LIFE CONTINGENCY MODELS II

VLADISLAV KARGIN

Contents

1. Policy Values and Reserves

If an insurance product is priced by using the equivalence principle, then the expected net inflow at the time of the issue is 0. However, later it can become different from zero.

Example 1.1. An insurer has issued a policy paying 1 unit at the end of the year of death in exchange for the payment of a premium P at the beginning of each year, provided the life survives. The premium P is determined using the equivalence principle.

Assume that the insured is still alive 1 year after entering into the contract. Further, assume that the insurer continues to use $i = 0.06$ and the following mortality assumption:

$$_{k|}q_0 = 0.2 \quad k = 0, 1, 2, 3, 4.$$

Find the difference between expected liabilities and assets at that time.

Definition 1.2. If the insurance contract is in force at time t, that is, if the insuree is alive at that time, then the *loss* at time t is the difference between the present values of future benefit payments and premiums evaluated at that time.

Definition 1.3. The *policy value* of an insurance contract at time t is the expected value of the loss at time t.

Indeed, typically the expected future benefit payments are larger than expected future premium payments and therefore the policy has a value for the insuree.

Consider, for example, the value of the policy for a very old person. The expected future premium payments are much smaller than the expected future benefit payment.

The policy value has also another name. It is often called *benefit reserve*. Indeed, the company need to have a certain sum of money on hand since the expected present value of the future benefit payment exceeds the expected present value of the future premiums. The insurance company is usually required to have sufficient reserves to cover this difference.

Usually these reserves are obtained by investing the received premiums.

At each moment the insurance company can estimate the value of its investments. The profit of the company is the difference between the value of all investments and the sum of benefit reserves. If the profit is positive it can be distributed to shareholders. If the profit is negative, the company needs to borrow funds to cover reserve requirements.

This explains the importance of the policy values/benefit reserves.

The policy values can be net and gross. The *net policy values* are computed using the net benefit premiums, that is, the premiums computed by using the equivalence principle.

The *gross policy values* are computed using the gross benefit premiums, that is, the premiums actually written in the contract.

The policy value can be calculated just before a premium payment is made or immediately after this payment. Depending on this, we call this policy value either the t-th *terminal policy value* or $(t + 1)$-th *initial policy value*.

In this chapter, we study policy values (benefit reserves) of some insurance liabilities.

1.1. Policy value of the whole life insurance.

1.1.1. *Formulas for calculating policy values.* The t-th terminal loss for a whole life insurance on (x) is the present value of the random future stream of benefit and premium payments,

$${}_tL_x = v^{K_{x+t}} - P_x \ddot{a}_{\overline{K_{x+t}}|} = Z_{x+t} - P_x \ddot{Y}_{x+t}.$$

The t-th *terminal* policy value of a fully discrete unit whole life insurance is the expectation of the loss. It is denoted by ${}_tV_x$,

$${}_tV_x = \mathbb{E}({}_tL_x) = A_{x+t} - P_x \ddot{a}_{x+t}.$$

This formula is called the prospective formula for the policy value because at time $x + t$ it depends on estimates of the future streams of payments.

Note that at that time the company can use an updated information about the interest rates and mortality and therefore an updated values of A_{x+t} and $\ddot{a}_{x+t}$.

The $(t + 1)$-st *initial* policy value of a fully discrete unit whole life insurance is denoted by ${}_{t+1}I_x$,

$${}_{t+1}I_x = {}_tV_x + P_x.$$

We will usually omit the word "terminal" in the future.

Since the t-th policy value measures the expected value of insurance at time t to the insuree, it can be used to calculate the refund in the case when the insurance is cancelled.

That is, if a policy is canceled t years after issue, before the benefit premium is paid, the amount to be returned to an insuree, not including expenses, is the t-th policy value.

Example 1.4. Ten years ago, Joan entered a fully discrete whole life insurance contract with face value $\$100,000$. Joan was 40 years old when she entered this insurance. Suppose that $i = 5\%$, $A_{40} = 0.13$ and $A_{50} = 0.20$. The insurance allows the insuree to cancel this policy any time five years after issue. If the insuree cancels her insurance at time t, insuree will receive the t-th terminal (net) benefit reserve as compensation for canceling this insurance.

(1) Calculate the annual benefit premium paid by Joan in each of these 10 years.
(2) Calculate the amount that Joan can receive if she cancels her insurance.

Solution.
(i) We have that

$$100,000 P_{40} = \frac{dA_{40}}{1 - A_{40}} = 711.55.$$

(ii)

$$\ddot{a}_{50} = \frac{1 - A_{50}}{d} = 16.8,$$
$$100,000 {}_{40}V_{10} = 100,000(A_{50} - P_{40}\ddot{a}_{50}) = 8045.98$$

This is the amount that Joan can receive if she cancels her insurance.

We can also calculate the policy value in several different ways, formulated in the following theorem.

Theorem 1.5. *We have that*

(1)
$${}_tV_x = (P_{x+t} - P_x)\ddot{a}_{x+t}. \textit{ [Premium difference formula]}$$

(2)
$${}_tV_x = \Big(1 - \frac{P_x}{P_{x+t}}\Big) A_{x+t}. \textit{ [Uncovered benefit formula]}$$

(3)
$${}_tV_x = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}. \textit{ [Ratio of annuities formula]}$$

(4)
$${}_tV_x = 1 - \frac{P_x + d}{P_{x+t} + d}. \textit{ [Benefit premiums formula]}$$

(5)
$${}_tV_x = 1 - \frac{1 - A_{x+t}}{1 - A_x}. \textit{ [Life insurance formula]}$$

Proof.

(1)

$$ {}_tV_x = A_{x+t} - P_x\ddot{a}_{x+t} = P_{x+t}\ddot{a}_{x+t} - P_x\ddot{a}_{x+t} = \big(P_{x+t} - P_x\big)\ddot{a}_{x+t}. $$

(2)

$$ {}_tV_x = A_{x+t} - P_x\ddot{a}_{x+t} = A_{x+t} - P_x\frac{A_{x+t}}{P_{x+t}} = \Big(1 - \frac{P_x}{P_{x+t}}\Big)A_{x+t}. $$

(3) Let us use the facts that $A_{x+t} = 1 - d\ddot{a}_{x+t}$ and $P_x = 1/\ddot{a}_x - d$. Then

$$ {}_tV_x = A_{x+t} - P_x\ddot{a}_{x+t} = 1 - d\ddot{a}_{x+t} - \Big(\frac{1}{\ddot{a}_x - d} - d\Big)\ddot{a}_{x+t} = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}. $$

(4) This formula immediately follows from the previous one since $P_x + d = 1/\ddot{a}_x$ and $P_{x+t} + d = 1/\ddot{a}_{x+t}$.

(5) This formula also follows from the ratio of annuities formula, since $\ddot{a}_x = (1 - A_x)/d$ and $\ddot{a}_{x+t} = (1 - A_{x+t})/d$.

□

One consequence of this theorem is that usually the policy value is positive. Indeed, usually the force of mortality of $(x + t)$ is bigger than that of (x). Hence, $P_{x+t} > P_x$ and ${}_tV_x > 0$.

Theorem 1.6. *Under constant force of mortality* ${}_tV_x = 0$.

Proof. Under constant force of mortality, the survival function ${}_tp_x$ does not depend on x. Hence, A_x, $\ddot{a}_x$ and P_x do not depend on x. Thus, by the previous theorem, ${}_tV_x = 0$. □

Suppose a whole life insurance contract allows the insuree to modify the policy at time t in the following way. The insuree can stop paying annual benefit premiums and receive a whole life insurance with a face value of F.

This situation is equivalent to paying refund to the insuree, which the insuree uses to buy a new insurance with face value F.

Hence, we have

$$FA_{x+t} = {}_tV_x = \Big(1 - \frac{P_x}{P_{x+t}}\Big)A_{x+t},$$
$$F = 1 - \frac{P_x}{P_{x+t}}$$

Example 1.7. Ten years ago, Joan entered a fully discrete whole life insurance contract with face value $\$100,000$. Joan was 40 years old when she entered this insurance. Suppose that $i = 5\%$, $A_{40} = 0.13$ and $A_{50} = 0.20$. This insurance allows the insuree to stop paying benefit premiums payments and receive a whole life insurance with a face value of F. Calculate F if Joan decides to stop paying benefit premiums.

Solution. We have that

$$P_{40} = \frac{dA_{40}}{1 - A_{40}} = 0.00711549,$$
$$P_{50} = \frac{dA_{50}}{1 - A_{50}} = 0.01190476,$$
$$F = (100,000) \times \Big(1 - \frac{P_{40}}{P_{50}}\Big) = 40,229.89$$

Example 1.8. An insurer issues a whole life insurance policy to a life aged 50. The sum insured of $\$100,000$ is payable at the end of the year of death. Level premiums of $2,000$ are payable annually in advance throughout the term of the contract. The expenses are 12.5% of each premium. The interest rate is 6% and the mortality follows the Standard Mortality table.

(1) Calculate the net premium policy value five years after the issue of the contract, assuming that the policy is still in force.
(2) Calculate the gross premium policy value five years after the inception of the contract, assuming that the policy is still in force.

Solution. (1) First, we find P_{50},

$$1000P_{50} = \frac{1000A_{50}}{\ddot{a}_{50}} = \frac{249.05}{13.2668} = 18.7724$$

Then,

$$\begin{aligned} 1000 \times {}_5V_{50} &= 1000A_{55} - 1000P_{50}\ddot{a}_{55} \\ &= 305.14 - 18.7724 \times 12.2758 = 74.6938 \end{aligned}$$

and

$$100,000 \times {}_5V_{50} = 7,469.38$$

(2) Here we have

$$\begin{aligned} 100,000 \times {}_5V_{50}^g &= 100,000A_{55} - (1 - 12.5\%)G\ddot{a}_{55} \\ &= 30,514 - 0.875 \times 2,000 \times 12.2758 = 9,031.35 \end{aligned}$$

1.2. **Recursive formulas for policy values.**

Theorem 1.9. *We have that*

$$_tV_x + P_x = v\big(q_{x+t} + {}_{t+1}V_x\, p_{x+t}\big).$$

Proof. Note that $_tV_x + P_x = {}_{t+1}I_x$, the policy value at the beginning of year $t+1$. This value is equal to the discounted expected policy value at the end of this year. The value of the policy if the individual died in year $t+1$ is 1, which is the benefit value. The policy value if the individual is alive at the end of year $t+1$ is $_{t+1}V_x$. Hence the expected value is $1 \times q_{x+t} + {}_{t+1}V_x p_{x+t}$. □

Intuitively, the amount in the insurer's fund at the beginning of the $(t+1)$-st year is used to pay death benefits to the deceased and to fund insurance to the survivors at the end of the $(t+1)$-st year.

We can use this intuition to take into account the expenses.

Let e_t denote the premium related expense payable at time t, and let E_{t+1} denote the expense of the paying the benefit at time $t+1$. Recall that G_x denote the gross premium and $_tV_x^g$ denote the gross policy value.

Theorem 1.10. *We have that*

$$_tV_x^g + G_x - e_{x+t} = v\big(q_{x+t}(1 + E_{x+t+1}) + {}_{t+1}V_x\, p_{x+t}\big).$$

Example 1.11. Consider a 20-year endowment policy purchased by a life aged 50. Level premiums of $\$23,500$ per year are payable annually throughout the term of the policy. A sum insured of $\$700,000$ is payable at the end of the term if the life survives to age 70. On death before age 70 a sum insured is payable at the end of the year of death equal to the policy value at the start of the year in which the policyholder dies.

Assume that mortality follows the Standard Mortality Table, the interest rate 6% per year, and there are no expenses.

Calculate ${}_{15}V_{50}$, the policy value for a policy in force at the start of the 16-th year.

