras $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of algebra $\mathcal{A}$ (and operators that generate them) are said to be *freely independent* or *free*, if the following condition holds:

$$E\big(\overline{A}_{i(1)}\dots\overline{A}_{i(m)}\big)=0,$$

provided that $E(\overline{A}_{i(s)})=0$ and $i(s+1)\neq i(s)$ for every s. Two particular consequences of this definition is that (i) $E(AB)=E(A)E(B)$ if A and B are free, and (ii)

$$E(AX_1AX_2)=E\big(A^2\big)E(X_1)E(X_2), \tag{8}$$

if A is free from X_1 and X_2 and $E(A)=0$.

The *free Brownian motion*, or the *Wigner process*, is a family of operators W_t, where $t\geq 0$, that satisfies the following properties: (1) $W_0=0$; (2) the increments of W_t are free in the sense of Voiculescu, i.e., if $t>s$, then W_t-W_s is free from the subalgebra $\mathcal{W}_s$ which is generated by all W_τ with $\tau\leq s$, and (3) the spectral distribution of W_t-W_s is semicircle with zero expectation and variance $t-s$.

The choice of W_t is not unique, and in the rest of the paper we assume that a particular realization of W_t is fixed.

2.2 Free Stochastic Integral

Itô-style free stochastic integration with respect to the free Brownian motion was defined and studied in [9] and [3]. Their results show that under certain assumptions on the operator coefficients a_t and b_t, it is possible to define the integral

$$I=\int_0^1 a_t(dW_t)b_t,$$

where W_t is the free Brownian motion.

Let us briefly recall the construction of the integral. For details of the construction, the reader is advised to see Definition 2.2.1 and Sect. 3 in [3] and Sect. 3 and Theorem 14 in [1]. Suppose that a_t and b_t are functions of W_τ, $\tau\leq t$. That is, let a_t and b_t belong to the sub-algebra $\mathcal{W}_t$. Assume also that $\max\{\|a_t\|,\|b_t\|\}\leq C$ for all $t\in[0,1]$ and that $t\to a_t$ and $t\to b_t$ are continuous mappings in the operator norm. Let $t_0,\dots,t_n$ and $\tau_1,\dots,\tau_n$ be real numbers such that

$$0=t_0\leq t_1\leq\cdots\leq t_n=1,$$

and

$$0\leq\tau_k\leq t_{k-1}.$$

We denote the set of $t_0,\dots,t_n$ and $\tau_1,\dots,\tau_n$ as Δ. Let

$$d(\Delta)=\max_{1\leq k\leq n}(t_k-\tau_k).$$

Consider the sum

$$I(\Delta)=\sum_{i=1}^n a_{\tau_i}(W_{t_i}-W_{t_{i-1}})b_{\tau_i}.$$

It turns out that as $d(\Delta) \to 0$, the sums $I(\Delta)$ converge in operator norm and the limit does not depend on the choice of t_i and τ_i. The limit is called the free stochastic integral and denoted as $\int_0^1 a_t(dW_t)b_t$. An important point in the proof of convergence is that the convergence of sums in the operator norm depends on a free analogue of the Burkholder–Gundy martingale inequalities.

A very useful tool in the study of stochastic integrals is the Itô formula. A free probability analogue of the Itô formula was developed in [3]. In terms of formal rules, it can be written

$$\begin{aligned} a_t\,dt \cdot b_t\,dt = a_t\,dt \cdot b_t\,dW_t\,c_t = a_t\,dW_t\,b_t \cdot c_t\,dt = 0, \\ a_t\,dW_t\,b_t \cdot c_t\,dW_t\,d_t = E(b_t c_t)a_t d_t\,dt. \end{aligned} \tag{9}$$

Note that the rule in the second line is significantly different from the classical case. In terms of free stochastic integrals, the second rule can be written

$$\begin{aligned} \int_0^1 a_t\,dW_t\,b_t \cdot \int_0^1 c_t\,dW_t\,d_t = {} & \int_0^1 \left(\int_0^t a_\tau\,dW_\tau\,b_\tau\right) c_t\,dW_t\,d_t \\ & + \int_0^1 a_t\,dW_t\,b_t \left(\int_0^t c_\tau\,dW_\tau\,d_\tau\right) \\ & + \int_0^1 E(b_t c_t)a_t d_t\,dt. \end{aligned}$$

(Compare Theorem 4.1.2. in [3] or Proposition 8 and Corollary 10 in [1].) Here is an illustration (a particular case of Proposition 4.3.2 in [3]).

Let W_t be the free Brownian motion and define

$$\partial(W_t^n) := W_t^{n-1}\,dW_t + W_t^{n-2}(dW_t)W_t + \cdots + (dW_t)W_t^{n-1}.$$

Then,

$$\begin{aligned} (W_a)^n &= \int_0^\alpha d(W_t^n) \\ &= \int_0^\alpha \partial(W_t^n) + \int_0^\alpha \sum_{0\le k+l\le n-2} W_t^{n-k-l-2}(dW_t)W_t^k(dW_t)W_t^l, \end{aligned}$$

and it follows that

$$\int_0^a \partial(W_t^n) = (W_a)^n - \sum_{k=0}^{\lfloor n/2\rfloor -1}(n-2k-1)C_k \int_0^a W_t^{n-2k-2}t^k\,dt, \tag{10}$$

where C_k are the Catalan numbers,

$$C_k := \frac{1}{k+1}\binom{2k}{k} = E\big[(W_1)^{2k}\big].$$

An analogous formula for the classical Itô integral with respect to the Brownian motion B_t is quite different:

$$\int_0^a nB_t^{n-1}(dB_t) = B_a^n - \frac{n(n-1)}{2}\int_0^a B_t^{n-2}\,dt.$$

Below, we will use the free Itô formula in order to compute the moments of the variable X_t.