Solution. We will use Theorem 1.9 with a suitable modification. (The payment upon the death is not the face value but the policy value.) For the final year of this policy, the death benefit payable at the end of the year is ${}_{19}V_{50}$ and the survival benefit is the sum insured, $\$700,000$. So we have

$$ {}_{19}V_{50} + 23,500 = 1.06^{-1}({}_{19}V_{50} \times q_{69} + 700,000 p_{69}). $$

From the table $1000q_{69} = 30.37$ and we calculate:

$$ {}_{19}V_{50} = \frac{700000 \times 1.06^{-1} \times (1 - 0.03037) - 23,500}{1 - 1.06^{-1} \times 0.03037} = 635,015.49 $$

Next, we proceed by backward induction:

$$ {}_{18}V_{50} = \frac{635,015.49 \times 1.06^{-1} \times (1 - 0.02779) - 23,500}{1 - 1.06^{-1} \times 0.02779} = 573,970.81 $$
$$ {}_{17}V_{50} = \frac{573,970.81 \times 1.06^{-1} \times (1 - 0.02544) - 23,500}{1 - 1.06^{-1} \times 0.02544} = 516,605.11 $$
$$ {}_{16}V_{50} = \frac{516,605.11 \times 1.06^{-1} \times (1 - 0.02329) - 23,500}{1 - 1.06^{-1} \times 0.02329} = 462,678.45 $$
$$ {}_{15}V_{50} = \frac{462,678.45.11 \times 1.06^{-1} \times (1 - 0.02132) - 23,500}{1 - 1.06^{-1} \times 0.02132} = 411,969.18 $$

Hence, the answer is $\$411,969.18$.

We can write the formula in Theorem 1.9 as follows:

$$ ({}_tV_x + P_x)(1 + i) = q_{x+t} + {}_{t+1}V_x \times p_{x+t} $$

It has a clear interpretation: the funds available at the beginning of year $t + 1$ (after the premium payment is made) must be able to cover the expected need at the end of the year $t + 1$, which could be either death payment of 1 if the insuree has died or the necessary reserves if the insuree survived.

If the insuree has died yearly then the insurance company needs more money to pay off the death benefit than the reserve for this policy. On average, if the company has many insurees this need cancels out. However, it is an important quantity for the risk analysis if the number of insurees is not large or if the average mortality suddenly changes.

Definition 1.12. The *net amount at risk* is the difference between the face value of a life insurance policy and its policy value.

The net amount at risk is also called the *pure amount of protection*, the *Death Strain At Risk* (DSAR), or the Sum at Risk. This quantity is useful in determining risk management strategy, including reinsurance ? which is the insurance that an insurer buys to protect itself against adverse experience.

For a fully discrete unit whole life insurance, the net amount at risk during the t-th year is $1 - {}_tV_x$.

1.3. Retrospective formula.

Theorem 1.13 (Retrospective formula for the policy value)**.** *Then,*

$$ {}_tV_x = \frac{P_x \ddot{a}_{x:\overline{t}|} - A^1_{x:\overline{t}|}}{{}_tE_x} $$

Proof. We start with two identities,

$$ \begin{aligned} A_x &= A^1_{x:\overline{t}|} + {}_tE_x A_{x+t}, \\ \ddot{a}_x &= \ddot{a}_{x:\overline{t}|} + {}_tE_x \ddot{a}_{x+t}. \end{aligned} $$

It follows that

$$ A_x - P_x \ddot{a}_x = A^1_{x:\overline{t}|} - P_x \ddot{a}_{x:\overline{t}|} + {}_tE_x \big(A_{x+t} - P_x \ddot{a}_{x+t} \big). $$

The left-hand side is zero because the premium P_x is calculated by using the equivalence premium. Hence,

$$ P_x \ddot{a}_{x:\overline{t}|} - A^1_{x:\overline{t}|} = {}_tE_x \times {}_tV_x. $$

□

In the retrospective formula, $P_x \ddot{a}_{x:\overline{t}|}/{}_tE_x$ is called the *accumulated value of the premiums received*, and $A^1_{x:\overline{t}|}/{}_tE_x$ is called the *accumulated cost of insurance over the age interval* $(x, x+t]$. It is the value of the coverage by insurance over this time period.

At time $x+t$, the value obtained by using the retrospective formula coincides with the value given by the prospective formula if the assumptions about interest rates and mortality are not updated at time $x+t$.

From the point of view of the insurance company, the prospective formula provides a more accurate estimate of the needed reserves since it allows for the update of the information at time $x+t$.

Now, let us write $P^1_{x:\overline{t}|}$ and $P_{x:\overset{1}{\overline{t}|}}$ to denote the annual benefit premiums for the term and pure endowment life insurances, respectively. That is,

$$P^1_{x:\overline{t}|} = \frac{A^1_{x:\overline{t}|}}{\ddot{a}_{x:\overline{t}|}} \text{ and } P_{x:\overset{1}{\overline{t}|}} = \frac{A_{x:\overset{1}{\overline{t}|}}}{\ddot{a}_{x:\overline{t}|}} = \frac{{}_xE_t}{\ddot{a}_{x:\overline{t}|}}.$$

Theorem 1.14.

$$P_x = P^1_{x:\overline{t}|} + P_{x:\overset{1}{\overline{t}|}} \times {}_tV_x.$$

Proof. The formula in Theorem 1.13 can be written as follows:

$${}_tV_x \frac{{}_tE_x}{\ddot{a}_{x:\overline{t}|}} = P_x - \frac{A^1_{x:\overline{t}|}}{\ddot{a}_{x:\overline{t}|}},$$

which implies

$${}_tV_x \, P_{x:\overset{1}{\overline{t}|}} = P_x - P^1_{x:\overline{t}|}.$$

□

1.4. Sources of profit.

Example 1.15. A woman aged 60 purchases a 20-year endowment insurance with a sum insured of $\$100,000$ payable at the end of the year of death or on survival to age 80, whichever occurs first.

An annual premium of $\$5200$ is payable for at most 10 years. The insurer uses the following basis for the calculation of policy values:

- Survival model: Standard MLC Table.
- Interest: 6% per year

- Expenses: 10% of the first premium, 5% of subsequent premiums, and $\$200$ on payment of the sum insured.

Calculate ${}_0V_{60}$, ${}_5V_{60}$, and ${}_6V_{60}$, that is, the gross premium policy values for this policy at times $t = 0$, 5, and 6.

Solution. First, we need to calculate $A_{60:\overline{20}|}$, $A_{65:\overline{15}|}$, $A_{66:\overline{14}|}$ and $\ddot{a}_{60:\overline{10}|}$, $\ddot{a}_{65:\overline{5}|}$, and $\ddot{a}_{66:\overline{4}|}$. By using the table we get

$$\begin{aligned} A_{60:\overline{20}|} &= A^{1}_{60:\overline{20}|} + {}_{20}E_{60} = A_{60} - {}_{20}E_{60}A_{80} + {}_{20}E_{60} \\ &= 0.36913 + 0.14906 \times (1 - 0.66575) = 0.4190, \end{aligned}$$

$$\begin{aligned} A_{65:\overline{15}|} &= A^{1}_{65:\overline{15}|} + {}_{15}E_{65} = A_{65} - {}_{15}E_{65}A_{80} + {}_{15}E_{65} \\ &= 0.43980 + 0.39994 \times 0.54207 \times (1 - 0.66575) = 0.5123, \end{aligned}$$

$$A_{66:\overline{14}|} = A^{1}_{66:\overline{14}|} + {}_{14}E_{66} = A_{66} - {}_{14}E_{66}A_{80} + {}_{14}E_{66}.$$

We calculate

$$\begin{aligned} {}_{14}E_{66} &= {}_{15}E_{65}(1+i)/p_{65} \\ &= 0.39994 \times 0.54207 \times 1.06/(1 - 0.02132) = 0.2348 \end{aligned}$$

Hence,

$$A_{66:\overline{14}|} = 0.4546 + 0.2348 \times (1 - 0.66575) = 0.5331.$$

Similarly,

$$\ddot{a}_{60:\overline{10}|} = \ddot{a}_{60} - {}_{10}E_{60}\ddot{a}_{70} = 11.1454 - 0.4512 \times 8.5963 = 7.2667$$

$$\ddot{a}_{65:\overline{5}|} = \ddot{a}_{65} - {}_{5}E_{65}\ddot{a}_{70} = 9.8969 - 0.65623 \times 8.5963 = 4.2735$$

$$\ddot{a}_{66:\overline{4}|} = \ddot{a}_{66} - {}_{4}E_{66}\ddot{a}_{70} = 9.6354 - 0.7108 \times 8.5963 = 3.5455,$$

where we calculated

$$ {}_4E_{66} = {}_5E_{65}(1+i)/p_{65} = 0.65623 \times 1.06/(1-0.02132) = 0.7108 $$

To summarize we have:

$A_{60:\overline{20}\rvert}$	0.4190	$\ddot{a}_{60:\overline{10}\rvert}$	7.2667
$A_{65:\overline{15}\rvert}$	0.5123	$\ddot{a}_{65:\overline{5}\rvert}$	4.2735
$A_{66:\overline{14}\rvert}$	0.5331	$\ddot{a}_{66:\overline{4}\rvert}$	3.5455

Then we can compute the policy values. Let P denote the premium, $P = 5,200$.

$$
\begin{aligned}
{}_0V_{60} &= (100,000 + 200)A_{60:\overline{20}\rvert} + 5\%P + 5\%P\ddot{a}_{60:\overline{10}\rvert} - P\ddot{a}_{60:\overline{10}\rvert} \\
&= 100,200A_{60:\overline{20}\rvert} - (0.95\ddot{a}_{60:\overline{10}\rvert} - 0.05)P = 6,346.3
\end{aligned}
$$

Similarly, we calculate

$$
\begin{aligned}
{}_5V_{60} &= 100,200A_{65:\overline{15}\rvert} - 0.95\ddot{a}_{65:\overline{5}\rvert}P = 30,218. \\
{}_6V_{60} &= 35,900.
\end{aligned}
$$

In this example, the initial policy value, ${}_0V_x$, is greater than zero. This means that from the outset the insurer expects to make a loss on this policy. This sounds uncomfortable but is not uncommon in practice. The explanation is that the policy value basis may be more conservative than the premium basis. For example, the insurer may assume an interest rate of 7% in the premium calculation, but, for policy value calculations, assumes investments will earn only 6%.

Example 1.16. An insurer issued a large number of policies identical to the policy in Example 1.15 to women aged 60. Five years after they were issued, a total of 100 of these policies were still in force. In the following year,

- expenses of 6% of each premium paid were incurred,
- interest was earned at 7% on all assets,
- one policyholder died, and
- expenses of 250 were incurred on the payment of the sum insured for the policyholder who died.

(1) Calculate the profit or loss on this group of policies for this year.

(2) Determine how much of this profit/loss is attributable to profit/loss from mortality, from interest and from expenses.

Solution. (1) We assume that the reserves in the beginning of this year were $100 {}_5V_{60}$. (If they were less or greater, than it is because of losses and profits in the previous year).

Hence, after the premiums are received and the premium-related expenses are paid, the asset of the company are

$$100({}_5V_{60} + 0.94P) = 3,510,590.83$$

At the end of the year, this sum will grow to

$$3,510,590.83 \times 1.07 = 3,756,332.19$$

The death claim plus related expenses at the end of the year was $100,250$. A policy value equal to ${}_6V_{60}$ is required at the end of the year for each of the 99 policies still in force. Hence, the total amount the insurer requires at the end of the year is

$$100,250 + 99 \times {}_6V_{60} = 3,654,380.25.$$

Hence the insurer has made a profit in the sixth year of

$$100({}_5V_{60} + 0.94P) \times 1.07 - (100,250 + 99 \times {}_6V_{60}) = 102,335.55.$$

(2) We can attribute the total profit to three sources as follows.

(a) Interest: If expenses at the start the start of the year had been as assumed in the policy value basis, $0.05P$ per policy still in force, the additional interest earnedwould have been

$$(0.07 - 0.06) \times 100({}_5V_{60} + 0.95P) = 35,157.91$$

(b) Expenses: Now, we allow for the actual interest rate earned during the year (because the difference between actual and expected interest has already been accounted for in the interest profit above) but use the expected mortality. That is, we look at the loss arising from the expense experience given that the interest rate earned is 7%, but on the hypothesis that the number of deaths is $100q_{65}$.

The expected expenses on this basis, valued at the year end, are

$$100 \times (0.05P \times 1.07 + q_{65} \times 200) = 28,246.4$$

The actual expenses, if deaths were as expected, are

$$100 \times (0.06P \times 1.07 + q_{65} \times 250) = 33,917.$$

The loss from expenses, allowing for the actual interest rate earned in the year but allowing for the expected, rather than actual, mortality, was

$$33,917 - 28,246 = 5,670.6$$

(c) Mortality: Now, we use actual interest (7%) and actual expenses, and look at the difference between the expected cost from mortality and the actual cost. For each death, the cost to the insurer is the death strain at risk, in this case

$$100,000 + 250 - {}_6V_{60},$$

so the mortality profit is

$$(100 \times q_{65} - 1) \times (100,250 - {}_6V_{60}) = 72,848.24.$$

This gives a total profit of

$$102,335.55$$

which is the amount calculated earlier.

1.5. **Deferred Acquisition Cost and Full Preliminary Term approach.** The principles of reserve calculation and how to determine the appropriate basis are established by insurance regulators. While most countries use a gross policy value approach, the regulators in the USA use the net policy value method.

If the expenses are distributed uniformly over the term of the insurance contract, then the gross and net policy values coincide. In this case the expenses in the gross policy value calculation are fully offset by the increase in the premium.

However, typically the initial expenses (acquisition expenses) are larger than the renewal expenses. In this case the net policy value leads to a significantly higher policy values (and therefore, reserve requirements). To mitigate this problem, the insurer can start with the net policy value calculation and adjust it by deferring the initial expenses. This will make the the policy value closer to the gross policy value.

Definition 1.17. The *expense loading* (or expense premium) is difference between the gross premium and the net premium,

$$P^e = P^g - P^n.$$

Example 1.18. An insurer issues a whole life insurance policy to a life aged 50. The sum insured of $100,000$ is payable at the end of the year of death. Level premiums are payable in advance throughout the term of the contract. All premiums and policy values are calculated using the data in the MLC Illustrative Life Table and the interest rate of 6% per year. Initial expenses are 50% of the gross premium plus $\$250$. Renewal expenses are 3% of the gross premium.

Calculate the expense loading P^e, and the net and gross policy values 10 years after the issue ${}_{10}V_{50}^n$ and ${}_{10}V_{50}^g$.

Solution. For P^n, we have

$$P^n = 100,000\frac{A_{50}}{\ddot{a}_{50}} = 100,000\frac{0.24905}{13.2668} = 1877.2$$

For P^g,

$$\begin{aligned} P^g \ddot{a}_{50} &= 100,000A_{50} + 250 + (0.5 - 0.03)P^g + 0.03P^g \ddot{a}_{50} \\ P^g &= \frac{100,000A_{50} + 250}{0.97\ddot{a}_{50} - 0.47} = \frac{100,000 \times 0.24905 + 250}{0.97 \times 13.2668 - 0.47} \\ P^g &= 2028.8 \end{aligned}$$

Hence, the expense loading is $P^e = 2028.8 - 1877.2 = 151.6$.

Note that this amount is significantly larger than the renewal expenses $0.03 \times 2028.8 = 60.86$. The other part of the loading is needed to compensate the initial expenses.

Next, we compute net and gross policy values.

$${}_{10}V_{50}^n = 100,000A_{60} - P^n \ddot{a}_{60} = 36,913 - 1877.2 \times 11.1454 = 15,991$$

$$\begin{aligned} {}_{10}V_{50}^g &= 100,000A_{60} - P^g \ddot{a}_{60} + 0.03P^g \ddot{a}_{60} \\ &= 36,913 - 0.97 \times 2028.8 \times 11.1454 = 14,980 \end{aligned}$$

The gross policy value significantly smaller than the net policy value. The intuition behind this is that the insurance charges higher premium to compensate for the initial expenses. However, this is not reflected in the net premium value.

Definition 1.19. The difference between the net and gross policy values is called the *expense reserve* or the *deferred acquisition cost* (DAC),

$${}_tV_x^e = {}_tV_x^n - {}_tV_x^g.$$

In order to adjust the net policy value, a useful approach is the *Full Preliminary Term* (FPT) method. The idea of the method is to split the policy into two (or more) parts which have the level expenses. For each of these policies the net policy value should equal the gross policy value. Hence we can expect that this is also true for the combination of the policies.

Technically, the method proceeds by assuming hypothetically that the premiums are not level.

The hypothetical premium in the first year, ${}_1P_x$, is set in such a way that it is sufficient to fund the expected benefit payment in the first year.

The hypothetical premiums in later years are level and set equal to P^n_{x+1}, the net premium for the insurance that was issued at the age $x+1$.

The policy value is evaluated using the net policy value method and this premium structure.

We illustrate this method by using the contract in Example 1.18. We calculate

$$
\begin{aligned}
{}_1P_{50} &= 100,000A^{1}_{50:\overline{1}|} = 100,000vq_{50} = \frac{592}{1.06} = 558.49 \\
P^n_{51} &= 100,000\frac{A_{51}}{\ddot{a}_{51}} = 100,000\frac{0.25961}{13.0803} = 1984.7
\end{aligned}
$$

Then, the policy value 10 years later is

$$
\begin{aligned}
{}_{10}V^{FPT}_{50} &= 100,000A_{60} - P_{51}\ddot{a}_{60} \\
&= 36,913 - 1984.7 \times 11.1454 = 14,793
\end{aligned}
$$

which is quite close to the gross policy value of $14,980$.

1.6. **Variance of the PV of the future loss.** The policy value at time t is the expectation of the future loss evaluated at time t,

$$ {}_tV_x = \mathbb{E}(Z_{x+t} - P\ddot{Y}_{x+t}) = A_{x+t} - P\ddot{a}_{x+t}. $$

Often, the insurer also wants to know the variance of the loss. For example, at the time of issue the insurer might want to set the premium so that it will ensure positive profit with some high probability.

To calculate variance of the loss at time t, we have the following result.

Theorem 1.20. *For a whole life insurance of* 1 *on* (x)*, the variance of the* t*-th terminal loss random variable is*

$$ \mathbb{V}\mathrm{ar}({}_tL_x) = \left(1 + \frac{P}{d}\right)^2 \left({}^2A_{x+t} - A_{x+t}^2\right) $$

Proof. Since

$$ \ddot{Y}_{x+t} = 1 + v + \ldots + v^{K_{x+t}-1} = \frac{1 - v^{K_{x+t}}}{1 - v} = \frac{1 - Z_{x+t}}{d}, $$

we have

$$ {}_tL_x = \left(1 + \frac{P}{d}\right) Z_{x+t} - \frac{P}{d}. $$

Consequently,

$$ \mathbb{V}\mathrm{ar}({}_tL_x) = \left(1 + \frac{P}{d}\right)^2 \mathbb{V}\mathrm{ar}(Z_{x+t}) = \left(1 + \frac{P}{d}\right)^2 \left({}^2A_{x+t} - A_{x+t}^2\right) $$

□

Example 1.21. Consider a whole life insurance policy on (35) with face value of $50,000$ payable at the end of the year of death. This policy will be funded by a level benefit annual premium at the beginning of each year while (35) is alive. Assume that $i = 6\%$ and the mortality follows the MLC illustrative table. There are no expenses and the premium is determined using the equivalence principle.

Find the variance and the standard deviation of the 10-th terminal loss random variable.

Solution. From the table, we have

$$ \ddot{a}_{35} = 15.3926, \; A_{35} = 0.12872, \; A_{45} = 0.20120, \; \text{and } {}^2A_{45} = 0.06802. $$

Hence,

$$P = 50,000\frac{A_{35}}{\ddot{a}_{35}} = \frac{0.12872}{15.3926} = 0.0083625,$$

and

$$\mathbb{V}\mathrm{ar}({}_{10}L_{35}) = \Big(1 + 0.0083625\frac{1.06}{0.6}\Big)^2\big(0.06802 - 0.20120^2\big) = 0.036276,$$

In order to get the variance of the insurance with face value $50,000$, we multiply $\mathbb{V}\mathrm{ar}({}_{10}L_{35})$ by $50,000^2$, and get $90,691,457$. The standard deviation is the square root of this quantity, and equals $9,523.20$.

For a fully continuous whole life insurance the calculations are similar and give the formula:

$$\mathbb{V}\mathrm{ar}({}_t\overline{L}_x) = \Big(1 + \frac{P}{\delta}\Big)^2\big({}^2\overline{A}_{x+t} - (\overline{A}_{x+t})^2\big)$$

1.7. **Thiele's differential equation.** Recall the recursive formula for the fully discrete life insurance:

$${}_tV_x + P = v\big(q_{x+t} + {}_{t+1}V_x \times p_{x+t}\big).$$

Consider now the discrete insurance that pays m times a year. Let $h = \frac{1}{m}$. For this insurance we have the following recursive relation:

$${}_{kh}V_x^{(m)} + Ph = v^{(m)}\big[q_{x+th}^{(m)} + {}_{(k+1)h}V_x^{(m)} \times (1 - q_{x+kh}^{(m)})\big],$$

where

$$v^{(m)} = (1+i)^{-1/m}.$$

and $q_{x+kh}^{(m)}$ is the probability of death in the time interval $[x+kh, x+(k+1)h]$. We can also rearrange this recursion and divide it by h, so that we get

$$\frac{{}_{kh}V_x^{(m)} - {}_{(k+1)h}V_x^{(m)}}{h}$$
$$= \frac{-Ph - (1 - v^{(m)})_{(k+1)h}V_x^{(m)} + v^{(m)}q_{x+kh}^{(m)}\big[1 - {}_{(k+1)h}V_x^{(m)}\big]}{h}.$$

Let $m \to \infty$, $h \to 0$ and $kh \to t$. Then the limit of the previous relation is

$$-\frac{d}{dt}\big({}_t\overline{V}_x\big) = -P - \delta_t\overline{V}_x + \mu_{t+x}\big[1 - {}_tV_x\big],$$

or

$$\frac{d}{dt}\left({}_t\overline{V}_x\right) = +P + \delta_t\overline{V}_x - \mu_{t+x}\left[1 - {}_tV_x\right].$$

This is called Thiele's differential equation. It has a clear intuitive meaning. The policy value grows by a constant inflow of payments P and the interest $\delta_t\overline{V}_x$. It declines by the outflow of benefit payments, μ_{t+x}, mitigated by the corresponding reduction in the required reserves, $\mu_{t+xt}V_x$.

1.8. **Surrender value and paid-up insurance.** A policy which is cancelled at the request of the policyholder before the end of its originally agreed term, is said to *lapse* or to be *surrendered*, and any lump sum payable by the insurance company for such a policy is called a *surrender value* or a *cash value*.

Typically, the surrender value is either pre-specified or determined on the basis of the policy value.

Since the surrender of the insurance policy leads to additional expenses for the insurance company, the surrender value is typically smaller than the policy value. The difference is called the *surrender charge*.

The policyholder can also wish to pay no more premiums without cancelling the policy, so that, in the case of an endowment insurance for example, a (reduced) sum insured is still payable on death or on survival to the end of the original term. Any policy for which no further premiums are payable is said to be *paid-up*, and the reduced sum insured is called a *paid-up sum insured*.

Example 1.22. For a fully discrete 30-year endowment insurance of 2000 on (35), you are given:

(1) At the end of 15 years, the policy is converted to a reduced paid-up endowment insurance of 800.
(2) At the time of conversion, the actuarial present value of the reduced paid-up endowment insurance equals the 15-th year net premium reserve on the original policy less the amount of the surrender charge (SC).
(3) $i = 5\%$
(4) $A_{35:\overline{30}|} = 0.255$
(5) $A_{50:\overline{15}|} = 0.506$

Calculate SC.