3 Free Stochastic Differential Equations

3.1 Existence and Uniqueness

A free stochastic differential equation (free SDE)

$$dX_t = a(X_t)\,dt + b(X_t)(dW_t)c(X_t) \tag{11}$$

is a convenient shortcut notation for the following integral equation:

$$X_t = X_0 + \int_0^t a(X_\tau)\,d\tau + \int_0^t b(X_\tau)\,dW_\tau c(X_\tau). \tag{12}$$

We consider only equations with the coefficients that do not depend explicitly on time, and we will always assume that $a(X_t)$, $b(X_t)$, and $c(X_t)$ are locally operator Lipschitz functions. (A function $f : \mathbb{R} \to \mathbb{C}$ is called locally operator-Lipschitz, if it is a locally bounded, measurable function, and if for all $A > 0$, there is a constant $K_A > 0$, such that

$$\left\| f(X) - f(Y)\right\| \le K_A \|X - Y\|$$

for all self-adjoint operators X and Y with the norm less than A. For example, all polynomials are locally operator-Lipschitz.)

Equations (11) and (12) are particular cases of the following more general equations:

$$dX_t = a(X_t)\,dt + \sum_{i=1}^m b_i(X_t)(dW_t)c_i(X_t) \tag{13}$$

and

$$X_t = X_0 + \int_0^t a(X_\tau)\,d\tau + \sum_{i=1}^m \int_0^t b_i(X_\tau)\,dW_\tau c_i(X_\tau). \tag{14}$$

Our results for the Cauchy transform of X_t can be extended to this more general setting at the expense of more cumbersome notation.

The existence of the solution of (14) may fail for large t if the norm of the solution approaches infinity in finite time. However, for sufficiently small $t > 0$ we have the following local existence result. (See Theorem 3.1 in [4] for a sufficient condition of the global existence in a simpler class of free SDEs, and Theorem 5.2.1 in [11] for an existence and uniqueness result in the case of classical SDEs.)

Theorem 3.1 *Suppose that a_i, b_i, and c_i are locally operator Lipschitz functions and $\overline{X}$ is bounded in operator norm. Then, there exist $t_0 > 0$ and a family of operators X_t defined for all $t \in [0, t_0)$ and bounded in operator norm, such that $X_0 = \overline{X}$, and X_t is a unique solution of* (14) *for $t < t_0$.*

Proof The proof proceeds by Picard's method of successive approximations. We will give the prove for the case $m = 1$. The general case is similar. Define $X_t^{(0)} = \overline{X}$, and

$$X_t^{(N+1)} = \overline{X} + \int_0^t a\bigl(X_\tau^{(N)}\bigr)\, d\tau + \int_0^t b\bigl(X_\tau^{(N)}\bigr)\, dW_\tau c\bigl(X_\tau^{(N)}\bigr). \tag{15}$$

We aim to show that this process converges for all sufficiently small t. For this, it is enough to show that for a sufficiently small $t_0 > 0$ and all $t < t_0$ and $N \geq 1$, the following two claims hold: (i)

$$\bigl\|X_t^{(N)} - X_t^{(N-1)}\bigr\|^2 \leq C \frac{R^N}{N!} t^N,$$

for some constant C and R, and (ii)

$$\bigl\|X_t^{(N)}\bigr\| \leq A$$

for a constant A.

Claim (ii) follows from claim (i) because (i) implies that

$$\bigl\|X_t^{(N)} - \overline{X}\bigr\| \leq C f(t),$$

where $f(t) := \sum_{k=1}^{\infty} \frac{R^{k/2}}{\sqrt{k!}} t^{k/2}$ is defined for all $t < 1/R$, is monotonically increasing, is differentiable at 0, and vanishes at zero. This implies that for all $A > \|\overline{X}\|$, there exists such $t_0 > 0$ that $\|X_t^{(N)}\| \leq A$ for all $t < t_0$. Moreover, this choice of t_0 is independent of N.

In order to prove (i), we proceed by induction. The case $N = 1$ is special and can be easily verified separately. Assume that (i) and (ii) hold for $X_t^{(N)}$ and $X_t^{(N-1)}$ and let us prove that (i) holds for $X_t^{(N+1)}$. We write

$$\begin{aligned} \bigl\|X_t^{(N+1)} - X_t^{(N)}\bigr\| \leq{}& \biggl\| \int_0^t \bigl[a\bigl(X_\tau^{(N)}\bigr) - a\bigl(X_\tau^{(N-1)}\bigr)\bigr] d\tau \biggr\| \\ &+ \biggl\| \int_0^t b\bigl(X_\tau^{(N)}\bigr) dW_\tau c\bigl(X_\tau^{(N)}\bigr) \\ &- \int_0^t b\bigl(X_\tau^{(N-1)}\bigr) dW_\tau c\bigl(X_\tau^{(N-1)}\bigr) \biggr\|. \end{aligned} \tag{16}$$

The second term in this expression can be estimated by the following sum:

$$\biggl\| \int_0^t \bigl[b\bigl(X_\tau^{(N)}\bigr) - b\bigl(X_\tau^{(N-1)}\bigr)\bigr] dW_\tau\, c\bigl(X_\tau^{(N)}\bigr) \biggr\|$$

$$+\left\|\int_0^t b\big(X_\tau^{(N-1)}\big)dW_\tau\big[c\big(X_\tau^{(N)}\big)-c\big(X_\tau^{(N-1)}\big)\big]\right\|$$

$$\le 2\sqrt{2}\left(\int_0^t \big\|b\big(X_\tau^{(N)}\big)-b\big(X_\tau^{(N-1)}\big)\big\|^2\big\|c\big(X_\tau^{(N)}\big)\big\|^2\,d\tau\right)^{1/2}$$

$$+2\sqrt{2}\left(\int_0^t \big\|c\big(X_\tau^{(N)}\big)-c\big(X_\tau^{(N-1)}\big)\big\|^2\big\|b\big(X_\tau^{(N-1)}\big)\big\|^2\,d\tau\right)^{1/2},$$

where we used the free Burkholder–Gundy inequality (see Theorem 3.2.1 in [3]).