Solution. First we calculate the reserve,

$$ {}_{15}V = A_{50:\overline{15}|} - \frac{A_{35:\overline{30}|}}{\ddot{a}_{35:\overline{30}|}} \ddot{a}_{50:\overline{15}|}, $$

where

$$ \ddot{a}_{35:\overline{30}|} = \frac{1 - A_{35:\overline{30}|}}{d} = \frac{1.05}{0.5}(1 - 0.255) = 15.645, $$

and

$$ \ddot{a}_{50:\overline{15}|} = \frac{1 - A_{50:\overline{15}|}}{d} = \frac{1.05}{0.5}(1 - 0.506) = 10.374. $$

Therefore,

$$ {}_{15}V = 0.506 - \frac{0.255}{15.645} 10.374 = 0.33691. $$

Hence, we have equation,

$$ \frac{800}{2000} A_{50:\overline{15}|} = {}_{15}V - SC, $$

or

$$ SC = 0.33691 - 0.4 \times 0.506 = 0.13451. $$

For insurance of 2000, $SC = 269.03$.

2. Multiple State Models

2.1. Definition and Notation. Assume that we have a finite set of $n+1$ states labelled $0, 1, \ldots, n$. These states represent different conditions for an individual or groups of individuals. For each $t \geq 0$, the random variable $Y(t)$ takes one of the values $0, 1, \ldots, n$, and we interpret the event $Y(t) = s$ to mean that the individual is in state s at age $x + t$.

Definition 2.1. *A continuous-time stochastic process* is a collection of random variables indexed by a continuous time variable.

The set of random variables $\{Y(t)_{t\geq 0}\}$ is a continuous time stochastic process that takes values in the set $S = \{1, 2, \ldots, n\}$. This set is called the *state space*.

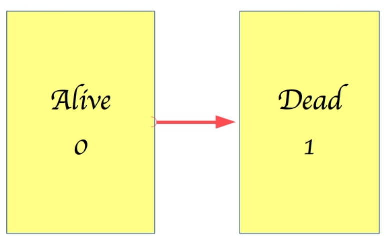

Figure 1. Standard Survival Model

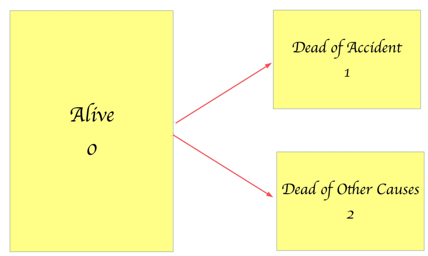

Figure 2. Accidental Death Model

Example 2.2 (Example in Figure 1)**.** The individual is, at any time, in one of two states, 'Alive' or 'Dead'. For convenience we label these states '0' and '1', respectively. Transition from state 0 to state 1 is allowed, but transitions in the opposite direction cannot occur. This is an example of a multiple state model with two states.

Example 2.3 (Example in Figure 2)**.** This model is an extension of the standard model. In both cases an individual starts by being alive, that is, starts in state 0, and, at some future time, dies. The difference is that we now need to distinguish between deaths due to accident and deaths due to other causes since the sum insured is different in the two cases.

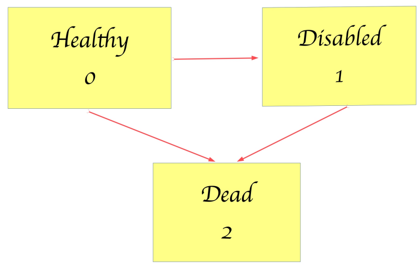

FIGURE 3. Permanent Disability Model

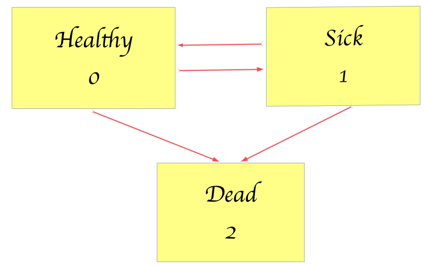

FIGURE 4. Temporary Disability Model

Example 2.4 (Example in Figure 3)**.** This Figure describes the permanent disability model, which is the basis for a policy with the following benefits:

(1) an annuity while permanently disabled,
(2) a lump sum on becoming permanently disabled, and,
(3) a lump sum on death,

with premiums payable while healthy. An important feature of this model is that disability is permanent – there is no arrow from state 1 back to state 0.

Example 2.5 (Example in Figure 4)**.** Disability income insurance pays a benefit during periods of sickness; the benefit ceases on recovery. In this model it is possible to transfer from state 1 to state 0, that is, to recover from an illness.

It is also possible to enter states 0 and 1 many times. In terms of our interpretation of the model, this means that several periods of sickness could occur before death, with healthy (premium paying) periods in between.

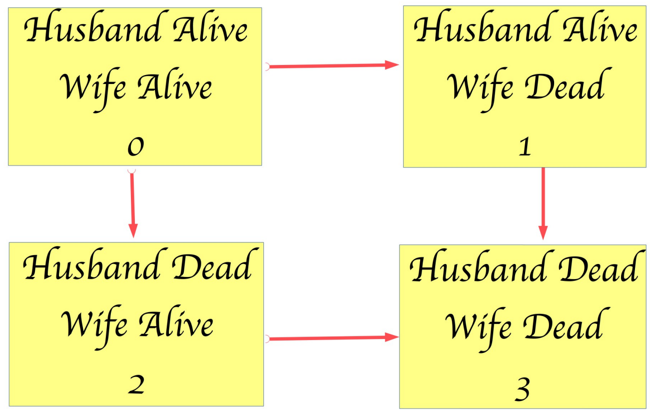

FIGURE 5. Two Lives Model

Example 2.6 (Example in Figure 5). The model in this example is useful for valuation of such policies as a *joint life annuity* which pays until the first death in a couple, or *last survivor annuity* which pays until the last death in a couple.

Another possibility is a *reversionary annuity* which is a life annuity that starts payment on the death of a specified life, if his or her spouse is alive, and continues through the spouse's lifetime.

Assumption 2.7 (Markov Property). For any states i and j and any times t and $t+s$, where $t, s \geq 0$,

$$\mathbb{P}\Big[Y(t+s) = j | Y(\tau), 0 \leq \tau \leq t\Big] = \mathbb{P}\Big[Y(t+s) = j | Y(t)\Big].$$

In other words, the future values of the process depend only on the present and not on past values.

Definition 2.8. A continuous-time stochastic process $Y(t), t \geq 0$, that satisfies Assumpton 2.7 is called *Markov process*.

The theory of continuous-time Markov processes is essentially due to Kolmogorov.

A change of the state of the process $Y(t)$ is called a *transition*. We are not very interested in the processes that can make an infinite number of transition in finite amount of time. To rule out this possibility, the following assumption is useful.

Assumption 2.9. For any time t,

$$\mathbb{P}\{\text{There are > 0 transition in } (t-h, t+h)\} \leq Ch,$$

for some constant $C > 0$.

$$\lim_{h\to 0} \frac{1}{h}\mathbb{P}\{\text{There are >1 transition in } (t-h, t+h)\} = 0.$$

We will use the following notation for transitional probabilities:

$$_tp_x^{ij} = \mathbb{P}\{Y(x+t) = j | Y(x) = i\}.$$

That is, this is the probability to transit from state i at time x to state j at time $x + t$.

Assumption 2.10. The transitional probabilities are differentiable in t.

Definition 2.11. The *force of transition* $i \to j$ at time x or the *transition intensity* is

$$\mu_x^{ij} = \frac{d}{dt} {}_tp_x^{ij}\Big|_{t=0} = \lim_{t\to 0^+} \frac{{}_tp_x^{ij}}{t}.$$

Example 2.12. For the standard survival model, we have

$$_tp_x^{01} = \mathbb{P}\{Y(x+t) = 1 | Y(x) = 0\} = {}_tq_x,$$

and, correspondingly ${}_tp_x^{00} = {}_tp_x$. Two other transition probabilities are ${}_tp_x^{10} = 0$ and ${}_tp_x^{11} = 1$.

In addition,

$$\mu_x^{01} = \frac{d}{dt}({}_tp_x^{01})\Big|_{t=0} = \frac{d}{dt}\frac{S(x) - S(x+t)}{S(x)}\Big|_{t=0} = -\frac{S'(x)}{S(x)} = \mu_x,$$

and

$$\mu_x^{00} = \frac{d}{dt}(1 - {}_tp_x^{01})\Big|_{t=0} = -\mu_x.$$

□

In general, we can write the following approximation

$$_tp_x^{ij} = \begin{cases} (\mu_x^{ij})t + o(t) & \text{if } j \neq i, \\ 1 + (\mu_x^{ii})t + o(t) & \text{if } j = i. \end{cases}$$

Another useful quantity is the probability to stay in the state i for the entire duration of the interval $[x, x+t]$.

$$ {}_tp_x^{\overline{ii}} = \mathbb{P}\{Y(\tau) = i \text{ for all } \tau \in [x, x+t] | Y(x) = i\}. $$

Note, that in general ${}_tp_x^{\overline{ii}} \neq {}_tp_x^{ii}$. In the second case the process can exit i after time x and return back to i before time $x+t$. So ${}_tp_x^{\overline{ii}} \leq {}_tp_x^{ii}$ with a possible strict inequality.

Theorem 2.13. *The probability ${}_tp_x^{\overline{ii}}$ satisfies the differential equation*

$$ \frac{d}{dt}({}_tp_x^{\overline{ii}}) = (\mu_{x+t}^{ii}){}_tp_x^{\overline{ii}}. $$

Proof. When we talk about the Markov processes it useful to think about the process as a random movement of a particle which can jump from state to state.

Consider the interval $[x, x+t+h]$. The probability that the particle stays in the state i during this interval is the product of the probabilities that it stays there during the interval $[x, x+t]$ and then stays at i during the interval $[x+t, x+t+h]$. So,

$$ {}_{t+h}p_x^{\overline{ii}} = {}_tp_x^{\overline{ii}}({}_hp_{x+t}^{\overline{ii}}), $$

and

$$ {}_{t+h}p_x^{\overline{ii}} - {}_tp_x^{\overline{ii}} = {}_tp_x^{\overline{ii}}({}_hp_{x+t}^{\overline{ii}} - 1), $$

Note that

$$ {}_hp_{x+t}^{\overline{ii}} = 1 - \left(\sum_{j \neq i} \mu_{x+t}^{ij}\right) h + o(h), $$

because by one of our assumptions the probability that there are more than one transitions in the interval $[x+t, x+t+h]$ is $o(h)$ and the probability that there is a transition to $j \neq i$ is $\mu_{x+t}^{ij}h + o(h)$.

So we get

$$ \frac{{}_{t+h}p_x^{\overline{ii}} - {}_tp_x^{\overline{ii}}}{h} = -\left(\sum_{j \neq i} \mu_{x+t}^{ij}\right) {}_tp_x^{\overline{ii}} + \frac{o(h)}{h}, $$

and by taking the limit $h \to 0$ we get

$$\frac{d}{dt}({}_tp_x^{\overline{ii}}) = -\left(\sum_{j \neq i} \mu_{x+t}^{ij}\right) {}_tp_x^{\overline{ii}}.$$

It remains to notice that

$$\sum_{j \neq i} \mu_{x+t}^{ij} = -\mu_{x+t}^{ii}.$$

□

Corollary 2.14.

$${}_tp_x^{\overline{ii}} = \exp\left\{ \int_0^t \mu_{x+s}^{ii}\, ds \right\} = \exp\left\{ -\int_0^t \Big(\sum_{j \neq i} \mu_{x+t}^{ij} \Big)\, ds \right\}.$$

Proof. This is the solution of the differential equation in Theorem 2.13.
□

We can also derive the differential equations for the transition probabilities ${}_tp_x^{ij}$.

Theorem 2.15 (Kolmogorov's forward equations (1938))**.** *The transition probabilities satisfy the following equation,*

$$\frac{d}{dt}({}_tp_x^{ij}) = \sum_k {}_tp_x^{ik} \mu_{x+t}^{kj}.$$

Proof. The idea of the proof is the same as in the previous theorem. Consider ${}_{t+h}p_x^{ij}$. We can write this probability as the sum of the probabilities of the events which we obtain by adding the condition that the particle was at a state k at time $x+t$. Then, we have the following equation:

$${}_{t+h}p_x^{ij} = \sum_k {}_tp_x^{ik} ({}_hp_{x+t}^{kj}),$$

and therefore,

$${}_{t+h}p_x^{ij} - {}_tp_x^{ij} = \sum_{k \neq j} {}_tp_x^{ik} ({}_hp_{x+t}^{kj}) + {}_tp_x^{ij} ({}_hp_{x+t}^{jj} - 1).$$

Note that the sum of the transition probabilities from j to all other possible states is 1, and therefore,

$${}_hp_{x+t}^{jj} = 1 - \sum_{k \neq j} {}_hp_{x+t}^{jk}$$

Hence, we get

$$\frac{{}_{t+h}p_x^{ij} - {}_tp_x^{ij}}{h} = \sum_{k\neq j} \left({}_tp_x^{ik}\mu_{x+t}^{kj} - {}_tp_x^{ij}\mu_{x+t}^{jk} \right) + \frac{o(h)}{h},$$

and by taking the limit,

$$\frac{d}{dt}({}_tp_x^{ij}) = \sum_{k\neq j} \left({}_tp_x^{ik}\mu_{x+t}^{kj} - {}_tp_x^{ij}\mu_{x+t}^{jk} \right).$$

We can remove the condition $k \neq j$ because the term with $k = j$ gives zero contribution. Finally we note that

$$\sum_k \mu_{x+t}^{jk} = \frac{d}{dt} \sum_k {}_tp_x^{jk} = \frac{d}{dt} 1 = 0.$$

□

We can write this result in a more appealing form if we use the matrix notation.

Let ${}_tP_x$ be a matrix with the entries $({}_tP_x)_{ij} = {}_tp_x^{ij}$ and let Γ_x be a matrix with the entries $(\Gamma_x)_{ij} = \mu_x^{ij}$. (The matrix Γ_x is called the *generator* of the Markov process.)

Then, we have the following differential equation,

$$\frac{d}{dt}{}_tP_x = ({}_tP_x)\Gamma_{x+t}.$$

Similarly, we can derive Kolmogorov's backward equations.

Theorem 2.16 (Kolmogorov's backward equations (1938))**.** *The transition probabilities satisfy the following equation,*

$$\frac{d}{dx}{}_tP_x = -\Gamma_x({}_tP_x).$$

2.2. Evaluation of probabilities - Examples.

Example 2.17.

Consider the permanent disability model illustrated in Figure 6.

(a) Suppose the transition intensities for this model are all constants, as follows: $\mu^{01} = 0.0279$, $\mu^{02} = 0.0229$, $\mu^{12} = \mu^{02}$.

Calculate ${}_{10}p^{00}$ and ${}_{10}p^{01}$.

Before solving this problem, we formulate and prove the general result.

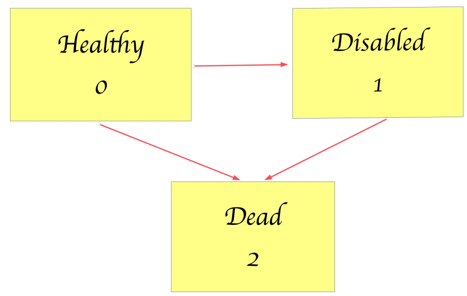

FIGURE 6. Permanent Disability Model

Theorem 2.18. *For the permanent disability model with constant transition probabilities we have*

$$
\begin{aligned}
{}_tp^{00} &= \exp\{(\mu^{00})t\} \\
{}_tp^{01} &= \frac{\mu^{01}}{\mu^{00} - \mu^{11}}\left(e^{(\mu^{00})t} - e^{(\mu^{11})t}\right) \\
{}_tp^{11} &= \exp\{(\mu^{11})t\}
\end{aligned}
$$

Proof. Since none of the states can be entered twice, so

$$ {}_tp_x^{ii} = {}_tp_x^{\overline{ii}}. $$

Hence we can use the formula for ${}_tp_x^{\overline{ii}}$ to calculate ${}_tp_x^{ii}$.

$$ {}_tp^{00} = \exp\left\{-\int_0^t (\mu^{01} + \mu^{02})\, ds\right\} = \exp\{-(\mu^{01} + \mu^{02})t\} = \exp\{(\mu^{00})t\}. $$

$$ {}_tp^{11} = \exp\left\{-\int_0^t \mu^{12}\, ds\right\} = \exp\{-(\mu^{12})t\} = \exp\{(\mu^{11})t\} $$

For $f(t) = {}_tp^{01}$, we have Kolmogorov's forward differential equation

$$ \frac{d}{dt}({}_tp^{01}) = {}_tp^{00}\mu^{01} + {}_tp^{01}\mu^{11}. $$

The homogeneous part of this equation is

$$\frac{d}{dt}\big(f(t)\big) = f(t)\mu^{11},$$

which we can solve as

$$f(t) = Ae^{(\mu^{11})t},$$

so we seek for the solution in the form

$${}_tp^{01} = A(t)e^{(\mu^{11})t}.$$

This leads to a differential equation on $A(t)$,

$$A'(t)e^{(\mu^{11})t} = {}_tp^{00}\mu^{01} = e^{(\mu^{00})t}\mu^{01},$$

or

$$A'(t) = e^{(\mu^{00}-\mu^{11})t}\mu^{01},$$

which we solve as

$$A(t) = e^{(\mu^{00}-\mu^{11})t}\frac{\mu^{01}}{\mu^{00}-\mu^{11}} + C.$$

From the initial condition ${}_0p^{01} = 0$, we find C, and conclude that

$$\begin{aligned} {}_tp^{01} &= \frac{\mu^{01}}{\mu^{00}-\mu^{11}}\big(e^{(\mu^{00}-\mu^{11})t} - 1\big)e^{(\mu^{11})t} \\ &= \frac{\mu^{01}}{\mu^{00}-\mu^{11}}\big(e^{(\mu^{00})t} - e^{(\mu^{11})t}\big) \end{aligned}$$

Since $\mu^{00} = -\mu^{01} - \mu^{02}$ and $\mu^{11} = -\mu^{12}$, we can also write this as

$${}_tp^{01} = \frac{\mu^{01}}{\mu^{12}-\mu^{01}-\mu^{02}}\big(e^{-(\mu^{01}+\mu^{02})t} - e^{-(\mu^{12})t}\big).$$

□

Solution (Solution of Example 2.17)**.** By the previous theorem,

$$\begin{aligned} {}_{10}p^{00} &= \exp\{-(\mu^{01}+\mu^{02})\times 10\} \\ &= \exp\{-0.0508\times 10\} = 0.60170 \\ {}_{10}p^{01} &= \frac{0.0279}{0.0229-0.0279-0.0229}\big(e^{-(0.0279+0.0229)\times 10} - e^{-0.0279\times 10}\big) \\ &= e^{-0.229}\Big(1 - e^{-0.279}\Big) = 0.1936. \end{aligned}$$

(b) Now suppose that in the previous example $x = 30$ and the transition probabilities are as follows.

(1) $\mu^{01}_{x+t} = 0.014 + 0.0007 \times 1.075^{(x+t)}$
(2) $\mu^{02}_{x+t} = 0.006$

Calculate ${}_{10}p^{00}_{30}$.

Solution. Since there is no re-entry to state 0 hence ${}_{10}p^{00}_{30} = {}_{10}p^{\overline{00}}_{30}$. Therefore,

$$\begin{aligned}
{}_{10}p^{00}_{30} &= \exp\left(- \int_0^{10} \left(\mu^{01}_{x+s} + \mu^{02}_{x+s}\right) ds \right) \\
&= \exp\left(- \int_0^{10} \left(0.014 + 0.0007 \times 1.075^{(30+s)} + 0.006\right) ds \right) \\
&= e^{-0.2} \exp\left(- 0.0007 \int_0^{10} 1.075^{(30+s)}\, ds \right).
\end{aligned}$$

We calculate

$$\int_0^{10} 1.075^{(30+s)}\, ds = \frac{1.075^{(30+s)}}{\ln 1.075}\Bigg|_0^{10} = \frac{1.075^{40} - 1.075^{30}}{\ln 1.075} = 128.4458,$$

and therefore,

$${}_{10}p^{00}_{30} = e^{-0.2} e^{-0.0007 \times 128.4458} = 0.748.$$

For ${}_{10}p^{01}_{30}$ we can write Kolmogorov's differential equation. However, it does not have a simple analytical solution and should be solved numerically.

Example 2.19. You are pricing a type of permanent disability insurance using the model in Figure 6.

The insurance will pay a benefit only if, by age 65, the insured had been disabled for a period of at least one year. You are given the following forces of transition:

(1) $\mu^{01} = 0.02$
(2) $\mu^{02} = 0.03$
(3) $\mu^{12} = 0.11$

Calculate the probability that a benefit will be paid for a Healthy life aged 50 who purchases this insurance.

Solution. The desired probability is

$$\begin{aligned}
&\int_0^{14} \exp\left\{-\int_0^u (\mu^{01}+\mu^{02})ds\right\}\cdot\mu^{01}\cdot\exp\left\{-\int_0^1 \mu^{12}ds\right\}du\\
&=\int_0^{14} e^{-0.05u}\times 0.02\times e^{-0.11}du\\
&=0.02\times e^{-0.11}\int_0^{14} e^{-0.05u}du\\
&=\frac{0.02}{0.05}\times e^{-0.11}\times\left(1-e^{-0.7}\right)\\
&=0.18
\end{aligned}$$

2.3. **Euler Method.** In many case, it is not possible to solve Kolmogorov's differential equations analytically. In this case, one is forced to use numerical methods for their solution. One of the simplest is the Euler method. Euler published it in his book *Institutionum calculi integralis* in 1768. It is certainly not the best numeric method for solving differential equations but it is very simple and easy to remember.

Example 2.20.

Consider the disability income insurance model illustrated in Figure 7.

Suppose the transition intensities for this model are as follows:

$\mu_x^{01} = a_1 + b_1 \exp(c_1 x)$,

$\mu_x^{10} = 0.1\mu_x^{01}$,

$\mu_x^{02} = a_2 + b_2 \exp(c_2 x)$,

$\mu_x^{12} = \mu_x^{02}$,

where $a_1 = 4 \times 10^{-4}$, $b_1 = 3.4674 \times 10^{-6}$, $c_1 = 0.138155$, $a_2 = 5 \times 10^{-4}$, $b_2 = 7.5858 \times 10^{-5}$, $c_2 = 0.087498$.

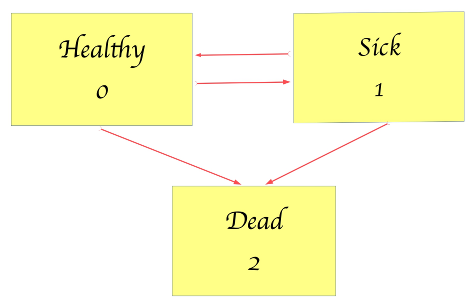

FIGURE 7. Disability Income Insurance Model

Calculate ${}_{10}p_{60}^{00}$ and ${}_{10}p_{60}^{01}$ using the Euler method with a step size of $h = 1/12$ to solve the Kolmogorov's forward equations.