By using the assumption that b and c are operator Lipschitz and claim (i), we see that this expression is bounded by

$$4\sqrt{2}\left[K_A^2C\frac{R^N}{N!}A^2\int_0^t\tau^N\,d\tau\right]^{1/2}=\left[32K_A^2A^2C\frac{R^N}{(N+1)!}t^{N+1}\right]^{1/2}.$$

A similar estimate can be obtained for the first part of (16), and by worsening a constant, we can obtain the following inequality:

$$\big\|X_t^{(N+1)}-X_t^{(N)}\big\|^2\le 64K_A^2A^2C\frac{R^N}{(N+1)!}t^{N+1}$$

provided that $t<t_0$. This shows that claim (i) holds for $X_t^{(N+1)}$ provided that $R>64K_A^2A^2$.

Hence, the sequence $X_t^{(N)}=(X_t^{(N)}-X_t^{(N-1)})+\cdots+(X_t^{(1)}-\overline{X})+\overline{X}$ is convergent in operator norm for every $t<t_0$. Let the limit be denoted by X_t. By using the free Burkholder–Gundy inequality, we can take limits on both sides of (15) and check that X_t is a solution of (12).

Next, suppose that X_t and X_t' are two different solutions of (12) for $t<t_0$. Let $v(t)=\|X_t-X_t'\|$. By using the assumption that the coefficients are operator Lipschitz and by using the free Burkholder–Gundy inequality, we obtain:

$$v(t)\le c_1\int_0^t v(\tau)\,d\tau+c_2\left(\int_0^t v(\tau)^2\,d\tau\right)^{1/2}$$

$$\le\big(c_1\sqrt{t_0}+c_2\big)\left(\int_0^t v(\tau)^2\,d\tau\right)^{1/2},$$

where c_1 and c_2 are certain positive constants that depend on Lipschitz constants. (The second inequality follows by the Cauchy–Schwarz inequality.) By the Gronwall inequality (see [11], exercise 5.17 on p. 80) it follows that $v(t)^2=0$ for all $t<t_0$. Hence $X_t=X_t'$ and we established the uniqueness of the solution. □

3.2 Equations for the Cauchy Transform

Theorem 3.2 *Assume that a, b, and c are locally operator-Lipschitz functions and let X_t be a solution of* (11) *bounded in operator norm for all $t\in[0,t_0)$. Let G_t and*

g_t denote the resolvent of X_t and the expectation of the resolvent, respectively, and let $a_t = a(X_t)$, $b_t = b(X_t)$, and $c_t = c(X_t)$. Then, for all $t \in [0, t_0)$,

$$\frac{dg_t}{dt} = -E\big(a_t G_t^2\big) + E(b_t c_t G_t) E\big(b_t c_t G_t^2\big). \tag{17}$$

Let us mention an important particular case, when the product bc does not depend on X_t. In this case, the equation simplifies to the following:

$$\frac{dg_t}{dt} = -E\big(a_t G_t^2\big) + \big[E(bc)\big]^2 g_t \frac{\partial g_t}{\partial z}. \tag{18}$$

If we assume in addition that $b = c = 1$ and $a(x)$ is a polynomial then (18) implies the free Fokker–Planck equation (2) of Biane and Speicher. We will demonstrate this in Corollary 3.6 below.

In the proof of Theorem 3.2 we need the following lemma.

Lemma 3.3 *Let operators H_1 and H_2 belong to the subalgebra $\mathcal{W}_a$ which is generated by $\{W_\tau\}$ where $\tau \le a$. Then*

$$E\bigg[\bigg(\int_a^b b_\tau (dW_\tau) c_\tau\bigg) H_1 \bigg(\int_a^b b_\tau (dW_\tau) c_\tau\bigg) H_2\bigg] = \int_a^b E(c_\tau H_1 b_\tau) E(c_\tau H_2 b_\tau)\, d\tau.$$

This result follows if we write the integral as the limit of sums and use formula (8).

Proof of Theorem For conciseness of the following formulas, let us use the following notation:

$$A = \int_t^{t+\Delta t} a_\tau \, d\tau,$$

and

$$B = \int_t^{t+\Delta t} b_\tau (dW_\tau) c_\tau.$$

Note that $\|A\|_2 = O(\Delta t)$ and $\|B\|_2 = O(\sqrt{\Delta t})$ for small Δt.

By using the resolvent identity twice, we can write:

$$\begin{aligned} G_{t+\Delta t} - G_t &= -G_{t+\Delta t}(A+B)G_t \\ &= -G_t A G_t - G_t B G_t + G_{t+\Delta t}(A+B)G_t(A+B)G_t. \end{aligned}$$

Note that

$$\|G_{t+\Delta t} A G_t A G_t + G_{t+\Delta t} A G_t B G_t + G_{t+\Delta t} B G_t A G_t\|_2 = o(\Delta t).$$

In addition,

$$\|G_{t+\Delta t} - G_t\|_2 = O\big(\sqrt{\Delta t}\big),$$

which implies

$$\|G_{t+\Delta t}BG_tBG_t - G_tBG_tBG_t\|_2 = o(\Delta t)$$

Hence, we can write

$$E(G_{t+\Delta t} - G_t) = E(-G_tAG_t - G_tBG_t + G_tBG_tBG_t) + o(\Delta t). \tag{19}$$