Solution. Kolmogorov's forward equations are

$$\frac{d}{dt} {}_tP_x = ({}_tP_x)\Gamma_{x+t}.$$

The matrix of intensities is

$$\Gamma_x = \begin{bmatrix} \mu_x^{00} & \mu_x^{01} & \mu_x^{02} \\ \mu_x^{10} & \mu_x^{11} & \mu_x^{12} \\ 0 & 0 & 0 \end{bmatrix}$$

and therefore we get the following system of ODEs:

$$\frac{d}{dt}{}_tp_{60}^{00} = {}_tp_{60}^{00} \times \mu_{x+t}^{00} + {}_tp_{60}^{01} \times \mu_{x+t}^{10}$$
$$\frac{d}{dt}{}_tp_{60}^{01} = {}_tp_{60}^{00} \times \mu_{x+t}^{01} + {}_tp_{60}^{01} \times \mu_{x+t}^{11}$$

According to the Euler method we write a system of discrete-time equations which approximate the continuous time system,

$${}_{t+h}p_{60}^{00} = {}_tp_{60}^{00} + \left({}_tp_{60}^{00} \times \mu_{x+t}^{00} + {}_tp_{60}^{01} \times \mu_{x+t}^{10}\right)h$$
$${}_{t+h}p_{60}^{01} = {}_tp_{60}^{01} + \left({}_tp_{60}^{00} \times \mu_{x+t}^{01} + {}_tp_{60}^{01} \times \mu_{x+t}^{11}\right)h,$$

and solve it step by step.

2.4. **Calculation of Premiums.** Suppose we have a life aged x currently in state i. Let $\overline{a}_x^{ij}$ denote the expected present value of the annuity that pays continuously $\$1$ per year while the life is in some state j (which may be equal to i). Then

$$\begin{aligned}\overline{a}_x^{ij} &= \mathbb{E}\Big[\int_0^\infty e^{-\delta t}\mathbb{1}_{Y_t=j}\,dt\big|Y_0=i\Big]\\ &= \int_0^\infty e^{-\delta t}\mathbb{E}\Big[\mathbb{1}_{Y_t=j}\big|Y_0=i\Big]\,dt\\ &= \int_0^\infty e^{-\delta t}{}_tp_x^{ij}\,dt\end{aligned}$$

If annuity is discrete and payable at the beginning of each year, conditional that the life at that moment is in state j, then we can similarly derive the following formula.

$$\ddot{a}_x^{ij} = \sum_{k=0}^{\infty} v^k{}_kp_x^{ij}.$$

For insurance benefits, the payment is usually conditional on making a transition. So, suppose a unit benefit is payable immediately on each future transfer into state j, given that the life is currently in state i (which may be equal to j). Then the expected present value of the benefit is

$$\overline{A}_x^{ij} = \int_0^\infty e^{-\delta t}\Big(\sum_{k\neq j}{}_tp_x^{ik}\mu_{x+t}^{kj}\Big)\,dt$$

To derive this, consider payment in the interval $(t, t+dt)$.

(1) The amount of the payment is 1,
(2) The discount factor is $e^{-\delta t}$, and
(3) The probability that the benefit is paid is the probability that the life transfers into state j in $(t, t+dt)$, given that the life is in state i at time 0.

In order to transfer into state j in $(t, t+dt)$, the life must be in some state $k \neq j$ at time t, since the probability of two transitions in infinitesimal time dt is negligible.

This happens with probability ${}_tp_x^{ik}$. In addition, the life should then transfer from k to j during the interval $(t, t+dt)$, which happens with probability $\mu_{x+t}dt$.

Example 2.21. An insurer issues a 10-year disability income insurance policy to a healthy life aged 60. Calculate the premiums for the following policy design using the model and parameters from Example 2.20.

Assume an interest rate of 5% per year effective, and that there are no expenses.

Premiums are payable monthly in advance conditional on the life being in the healthy state at the premium date.

The sickness benefit of $\$20,000$ per year is payable monthly in arrear, if the life is in the sick state at the payment date. A death benefit of $\$50,000$ is payable at the end of the month of death.

Solution. The EPV of monthly annuity payable to the insurance company is

$$P\ddot{a}^{00}_{60:\overline{10}|} = P\frac{1}{12}\sum_{k=0}^{120}(1.05)^{-k/12} \times {}_{k/12}p_{60}^{00} = 6.628P.$$

The EPV of the sickness benefit is

$$20,000\ddot{a}^{01}_{60:\overline{10}|} = 20,000 \times \frac{1}{12}\sum_{k=1}^{120}(1.05)^{-k/12} \times {}_{k/12}p_{60}^{01} = 13,375.42.$$

And the EPV of the death benefit is

$$50,000A^{02}_{60:\overline{10}|} = 50,000\sum_{k=1}^{120}(1.05)^{-k/12}\Big({}_{(k-1)/12}p_{60}^{00} \times \mu^{02}_{60+(k-1)/12} \times \frac{1}{12}$$
$$+ {}_{(k-1)/12}p_{60}^{01} \times \mu^{12}_{60+(k-1)/12} \times \frac{1}{12}\Big) = 8,078.41$$

By equivalence principle, we have equation

$$6.628P = 13,375.42 + 8,078.41,$$

and therefore $P = 3236.80$ or 269.73 per month.

Example 2.22. A life insurance company needs to calculate premiums for a 3-year sickness policy issued to Healthy lives.

The company will pay a benefit of $20,000$ at the end of each year if the policyholder is Sick at that time.

The insurance company uses the following transition probabilities, applicable in each of the three years:

	H	S	D
H	0.950	0.025	0.025
S	0.300	0.600	0.100
D	0	0	1

Calculate the expected present value at issue of sickness benefit payments using an interest rate of 6%.

Solution. The probabilities are:

$$\begin{aligned}
&\text{Sick at } t=1: && {}_1p^{01} = 0.025 \\
&\text{Sick at } t=2: && {}_2p^{01} = 0.95 \times 0.025 + 0.025 \times 0.6 = 0.03875 \\
&\text{Sick at } t=3: && {}_3p^{01} = 0.95 \times 0.95 \times 0.025 + 0.95 \times 0.025 \times 0.6 \\
& && + 0.025 \times 0.6 \times 0.6 + 0.025 \times 0.3 \times 0.025 \\
& && = 0.046
\end{aligned}$$

Hence, the EPV is

$$20,000\Big(0.025 \times \frac{1}{1.06} + 0.03875 \times \frac{1}{1.06^2} + 0.046 \times \frac{1}{1.06^3}\Big) = 1934$$

2.5. **Policy values and Thiele's differential equation.**

Definition 2.23. The *policy value* ${}_tV^i$ at time t is the expectation of the PV of future loss for a policy which is in state i at time t.

The key to calculating policy values is Thiele's differential equation, which can be solved numerically using Euler's, or some more sophisticated, method.

Example 2.24 (Policy values in the disability income model)**.** Consider a policy with a term of n years issued to a life aged x. Premiums are payable continuously throughout the term at rate P per year while the life is healthy.

An annuity benefit is payable continuously at rate B per year while the life is sick, and a lump sum, S, is payable immediately on death within the term. Recovery from sick to healthy is possible.

We are interested in calculating policy values for this policy.

For simplicity we ignore expenses in this example, but these could be included as extra 'benefits' or negative 'premiums' provided only that they are payable continuously at a constant rate while the life is in a given state or are payable as lump sums immediately on transition between pairs of states.

Also for simplicity, we assume that the premium, the benefits and the force of interest, δ per year, are constants rather than functions of time.

Theorem 2.25 (Thiele's equation for Disability Income Model)**.** *The policy values in the Disability Income model satisfy the following equations for* $0 < t < n$,

$$\frac{d}{dt}{}_tV^0 = \delta \times {}_tV^0 + P - \mu^{01}_{x+t}\Big({}_tV^1 - {}_tV^0\Big) - \mu^{02}_{x+t}\Big(S - {}_tV^0\Big)$$
$$\frac{d}{dt}{}_tV^1 = \delta \times {}_tV^1 - B - \mu^{10}_{x+t}\Big({}_tV^0 - {}_tV^1\Big) - \mu^{12}_{x+t}\Big(S - {}_tV^1\Big).$$

Proof #1. By definition,

$${}_tV^0 = S\overline{A}^{02}_t + B\overline{a}^{01}_t - P\overline{a}^{00}_t.$$

We can write this the following way (suppressing age (x) everywhere)

$$\begin{aligned}
{}_tV^0 &= \int_0^\infty e^{-\delta\tau}\Big(- P_\tau p_t^{00} + B_\tau p_t^{01} + S \sum_{j=0,1} {}_\tau p_t^{0j} \mu_{t+\tau}^{j2}\Big)\, d\tau \\
&= -Pdt + S\mu_t^{02} dt \\
&+ (1-\delta dt) \times \Big((1 - \mu_t^{01} dt - \mu_t^{02} dt) \times {}_{t+dt}V^0 + \mu_t^{01} dt \times {}_{t+dt}V^1\Big) \\
&+ o(dt).
\end{aligned}$$

After subtracting ${}_tV^0$ on both sides, we get

$$\begin{aligned}
0 &= -Pdt + S\mu_t^{02} dt \\
&\quad + ({}_{t+dt}V^0 - {}_tV^0) - \delta dt \times {}_{t+dt}V^0 - (\mu_t^{01} + \mu_t^{02}) dt \times {}_{t+dt}V^0 \\
&\quad + \mu_t^{01} dt \times {}_{t+dt}V^1 + o(dt).
\end{aligned}$$

Then, after dividing by dt and taking the limit $dt \to 0$, we find

$$\frac{d}{dt} {}_tV^0 = \delta_t V^0 + P - \mu_t^{01}\Big({}_tV^1 - {}_tV^0\Big) - \mu_t^{01}\Big(S - {}_tV^0\Big)$$

The second equation can be derived similarly. □

Proof #2. Suppose an entrepreneur borrows ${}_tV^{00}$ and buys a policy of a healthy individual, which is now in its t-th year. This entitles the entrepreneur to all payments that the insurance company and the individual make to each other.

So, the portfolio consists of a disability income policy and a debt obligation. The amount of money needed to create this portfolio is 0. By a fundamental theorem of theoretical economics, it follows that the expected profit of this portfolio should be equal to zero.

Now, what happens in the interval of time $[t, t+h]$? What is the profit?

The debt increases due to interest and the entrepreneur needs to pay the premiums. Then, there are three possibilities: the individual remains healthy, the individual get sick and the individual dies.

In the first case the value of the portfolio grows by ${}_{t+h}V^{00} - {}_tV^{00}$. In the second case, the value of the policy increases by ${}_{t+h}V^1 - {}_tV^0$ and in the third case it increases by $S - {}_tV^0$.

So altogether, the expected profit is

$$
\begin{aligned}
&-\delta h \times {}_tV^0 - Ph + (1 - \mu_t^{01}h - \mu_t^{02}h) \times ({}_{t+h}V^0 - {}_tV^0) \\
&+ \mu_t^{01}h({}_tV^1 - {}_tV^0) + \mu_t^{02}h(S - {}_tV^0) + o(h)
\end{aligned}
$$

up to the error term of higher order in h. Since this amount must be equal to 0, we find that

$$
\frac{d}{dt}\Big({}_tV^0\Big) = \delta \times {}_tV^0 + P - \mu_t^{01}({}_tV^1 - {}_tV^0) - \mu_t^{02}(S - {}_tV^0).
$$

The second equation can be derived similarly. □

2.6. **Multiple decrement models.** Multiple decrement models are special types of multiple state models which occur frequently in actuarial applications.

Definition 2.26. A *multiple decrement model* is the multiple state model with a single starting state and several exit states. There is a possible transition from the starting state to any of the exit states, but no further transitions.

Figure 8 illustrates a general multiple decrement model. The accidental death model is an example of such a model with two exit states.

Calculating probabilities for a multiple decrement model is relatively easy since only one transition can ever take place. For such a model we have

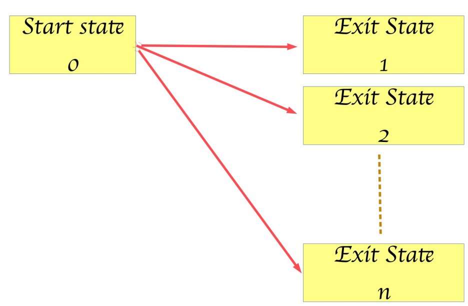

Figure 8. Multiple Decrement Model

$$
{}_tp_x^{00} = {}_tp_x^{\overline{00}} = \exp\left\{ -\int_0^t \Big(\sum_{i=1}^{n} \mu_{x+s}^{0i}\Big)\, ds \right\},
$$

$$
{}_tp_x^{01} = \int_0^t {}_sp_x^{00}\, \mu_{x+s}^{0i}\, ds,
$$

Example 2.27. A 10-year term insurance policy is issued to a life aged 50. The sum insured, payable immediately on death, is \$200,000 and premiums are payable continuously at a constant rate throughout the term.

No benefit is payable if the policyholder lapses, that is, cancels the policy during the term.

Assume that the force of interest is 2.5% per year. The force of mortality is given by $\mu_x = 0.002 + 0.0005(x - 50)$ and there are no expenses.

Calculate the annual premium rate using the following two sets of assumptions.

(a) There are no lapses.

(b) The transition intensity for lapses is a constant equal to 0.05.

Solution. (a) We use the model with two states: 0 – alive, and 1 – dead, and calculate

$$ {}_tp^{00}_{50} = \exp\big\{ - 0.002t - 0.00025t^2 \big\}. $$

$$ \begin{aligned} \overline{a}^{00}_{50:\overline{10}|} &= \int_0^{10} e^{-\delta t}\, {}_tp^{00}_{50}\, dt = 8.6961, \\ \overline{A}^{00}_{50:\overline{10}|} &= \int_0^{10} e^{-\delta t}\, {}_tp^{00}_{50}\mu_{50+t}\, dt = 0.03807. \end{aligned} $$

Therefore,

$$ P = 200,000 \times \frac{0.03807}{8.6961} = \$875.49. $$

(b) Here we add another state to the model: 2 – lapsed.

Then we get different value for the transition probability

$$ {}_tp^{00}_{50} = \exp\big\{ - 0.052t - 0.00025t^2 \big\}, $$

which gives a different value for the premium,

$$ P = 200,000 \times \frac{0.02890}{6.9269} = \$834.54. $$

Example 2.28. P&C Insurance Company is pricing a special fully discrete 3-year term insurance policy on (70). The policy will pay a benefit if and only if the insured dies as a result of an automobile accident.

You are given:

(1)

x	$l_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$	Benefit
70	1000	80	10	40	5,000
71	870	94	15	60	7,500
72	701	108	18	82	10,000

where $d_x^{(1)}$ represents deaths from cancer, $d_x^{(2)}$ represents deaths from automobile accidents, and $d_x^{(3)}$ represents death from all other causes.

(2) $i = 0.06$

(3) Level premiums are determined using the equivalence principle.

Calculate the annual premium.

Solution. Expected Present Value of Benefits:

$$5,000 \times \frac{10}{1000} \times \frac{1}{1.06} + 7,500 \times \frac{15}{1000} \times \frac{1}{1.06^2} + 10,000 \times \frac{18}{1000} \times \frac{1}{1.06^3}$$
$$= 47.17 + 100.12 + 151.13 = 298.42.$$

Expected Present Value of Premiums:

$$\left[1 + \frac{870}{1000} \times \frac{1}{1.06} + \frac{701}{1000} \times \frac{1}{1.06^2}\right] P = 2.4446P$$

The annual premium is

$$P = \frac{298.42}{2.4446} = 122$$

2.6.1. *Associated Single Decrement Rates.* For each of the causes of decrement recognized in a multiple decrement model, it is possible to define a single decrement model that depends only on the particular cause of decrement. This model essentially shows what would happen if all other sources of the decrement were absent.

We define the *associated single decrement model* as follows:

$$
\begin{aligned}
{}_tp_x^{\prime(j)} &= \exp\left[-\int_0^t \mu_x^{(j)}(s)\,ds\right],\\
{}_tq_x^{\prime(j)} &= 1 - {}_tp_x^{\prime(j)}.
\end{aligned}
$$

(A multiple decrement model is a particular case of the multiple state model and an alternative notation is often used: $\mu_x^{(j)}(t) := \mu_{x+t}^{0j}$ for $j \neq 0$ and $\mu_x^{(\tau)}(t) := -\mu_{x+t}^{00}$. We will use this notation in the rest of this section.)

Quantities such as ${}_tq_x^{\prime(j)}$ are also often called *net probabilities of decrement* because they are net of other causes of decrement.

Since the other causes of decrement are absent in this new model, the transition probability due to decrement j is in fact larger than in the original model.

Theorem 2.29. *For all* $t \geq 0$*,*

$$
{}_tq_x^{\prime(j)} \geq {}_tq_x^{(j)}.
$$

Proof. Since

$$
\exp\left[-\int_0^t \mu_x^{(j)}(s)\,ds\right] \geq \exp\left[-\int_0^t \sum_{i=1}^m \mu_x^{(i)}(s)\,ds\right],
$$

therefore

$$
{}_tp_x^{\prime(j)} \geq {}_tp_x^{(\tau)}.
$$

Hence,

$$
{}_tq_x^{\prime(j)} = \int_0^t {}_sp_x^{\prime(j)}\mu_x^{(j)}(s)\,ds \geq \int_0^t {}_sp_x^{(\tau)}\mu_x^{(j)}(s)\,ds = {}_tq_x^{(j)}.
$$

$\square$

This difference between ${}_tq_x^{'(j)}$ and ${}_tq_x^{(j)}$ can be considerable.

How can we compute the associated single decrement probabilities?

Let us say that decrement j satisfies the *constant force assumption* over the interval $(x, x+1)$ if

$$\mu_{x+t}^{(j)} = \mu_x^{(j)} \text{ and } \mu_{x+t}^{(\tau)} = \mu_x^{(\tau)}$$

for all $t \in [0,1]$.

We will say that decrement j satisfies the *uniform distribution assumption* over the interval $(x, x+1)$ if

$${}_tq_x^{(j)} = t \times q_x^{(j)} \text{ and } {}_tq_x^{(\tau)} = t \times q_x^{(\tau)}.$$

for all $t \in [0,1]$.

Theorem 2.30. *Suppose that decrement j satisfies either the constant force or the uniform distribution assumption over the interval $(x, x+1)$. Then,*

$$q_x^{'(j)} = 1 - (p_x^{(\tau)})^{q_x^{(j)}/q_x^{(\tau)}}.$$

Proof. (a) Let the decrement j satisfy the constant force assumption. Then, for $t \in [0,1]$,

$${}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)} \mu_{x+s}^{(j)}\, ds = \frac{\mu_x^{(j)}}{\mu_x^{(\tau)}} \int_0^t {}_sp_x^{(\tau)} \mu_{x+s}^{(\tau)}\, ds = \frac{\mu_x^{(j)}}{\mu_x^{(\tau)}} {}_tq_x^{(\tau)}$$

On the other hand, for any $r \in [0,1]$ we have

$$\mu_x^{(\tau)} = -\frac{1}{r}\log\left({}_rp_x^{(\tau)}\right) \text{ and } \mu_x^{(j)} = -\frac{1}{r}\log\left({}_rp_x^{'(j)}\right).$$

So,

$$\text{(1)} \qquad {}_tq_x^{(j)} = \frac{\log\left({}_rp_x^{'(j)}\right)}{\log\left({}_rp_x^{(\tau)}\right)} {}_tq_x^{(\tau)}$$

After a rearrangement we get

$${}_rp_x^{'(j)} = \left({}_rp_x^{(\tau)}\right)^{{}_tq_x^{(j)}/{}_tq_x^{(\tau)}}$$

By setting $r = t = 1$, we get

$$p_x^{'(j)} = \left(p_x^{(\tau)}\right)^{q_x^{(j)}/q_x^{(\tau)}},$$

and the claim of the theorem follows immediately.

(b) Now suppose that the uniform distribution assumption holds for the decrement j.

By definition,

$$\mu_{x+t}^{(j)} = \frac{1}{{}_tp_x^{(\tau)}} \frac{d}{dt} {}_tq_x^{(j)}.$$

It follows that under UDD assumption,

$$\mu_{x+t}^{(j)} = \frac{q_x^{(j)}}{{}_tp_x^{(\tau)}} = \frac{q_x^{(j)}}{1 - tq_x^{(\tau)}}$$

Then we can compute

$$\begin{aligned}
{}_tp_x^{\prime(j)} &= \exp\left[-\int_0^t \mu_{x+s}^{(j)}\, ds\right] \\
&= \exp\left[-\int_0^t \frac{q_x^{(j)}}{1 - sq_x^{(\tau)}}\, ds\right] \\
&= \exp\left[\frac{q_x^{(j)}}{q_x^{(\tau)}} \log\left(1 - tq_x^{(\tau)}\right)\right] \\
&= \left({}_tp_x^{(\tau)}\right)^{q_x^{(j)}/q_x^{(\tau)}},
\end{aligned}$$

and after we substitute $t = 1$, the claim of the theorem follows. □

Example 2.31. You are given:

(i) The following excerpt from a triple decrement table:

x	$l_x^{(\tau)}$	$d_x^{(1)}$	$d_x^{(2)}$	$d_x^{(3)}$
50	100,000	490	8,045	1,100
51	90,365	-	8,200	-
52	80,000	-	-	-

(ii) All decrements are uniformly distributed over each year of age in the decrement table.

(iii) $q_x^{\prime(3)}$ is the same for all ages.

Calculate 10, 000 $q_{51}^{\prime(1)}$.

Solution. First, we can calculate $q_{50}^{'(3)}$.

$$p_{50}^{'(3)} = \left(p_{50}^{(\tau)}\right)^{q_{50}^{(3)}/q_{50}^{(\tau)}} = \left(\frac{l_{51}^{(\tau)}}{l_{50}^{(\tau)}}\right)^{\frac{d_{50}^{(3)}}{l_{50}^{(\tau)}-l_{51}^{(\tau)}}} = 0.9885$$

Hence

$$p_{51}^{'(3)} = p_{50}^{'(3)} = 0.9885.$$

Now we again use this formula and compute

$$\begin{aligned}
\frac{q_{51}^{(3)}}{q_{51}^{(\tau)}} &= \frac{\log p_{51}^{'(3)}}{\log p_{51}^{(\tau)}} \\
q_{51}^{(3)} &= \left(1 - \frac{l_{52}^{(\tau)}}{l_{51}^{(\tau)}}\right)\frac{\log(0.9885)}{\log l_{52}^{(\tau)} - \log l_{51}^{(\tau)}} \\
&= \left(1 - \frac{80,000}{90,365}\right)\frac{\log 0.9885}{\log 80,000 - \log 90,365} \\
&= 0.0109
\end{aligned}$$

and

$$d_{51}^{(3)} = q_{51}^{(3)} \times l_{51}^{(\tau)} = 984.$$

Hence,

$$d_{51}^{(1)} = l_{51}^{(\tau)} - l_{52}^{(\tau)} - d_{51}^{(2)} - d_{51}^{(3)} = 1181$$

Next,

$$p_{51}^{'(1)} = \left(p_{51}^{(\tau)}\right)^{q_{51}^{(1)}/q_{51}^{(\tau)}} = \left(\frac{l_{52}^{(\tau)}}{l_{51}^{(\tau)}}\right)^{\frac{d_{51}^{(1)}}{l_{51}^{(\tau)}-l_{52}^{(\tau)}}} = 0.9862$$

Next, we calculate 10, 000 $q_{51}^{'(1)}$:

$$10,000q_{51}^{'(1)} = 10,000(1 - p_{51}^{'(1)}) = 10,000(1 - 0.9862) = 138.$$

Theorem 2.30 gives us ability to compute the single decrements ${}_tq_x^{'(j)}$ if we know the multiple decrements ${}_tq_x^{(j)}$ and either the constant force or uniform distribution assumptions are satisfied.