Next, we use the facts that

$$\int_t^{t+\Delta t} a_\tau \, d\tau = a_t \Delta t + o(\Delta t),$$

that

$$E\left[\int_t^{t+\Delta t} G_t b_\tau (dW_\tau) c_\tau G_t\right] = 0,$$

and that

$$E\left[G_t\left(\int_t^{t+\Delta t} b_\tau (dW_\tau) c_\tau\right) G_t \left(\int_t^{t+\Delta t} b_\tau (dW_\tau) c_\tau\right) G_t\right]$$
$$= (\Delta t) E(c_t G_t b_t) E\left(c_t G_t^2 b_t\right) + o(\Delta t),$$

where the latter holds because of Lemma 3.3 and the assumption that b_t and c_t are Lipschitz. Hence, after taking the expectation in (19) we obtain

$$g_{t+\Delta t} - g_t = \Delta t\left\{-E\left(a_t G_t^2\right) + E(c_t G_t b_t) E\left(c_t G_t^2 b_t\right)\right\} + o(\Delta t),$$

which is equivalent to the statement of the theorem. □

In order to proceed and obtain a differential equation on g_t, we need to impose additional conditions on a_t, b_t, and c_t which would allow us to eliminate expectations from (17).

Proposition 3.4 *Let X_t be the solution of equation* (11), *and $G(t,z)$ and $g(t,z)$ be its resolvent and the expectation of the resolvent, respectively. Suppose that functions a and bc are polynomials in one variable and that their degrees are not greater than $k \geq 0$. Then,*

$$\frac{dg}{dt} = -\frac{\partial(ag)}{\partial z} - \sum_{j=0}^{k-2} \frac{(k-1-j)!}{k!} \frac{\partial^{j+2} a(z)}{(\partial z)^{j+2}} E\left(X^j\right) \tag{20}$$

$$+ \left[bcg + \sum_{j=0}^{k-1} \frac{(k-1-j)!}{k!} \frac{\partial^{j+1}[b(z)c(z)]}{(\partial z)^{j+1}} E\left(X_t^j\right)\right] \tag{21}$$

$$\times \left[\frac{\partial(bcg)}{\partial z} + \sum_{j=0}^{k-2} \frac{(k-1-j)!}{k!} \frac{\partial^{j+2}[b(z)c(z)]}{(\partial z)^{j+2}} E\left(X_t^j\right)\right]. \tag{22}$$

This equation is more useful than it might seem at the first sight. First of all, it is often possible to compute the expectations $E(X_t^j)$by using the Itô formula. Second, if these expectations are known, then the equation is a quasilinear PDE and the method of characteristics is applicable.

Proof Let $f(x)$ be a polynomial. If we expand $f(X)(X-z)^{-1}$ and $f(X)(X-z)^{-2}$ in partial fractions and then take the expectations, we obtain the formulas:

$$E\left(\frac{f(X)}{X-z}\right)=E\left(\frac{f(z)}{X-z}\right)+\sum_{j=0}^{k-1}\frac{(k-1-j)!}{k!}\frac{\partial^{j+1}f(z)}{(\partial z)^{j+1}}E\left(X^j\right),$$

and

$$E\left(\frac{f(X)}{(X-z)^2}\right)=E\left(\frac{f(z)}{(X-z)^2}\right)+E\left(\frac{f'(z)}{X-z}\right)$$
$$+\sum_{j=0}^{k-2}\frac{(k-1-j)!}{k!}\frac{\partial^{j+2}f(z)}{(\partial z)^{j+2}}E\left(X^j\right).$$

By using $a(z)$ or $b(z)c(z)$ as $f(z)$ it is easy to see that the statement of the proposition follows from Theorem 3.2. □

Corollary 3.5 *Suppose that a is a polynomial and that $bc=1$. Then,*

$$\frac{dg}{dt}=-\frac{\partial(ag)}{\partial z}+g\frac{\partial g}{\partial z} \tag{23}$$
$$-\sum_{j=0}^{k-2}\frac{(k-1-j)!}{k!}\frac{\partial^{j+2}a(z)}{(\partial z)^{j+2}}E\left(X^j\right). \tag{24}$$

Corollary 3.6 *Suppose that a is a polynomial with real coefficients, that $b=c=1$, and that X_0 is self-adjoint. Assume that the spectral distribution of X_t is absolutely continuous and bounded with the density $p(x,t)$. Then, at all points where $\partial p/\partial x$ is defined, it is true that*

$$\frac{dp}{dt}=-\frac{\partial}{\partial x}\{ap+p\cdot Hp\},$$

where Hp is the Hilbert transform of p.

(This is the free Fokker–Planck equation (2) of Biane and Speicher.)

Proof of Corollary 3.6 Note that X_t are self-adjoint for all t. Let us take the imaginary part on both sides of the formula in Corollary 3.5, and then pass to the limit $y\to 0$, where $y:=\operatorname{Im} z$. Assume that $g(z)$ is analytic at $z=x$ and therefore taking the limit commutes with operations of differentiation with respect to t and z.

Since X_t is self-adjoint, therefore $\mathrm{Im}(E(X_t^j)) = 0$. Hence, the formula in Corollary 3.5 simplifies to

$$\pi \frac{dp}{dt} = \lim_{y\to 0}\left\{-\frac{\partial}{\partial z}\mathrm{Im}[ag] + \mathrm{Im}\left[g\frac{\partial g}{\partial z}\right]\right\}, \tag{25}$$

where we used the Stieltjes inversion formula. Note that

$$\begin{aligned}\lim_{y\to 0}\mathrm{Im}\left(g\frac{\partial g}{\partial z}\right) &= \lim_{y\to 0}\left\{\mathrm{Im}\, g\,\mathrm{Re}\,\frac{\partial g}{\partial z} + \mathrm{Re}\, g\,\mathrm{Im}\,\frac{\partial g}{\partial z}\right\}\\ &= \pi\left(p\frac{\partial}{\partial x}(-Hp) - (Hp)\frac{\partial}{\partial x}p\right)\\ &= -\pi\frac{\partial}{\partial x}[p\cdot Hp].\end{aligned}$$

Similarly,

$$\lim_{y\to 0}\frac{\partial}{\partial z}\mathrm{Im}[ag] = \pi\frac{\partial}{\partial x}[ap]$$

because $\mathrm{Im}\, a(x) = 0$ and $\mathrm{Re}\, a(x) = a(x)$. Hence, (25) simplifies to

$$\frac{dp}{dt} = -\frac{\partial}{\partial x}[ap + p\cdot Hp].$$

□

3.3 Examples

In this section, we calculate explicit solutions in several particular cases.

3.3.1 *Ornstein–Uhlenbeck*

Proposition 3.7 *Suppose that X_t satisfies the equation of the free Ornstein–Uhlenbeck process*:

$$dX_t = \theta X_t\, dt + \sigma\, dW_t.$$

Suppose that $X_0 = 0$. Then the spectral distribution of X_t is the semicircle distribution supported on the interval I_θ, where

(1)

$$I_\theta = \left[-\sqrt{\frac{2\sigma^2}{\theta}\left(e^{2\theta t} - 1\right)}, +\sqrt{\frac{2\sigma^2}{\theta}\left(e^{2\theta t} - 1\right)}\right],$$

if $\theta > 0$,

(2)

$$I_\theta = \left[-2\sigma\sqrt{t}, 2\sigma\sqrt{t}\right]$$

if $\theta = 0$, and

(3)

$$I_\theta = \left[-\sqrt{\frac{2\sigma^2}{|\theta|}\left(1 - e^{-2|\theta|t}\right)}, +\sqrt{\frac{2\sigma^2}{|\theta|}\left(1 - e^{-2|\theta|t}\right)} \right],$$

if $\theta < 0$.

Hence, if $\theta > 0$, then the support of the distribution grows exponentially; if $\theta = 0$, then the support grows linearly, and if $\theta < 0$, the spectral distribution converges to the semicircle distribution supported on the interval $[-\sigma\sqrt{2/|\theta|}, \sigma\sqrt{2/|\theta|}]$.

Proof In this case $a_t = \theta X_t$, $b_t = \sqrt{\sigma}$. Note that

$$a_t = \theta X_t = \theta\left(z + G_t^{-1}\right).$$

Hence,

$$a_t G_t^2 = \theta\left(z G_t^2 + G_t\right)$$

and

$$E\left(a_t G_t^2\right) = \theta\left(z_t \frac{\partial g_t}{\partial z} + g_t\right).$$

Therefore, the differential equation for g_t is

$$\frac{\partial g}{\partial t} + \left(\theta z - \sigma^2 g\right)\frac{\partial g}{\partial z} = -\theta g_t. \tag{26}$$

The initial condition $X_0 = 0$ corresponds to $g(0, z) = -z^{-1}$, and we can solve this partial differential equation by using the method of characteristics (see pp. 9–19 in [8]).

Indeed the equations of characteristic curves are

$$\frac{dt}{d\xi} = 1, \tag{27}$$

$$\frac{dz}{d\xi} = \theta z - \sigma^2 g, \tag{28}$$

$$\frac{dg}{d\xi} = -\theta g. \tag{29}$$

By using (27), we can set $\xi = t$. Then (29) implies that

$$g(t) = A e^{-\theta t},$$

and then we can solve (28) as

$$z(t) = C e^{\theta t} + \frac{\sigma^2 A}{2\theta} e^{-\theta t}.$$

It follows that the initial point of a characteristic curve is given by the equations

$$g(0) = A,$$
$$z(0) = C + \frac{\sigma^2 A}{2\theta}.$$

On the other hand we can parameterize the initial condition of the PDE as

$$z(s) = s, \qquad g(s) = -1/s.$$

Hence, we obtain the following parameterization for A and C:

$$A = -1/s, \qquad C = s + \frac{\sigma^2}{2\theta}\frac{1}{s}.$$

Therefore, the equations of the characteristic surface are

$$g(s,t) = -\frac{1}{s}e^{-\theta t}, \tag{30}$$

$$z(s,t) = \left(s + \frac{\sigma^2}{2\theta}\frac{1}{s}\right)e^{\theta t} - \frac{\sigma^2}{2\theta}\frac{1}{s}e^{-\theta t}. \tag{31}$$

From (30) we have

$$s = -\frac{1}{g e^{\theta t}}.$$

After we substitute this in (31) and re-arrange, we obtain the following functional equation for $g(t,z)$:

$$g^2 + \frac{2\theta z}{\sigma^2(e^{2\theta t} - 1)} g + \frac{2\theta}{\sigma^2(e^{2\theta t} - 1)} = 0,$$

provided that $\theta \neq 0$. We can easily solve this quadratic equation for g. Note that by the Stieltjes inversion formula the density of the corresponding distribution is given by the imaginary part of the Cauchy transform g. We can check that in our case this density corresponds to the density of the semicircle distribution. The radius of the semicircle distribution is

$$\sqrt{\frac{2\sigma^2}{\theta}\left(e^{2\theta t} - 1\right)},$$

if $\theta > 0$, and

$$\sqrt{\frac{2\sigma^2}{|\theta|}\left(1 - e^{-2|\theta| t}\right)},$$

if $\theta < 0$. This implies the statement of the proposition for $\theta \neq 0$. The case $\theta = 0$ can be analyzed similarly. □

3.3.2 Geometric Brownian Motion

Now let us consider the case when the coefficient b_t explicitly depends on X_t. Namely, let $a_t = \theta X_t$, and $b_t = X_t^{1/2}$.

In this example we deal with the equation

$$dX_t = \theta X_t\,dt + X_t^{1/2}(dW_t)X_t^{1/2},$$

which is an analog of the classical equation for the "geometric Brownian motion", $dx_t = \theta x_t\,dt + x_t\,dB_t$.

Let us assume that $X_0 = I$ and use the free Itô formula to study the moments of the solution. Clearly, $E(X_t) = e^{\theta t}$. In order to calculate the second moment, we write

$$\begin{aligned} d(X_t^2) &= (X_t + dX_t)^2 - X_t^2 \\ &= \big(2\theta X_t^2 + e^{\theta t}X_t\big)dt + X_t^{3/2}(dW_t)X_t^{1/2} + X_t^{1/2}(dW_t)X_t^{3/2}, \end{aligned}$$

where we used the free Itô formula to calculate

$$\begin{aligned} dX_t\,dX_t &= X_t^{1/2}(dW_t)X_t(dW_t)X_t^{1/2} \\ &= E(X_t)X_t\,dt. \end{aligned}$$

Let h_t denote $E(X_t^2)$. Then we have the following equation:

$$\frac{dh_t}{dt} = 2\theta h_t + e^{2\theta t}$$

with the initial condition $h_0 = 1$. The solution is

$$h_t = (t+1)e^{2\theta t}.$$

Hence, the variance of the spectral distribution of X_t is $te^{2\theta t}$. The ratio of the standard deviation to the expectation of X_t is $\sqrt{t}$.

In order to recover the entire spectral distribution, we use Theorem 3.2, and obtain the following result.

Proposition 3.8 *Suppose that X_t satisfies the following equation*:

$$dX_t = \theta X_t\,dt + X_t^{1/2}(dW_t)X_t^{1/2},$$

and that $X_0 = I$. Then, the expectation of the resolvent satisfies the following functional equation:

$$z + g^{-1} = e^{(\alpha - zg)t}, \tag{32}$$

where $\alpha = \theta - 1$. The density of the spectral distribution of X_t is supported on the interval

$$I = \left[\frac{r_1(t)}{1 + r_1(t)}e^{(\alpha - r_1(t))t}, \frac{r_2(t)}{1 + r_2(t)}e^{(\alpha - r_2(t))t}\right],$$

where

$$r_{1,2}(t) = \frac{-1 \pm \sqrt{1+4/t}}{2}.$$

We can see from this proposition that the solution of the free SDE exists remains positive definite for all $t > 0$.

If $t \to \infty$, then $r_{1,2}(t)$ are asymptotically $1/t$ and $-1 - 1/t$. Hence, as $t \to \infty$ the support of the solution becomes asymptotically close to

$$\left[\frac{1}{et} e^{(\theta-1)t}, ete^{\theta t}\right].$$

In particular, if $\theta < 0$, then both the lower and the upper bound of the spectral distribution shrink to zero exponentially fast, although at different rates ($\theta - 1$ and θ). If $\theta = 0$, then the lower bound shrinks to zero exponentially and the upper bound grows linearly. If $\theta \in (0, 1)$, then the lower bound shrinks exponentially and the upper bound grows exponentially. If $\theta = 1$, then the lower bound declines as $(et)^{-1}$ and the upper bound grows exponentially. If $\theta > 1$, then both the upper and lower bounds grow exponentially.

Proof of Proposition 3.8 We have

$$E(b_t^* G_t b_t) = E(G_t X_t) = 1 + z g_t,$$

and

$$E\big(b_t^* G_t^2 b_t\big) = E\big(G_t^2 X_t\big) = g_t + z\frac{\partial g_t}{\partial z}.$$

Hence the differential equation is

$$\frac{\partial g}{\partial t} = -\theta\left(g + z\frac{\partial g}{\partial z}\right) + (1 + z g_t)\left(g_t + z\frac{\partial g_t}{\partial z}\right),$$

or

$$\frac{\partial g}{\partial t} + z\big((\theta-1) - zg\big)\frac{\partial g}{\partial z} = -g\big((\theta-1) - zg\big). \tag{33}$$

By assumption, the initial condition is $g(0, z) = (1 - z)^{-1}$.

The equations of characteristic curves are

$$\frac{dt}{d\xi} = 1, \tag{34}$$

$$\frac{dz}{d\xi} = z(\theta - 1 - zg), \tag{35}$$

$$\frac{dg}{d\xi} = -g(\theta - 1 - zg). \tag{36}$$

From (34) we can set $\xi = t$. Then, if we divide (36) by (35), we obtain the following equation:

$$\frac{dg}{dz} = -\frac{g}{z},$$

which implies the following family of equations for the characteristic curves:

$$g = \frac{A}{z}.$$

If we substitute this in (35) and solve the resulting ODE, we find

$$z(t) = Ce^{(\theta-1-A)t}.$$

Hence,

$$g(t) = (A/C)e^{-(\theta-1-A)t}.$$

In particular, $z(0) = C,\ g(0) = A/C$.

On the other hand, we can parameterize the initial condition of (33) as

$$z(s) = s, \qquad g(s) = \frac{1}{1-s}.$$

This implies the following parameterization for A and C:

$$C = s, \qquad A = \frac{s}{1-s}.$$

Hence, the characteristic surface is

$$z(s,t) = s\exp\left\{\left(\theta - 1 - \frac{s}{1-s}\right)t\right\}, \tag{37}$$

$$g(s,t) = \frac{1}{1-s}\exp\left\{-\left(\theta - 1 - \frac{s}{1-s}\right)t\right\}. \tag{38}$$

We can eliminate s from these equations:

$$s = \frac{zg}{1+zg}.$$

After we substitute this expression for s in (37) and re-arrange the terms, then we obtain the following equation:

$$z + g^{-1} = \exp\{(\theta - 1 - zg)t\}.$$

Let us denote $\theta - 1$ as α for simplicity of notation. Then, the functional equation for the Cauchy transform $g(t,z)$ is

$$z + g^{-1} = e^{(\alpha - zg)t}. \tag{39}$$

If we take the differential of this equation, then we find that

$$dz\big(1+gte^{(\alpha-zg)t}\big)=dg\big(g^{-2}-zte^{(\alpha-zg)t}\big).$$

The branch points of the function $g(z)$ can be found from the equation $dz/dg=0$. Hence, at the branch points,

$$e^{(\alpha-zg)t}=\frac{1}{g^2zt}.$$

Substituting this into (39), we obtain the following equation for the branch points:

$$t(zg)^2+t(zg)-1=0.$$

Hence,

$$zg=\frac{-1\pm\sqrt{1+4/t}}{2}\equiv r_{1,2}(t). \tag{40}$$

Then (39) and (40) imply that at the branch points,

$$g_{1,2}=\big(1+r_{1,2}(t)\big)e^{-(\alpha-r_{1,2}(t))t}$$

and

$$z_{1,2}=\frac{r_{1,2}(t)}{1+r_{1,2}(t)}e^{(\alpha-r_{1,2}(t))t}.$$

Finally, note that branch points of the Cauchy transform are bounds for the support of the spectral probability distribution. □

3.3.3 Geometric Brownian Motion II

In our next example, we consider a different analog of the classical geometric Brownian motion equation, namely, the following free SDE:

$$dX_t=\theta X_t\,dt+X_t\,dW_t+(dW_t)X_t.$$

As in the previous example, assume that $X_0=I$, and note that $E(X_t)=e^{\theta t}$, the same as in the previous example.

It is possible to write a PDE for the expectation of the resolvent in this example similar to equations in Theorem 3.2 and Proposition 3.4. However, it seems that it is difficult to find an explicit solution of this equation and recover the spectral distribution function of X_t.

Still, it is possible to see that the behavior of the solution is quite different from the behavior of the solution in the previous example by studying the variance of the solution. By using the free Itô formula, we can write

$$\begin{aligned}d(X_t^2)&=\big[2\theta X_t^2+X_t^2+2E(X_t)X_t+E\big(X_t^2\big)\big]\,dt\\&\quad+X_t^2\,dW_t+2X_t(dW_t)X_t+(dW_t)X_t^2.\end{aligned}$$

Let h_t denote $E(X_t^2)$. Then we have the following ODE for h_t:

$$\frac{dh_t}{dt} = 2(\theta + 1)h_t + 2e^{2\theta t}.$$

The initial condition is $h_0 = 1$ and the solution is

$$h_t = 2e^{2(\theta+1)t} - e^{2\theta t}.$$

Hence the variance of X_t is $2e^{2\theta t}(e^{2t} - 1)$, and the ratio of the standard deviation to the expectation is $\sqrt{2(e^{2t} - 1)}$. This ratio grows exponentially fast with t, quite unlike the previous example, where this ratio equals $\sqrt{t}$.

3.3.4 *Explosive Equation*

In our final example, we will consider an equation whose solution explodes in finite time. By this we mean that the norm of the solution becomes infinite in finite time.

Proposition 3.9 *Suppose that X_t satisfies the following equation*:

$$dX_t = kX_t(dW_t)X_t,$$

and let the initial condition be $X_0 = aI$. Then the spectral distribution of X_t is defined for all $t \le (ak)^{-2}$ and it is supported on the interval

$$I = \left[\frac{(1 - ak\sqrt{t})^2}{(1 - a^2k^2t)^2}, \frac{(1 + ak\sqrt{t})^2}{(1 - a^2k^2t)^2}\right].$$

For $\tau \in (0, 1)$, the density of the spectral distribution of the operator $a^{-1}X_{(ak)^2\tau}$ is given by the formula

$$f(\xi) = \frac{\sqrt{-(1-\tau)^2\xi^2 + 2(1+\tau)\xi - 1}}{2\pi\xi^3\tau}.$$

Proof of Proposition 3.9 We can compute

$$\begin{aligned} E(G_t b_t^2) &= k\big[E\big(G_t^{-1}\big) + 2z + z^2 g_t\big] \\ &= k\big[a + z + z^2 g_t\big], \end{aligned}$$

where we used the fact that

$$E\big(G_t^{-1}\big) = E(X_t) - z = E(X_0) - z = a - z.$$

In addition,

$$E\big(G_t^2 b_t^2\big) = k\left[1 + 2zg_t + z^2\frac{\partial g_t}{\partial z}\right].$$

Hence, the differential equation for g_t is

$$\frac{\partial g}{\partial t} - k^2\big(a + z + z^2 g\big)z^2\frac{\partial g}{\partial z} = k^2\big(a + z + z^2 g\big)(1 + 2zg),$$

and the initial condition is $g_0 = (a - z)^{-1}$.

The equations for the characteristic curves are

$$\frac{dt}{d\xi} = 1, \tag{41}$$

$$\frac{dz}{d\xi} = -k^2\big(a + z + z^2 g\big)z^2, \tag{42}$$

$$\frac{dg}{d\xi} = k^2\big(a + z + z^2 g\big)(1 + 2zg). \tag{43}$$

From (41), we can set $\xi = t$, and the equations for the characteristic curves in (z, g)-plane become:

$$\frac{dz}{dt} = -k^2\big(a + z + z^2 g\big)z^2, \tag{44}$$

$$\frac{dg}{dt} = k^2\big(a + z + z^2 g\big)(1 + 2zg). \tag{45}$$

After dividing (45) by (44), we obtain

$$\frac{dg}{dz} = -\frac{1 + 2gz}{z^2}.$$

The general solution of this equation is

$$g(z) = -z^{-1} + Cz^{-2}. \tag{46}$$

If we substitute this expression in (44), we obtain

$$\frac{dz}{dt} = -k^2(a + C)z^2.$$

Hence,

$$z(t) = \frac{1}{k^2(a + C)t + A}. \tag{47}$$

By substituting this in (46), we obtain

$$g(t) = -(a + C)k^2 t - A + C\big((a + C)k^2 t + A\big)^2. \tag{48}$$

In particular, if $t = 0$, then

$$z(0) = 1/A, \tag{49}$$

$$g(0) = -A + CA^2.$$

On the other hand, the initial condition is $g(z) = (a - z)^{-1}$, which we can parameterize as

$$z(s) = s, \qquad g(s) = (a - s)^{-1}. \tag{50}$$

Comparing (49) and (50), we obtain the following parameterization for A and C:

$$A = \frac{1}{s}, \qquad C = \frac{as}{a - s}.$$

We substitute these expressions in (47) and (48) and obtain

$$z(t, s) = \frac{1}{\frac{a^2}{a-s}k^2 t + \frac{1}{s}}, \tag{51}$$

$$g(t, s) = \frac{a^5 s}{(a - s)^3}k^4 t^2 + \frac{a^2(a + s)}{(a - s)^2}k^2 t + \frac{1}{a - s}. \tag{52}$$

We are going to eliminate s from the pair of (51) and (52). For this reason, we write (52) as

$$g(t, s) = \frac{s}{a - s}\left(\frac{a^2}{a - s}k^2 t + \frac{1}{s}\right)\left(\frac{a^3}{a - s}k^2 t + 1\right),$$

and then we substitute (51) and obtain

$$g(t, s) = \frac{s}{a - s}\frac{1}{z}\left(\frac{a^3}{a - s}k^2 t + 1\right),$$

or

$$\frac{a - s}{s} zg = \frac{a^3}{a - s}k^2 t + 1. \tag{53}$$

By using (51) again, we note that

$$\frac{a^3}{a - s}k^2 t + 1 = \frac{a}{z} - \frac{a}{s} + 1.$$

Hence, (53) can be re-written as

$$(a - s)zg = \left(\frac{a}{z} + 1\right)s - a,$$

and, therefore,

$$s = a\frac{1 + zg}{1 + zg + \frac{a}{z}},$$

and

$$a - s = a\frac{\frac{a}{z}}{1 + zg + \frac{a}{z}}.$$

After substituting these expressions in equation (51), we obtain

$$\frac{1}{z} = \left(z + z^2 g + a\right)k^2 t + \frac{1}{a} + \frac{1}{z + z^2 g}.$$

After re-arranging the terms and dividing by z, we get the following equation:

$$k^2 t z^3 g^2 + \left(\frac{z}{a} - 1 + (a + 2z)zk^2 t\right)g + \frac{1}{a} + (z + a)k^2 t = 0.$$

This functional equation for $g(z,t)$ is quadratic and therefore it is easily solvable.

In particular, the branch points of $g_t(z)$ are the zeros of the discriminant of this equation, which can be computed as

$$D = \left(k^2 at - \frac{1}{a}\right)^2 z^2 - 2\left(k^2 at + \frac{1}{a}\right)z + 1.$$

Therefore, the branch points are

$$z_{\pm} = a\frac{(1 \pm ak\sqrt{t})^2}{(1 - a^2k^2t)^2}.$$

Note that as t approaches $(ak)^{-2}$, the branch points approach $a/4$ and ∞.

It follows that for $t < (ak)^{-2}$, the spectral distribution of X_t is supported on the interval $[z_-, z_+]$ and in this region it has the density

$$f(x)\,dx = \frac{1}{a}\frac{\sqrt{-(1 - k^2a^2t)^2(\frac{x}{a})^2 + 2(k^2a^2t + 1)(\frac{x}{a}) - 1}}{2\pi k^2a^2t(x/a)^3}\,dx.$$

If we use variables $\tau = k^2a^2t$, and $\xi = x/a$, then we can write this density as

$$f(\xi)\,d\xi = \frac{\sqrt{-(1 - \tau)^2\xi^2 + 2(1 + \tau)\xi - 1}}{2\pi\xi^3\tau}. \qquad \square$$

References

1. Anshelevich, M.: Itô formula for free stochastic integrals. J. Funct. Anal. **188**, 292–315 (2002)
2. Biane, P.: Free Brownian motion, free stochastic calculus and random matrices. In: Voiculescu, D.-V. (ed.) Free Probability Theory. Fields Institute Communications, vol. 12, pp. 1–19. Am. Math. Soc., Providence (1997)
3. Biane, P., Speicher, R.: Stochastic calculus with respect to free Brownian motion and analysis on Wigner space. Probab. Theory Relat. Fields **112**, 373–409 (1998)
4. Biane, P., Speicher, R.: Free diffusions, free entropy and free Fisher information. Ann. Inst. Henri Poincaré Probab. Stat. **37**, 581–606 (2001)
5. Capitaine, M., Donati-Martin, C.: Free Wishart processes. J. Theor. Probab. **18**, 413–438 (2005)
6. Demni, N.: Free Jacobi process. J. Theor. Probab. **21**, 118–143 (2008)
7. Gao, M.: Free Ornstein–Uhlenbeck processes. J. Math. Anal. Appl. **322**, 177–192 (2006)
8. John, F.: Partial Differential Equations. Applied Mathematical Sciences, vol. 1. Springer, Berlin (1981)

9. Kummerer, B., Speicher, R.: Stochastic integration on the Cuntz algebra O_∞. J. Funct. Anal. **103**, 372–408 (1992)
10. Nica, A., Speicher, R.: Lectures on the combinatorics of free probability. London Mathematical Society Lecture Note Series, vol. 335. Cambridge University Press, Cambridge (2006)
11. Oksendal, B.: Stochastic Differential Equations, 6th edn. Springer, Berlin (2003)
12. Speicher, R.: A new example of independence and white noise. Probab. Theory Relat. Fields **84**, 141–159 (1990)
13. Voiculescu, D., Dykema, K., Nica, A.: Free Random Variables. CRM Monograph Series, vol. 1. Am. Math. Soc., Providence (1992)