The converse calculation is also possible and becomes simple if impose the assumption of independence on the decrements.

Theorem 2.32. *Suppose that the transition probabilities ${}_tq_x^{(j)}$ are independent for different j. Then*

$$
{}_tp_x^{(\tau)} = \prod_{j=1}^{n} {}_tp_x^{\prime(j)}.
$$

Proof. By independence, we have

$$
\begin{aligned}
{}_tp_x^{(\tau)} &= \exp\Big[- \int_0^t \Big(\sum_{j=1}^{n} \mu_{x+s}^{(j)} \Big)\, ds \Big] \\
&= \prod_{j=1}^{n} \exp\Big[- \int_0^t \mu_{x+s}^{(j)}\, ds \Big] \\
&= \prod_{j=1}^{n} {}_tp_x^{\prime(j)}.
\end{aligned}
$$

□

Corollary 2.33. *If in addition to independence, it is assumed that decrement probabilities satisfy either the constant force or uniform distribution assumptions, then*

$$
{}_tq_x^{(j)} = \frac{\log\big(p_x^{\prime(j)}\big)}{\sum_{j=1}^{n} \log\big(p_x^{\prime(j)}\big)} \Big(1 - \prod_{j=1}^{n} {}_tp_x^{\prime(j)}\Big)
$$

Proof. This follows from the previous theorem and formula (1) on page 47. □

Example 2.34. For a double decrement table, you are given:

(1)

$$
\mu_{x+t}^{(1)} = 0.2\mu_{x+t}^{(\tau)},
$$

(2)

$$
\mu_{x+t}^{(\tau)} = kt^2,
$$

(3)

$$
q_x^{\prime(1)} = 0.04.
$$

Calculate ${}_2q_x^{(2)}$.

Solution. We have that

$$1 - q_x^{\prime(1)} = \exp\left[-\int_0^1 \mu_{x+t}^{(1)}\,dt\right] = \exp\left[-\int_0^1 0.2kt^2\,dt\right]$$
$$= \exp\left[-\frac{0.2}{3}k\right].$$

Hence,

$$k = -\frac{3}{0.2}\log(0.96) = -15\log(0.96) = 0.6123.$$

Next, we calculate

$${}_tp_x^{(\tau)} = \exp\left[-\int_0^t \mu_{x+s}^{(\tau)}\,ds\right] = \exp\left[-\int_0^t ks^2\,ds\right]$$
$$= \exp\left[-\frac{kt^3}{3}\right],$$

Then,

$${}_2q_x^{(2)} = \int_0^2 {}_sp_x^{(\tau)}\mu_{x+s}^{(2)}\,ds = \int_0^2 \exp\left[-\frac{ks^3}{3}\right]0.8ks^2\,ds$$
$$= -\int_0^2 0.8 \times d\left(\exp\left[-\frac{ks^3}{3}\right]\right)ds$$
$$= 0.8\left(1 - \exp\left[-\frac{k2^3}{3}\right]\right) = 0.6437$$

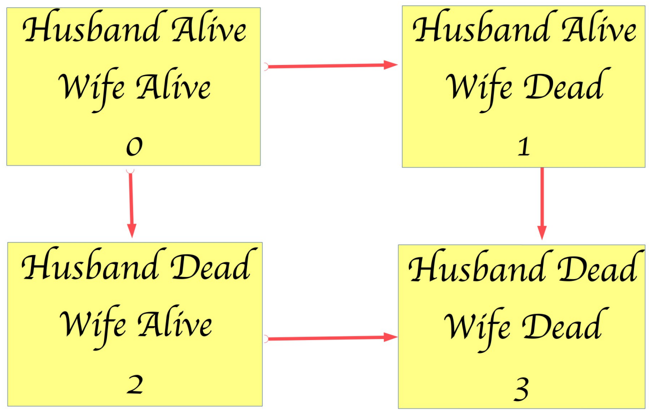

FIGURE 9. Two Lives Model

2.7. **Joint life and last survivor benefits.** We used the joint life model shown in Figure 9.

Notation:

(1) $\mu^{01}_{x+t:y+t}$ is the force of mortality for the wife when she is aged $y+t$ given that her husband is still alive and that he is aged $x+t$.
(2) $\mu^{02}_{x+t:y+t}$ is the force of mortality for the husband when he is aged $x+t$ given that his wife is still alive and that she is aged $y+t$.
(3) μ^{13}_{x+t} is the force of mortality for the husband at age $x+t$ given that his wife has already died.
(4) μ^{23}_{y+t} is the intensity of the mortality for the wife given that her husband has already died.

We have already introduced notation for transition probabilities and now we can relate it to the notation for multiple state models. Recall that xy is the *joint family status*. It ends when one of the individuals in the couple dies. And $\overline{xy}$ is the *last survivor status*. It ends when the last of individuals in the couple dies.

$${}_tp_{xy} = \mathbb{P}\{\text{both (x) and (y) are alive in } t \text{ years}\} = {}_tp^{00}_{xy},$$

(we will often suppress the subscript xy for shortness.)

$$\begin{aligned} {}_tq_{xy} &= \mathbb{P}\{\text{at least one of (x) and (y) is dead in } t \text{ years}\} \\ &= 1 - {}_tp_{xy} = 1 - {}_tp^{00} = {}_tp^{01} + {}_tp^{02} + {}_tp^{03}, \\ {}_tq_{\overline{xy}} &= \mathbb{P}\{\text{both (x) and (y) are dead in } t \text{ years}\} = {}_tp^{03}, \\ {}_tp_{\overline{xy}} &= \mathbb{P}\{\text{at least one of (x) and (y) is alive in } t \text{ years}\} \\ &= 1 - {}_tp^{03} = {}_tp^{00} + {}_tp^{01} + {}_tp^{02}. \end{aligned}$$

Sometimes the expressions are not so easy.

$$\begin{aligned} {}_tq^{1}_{xy} &= \mathbb{P}\{\text{husband (x) dies before wife (y) and within } t \text{ years}\} \\ \neq {}_tp^{02} &= \mathbb{P}\{\text{husband (x) dies before wife (y) and within } t \text{ years,} \\ &\quad \text{and wife is alive in } t \text{ years}\}. \end{aligned}$$

In this case we have the integral equation:

$$ {}_tq^{1}_{xy} = \int_0^t \mu^{02}_{x+r:y+r} \times {}_rp^{00}_{xy}\, dr. $$

Similarly,

$$\begin{aligned} {}_tq^{2}_{xy} &= \mathbb{P}\{\text{husband (x) dies after wife (y) and within } t \text{ years}\} \\ &= \int_0^t \mu^{13}_{x+r:y+r} \times {}_rp^{01}_{xy}\, dr. \end{aligned}$$

Example 2.35. You are using the multiple state model in Figure 9 for the future lifetimes of (30) and (30):

You are given:

(1) $\mu^{01}_{30+t:30+t} = 0.014 + 0.0007 \times 1.075^{30+t}$
(2) $\mu^{02}_{30+t:30+t} = 0.006$

Calculate ${}_{10}p^{00}_{30:30}$.

Solution.

$$
\begin{aligned}
{}_{10}p^{00}_{30:30} &= \exp\left\{ -\int_0^{10} \left(\mu^{01}_{30+t:30+t} + \mu^{02}_{30+t:30+t}\right) dt \right\} \\
&= \exp\left\{ -\int_0^{10} \left(0.006 + 0.014 + 0.0007 \times 1.075^{30+t}\right) dt \right\} \\
&= e^{-0.2} \times \exp\left\{ -0.0007 \times \frac{1.075^{30+t}}{\ln 1.075}\Big|_0^{10} \right\} \\
&= 0.748
\end{aligned}
$$

2.8. **Types of insurance products on multiple life.**

Joint life annuity:

$$\overline{a}_{xy} = \overline{a}^{00}_{xy},$$

a continuous payment at rate 1 per year while both husband and wife are still alive. If there is a maximum period, n years, for the annuity, then we refer to a *temporary joint life annuity* and the notation for the EPV is $\overline{a}_{xy:\overline{n}|}$.

If the annuity is payable m-thly in advance, with payments of $1/m$ every $1/m$ years, the EPV is denoted $\ddot{a}^{(m)}_{xy}$.

Joint life insurance:

$$\overline{A}_{xy},$$

a unit payment immediately on the death of the first to die of the husband and wife.

Last survivor annuity:

$$\overline{a}_{\overline{xy}} = \overline{a}^{00}_{xy} + \overline{a}^{01}_{xy} + \overline{a}^{02}_{xy},$$

Last survivor insurance:

$$\overline{A}_{\overline{xy}} = \overline{A}^{03}_{xy},$$

a unit payment immediately on the death of the second to die of the husband and wife.

Reversionary annuity:

$$\overline{a}_{x|y} = \overline{a}^{02}_{xy},$$

a continuous payment at unit rate per year while the wife is alive provided the husband has already died.

Contingent insurance:

$$\overline{A}^{1}_{xy},$$

a unit payment immediately on the death of the husband provided he dies before his wife. If there is a time limit on this payment, say n years, then it is called a *temporary contingent insurance* and the notation for the EPV is $\overline{A}^{1}_{xy:\overline{n}|}$.

Although the notation above is for the case of continuous benefits, it can easily be adapted for payments made at discrete points in time. For example, the EPV of a monthly joint life annuity-due would be denoted $\ddot{a}^{(12)}_{xy}$.

Theorem 2.36.

$$\begin{aligned}
\overline{a}_{\overline{xy}} &= \overline{a}_x + \overline{a}_y - \overline{a}_{xy}, \\
\overline{A}_{\overline{xy}} &= \overline{A}_x + \overline{A}_y - \overline{A}_{xy}, \\
\overline{a}_{x|y} &= \overline{a}_y - \overline{a}_{xy}.
\end{aligned}$$

Proof. For the first identity, it is enough to note that the portfolio of a joint life annuity and a last survivor annuity has the same payoffs as the portfolio of the annuity on the life of the wife and the annuity on the life of the husband. Hence,

$$\overline{a}_{\overline{xy}} + \overline{a}_{xy} = \overline{a}_x + \overline{a}_y.$$

The second identity is proved similarly.

For the third identity, we can note that the annuity on the life of the wife has the same payoffs as the portfolio of the annuity on the joint life and the reversionary insurance. Hence,

$$\overline{a}_{xy} + \overline{a}_{x|y} = \overline{a}_y.$$

□

Theorem 2.37.

$$\begin{aligned}
\overline{a}_{xy} &= \frac{1 - \overline{A}_{xy}}{\delta}, \\
\overline{a}_{\overline{xy}} &= \frac{1 - \overline{A}_{\overline{xy}}}{\delta}.
\end{aligned}$$

Proof. For the first identity, we have

$$\begin{aligned}
\overline{a}_{xy} &= \int_0^\infty e^{-\delta t}\,{}_tp_{xy}^{00}\,dt \\
&= -\frac{1}{\delta}\int_0^\infty {}_tp_{xy}^{00}\,d\Big(e^{-\delta t}\Big) \\
&= -\frac{1}{\delta}\Big(-1-\int_0^\infty e^{-\delta t}\,d\big({}_tp_{xy}^{00}\big)\Big) \\
&= -\frac{1}{\delta}\Big(-1+\int_0^\infty e^{-\delta t}\,{}_tp_{xy}^{00}\mu_{x+t:y+t}^{00}\,dt\Big) \\
&= \frac{1-\overline{A}_{xy}}{\delta}.
\end{aligned}$$

For the second identity,

$$\begin{aligned}
\overline{a}_{\overline{xy}} &= \overline{a}_x+\overline{a}_y-\overline{a}_{xy} \\
&= \frac{1-\overline{A}_x}{\delta}+\frac{1-\overline{A}_y}{\delta}-\frac{1-\overline{A}_{xy}}{\delta} \\
&= \frac{1-(\overline{A}_x+\overline{A}_y-\overline{A}_{xy})}{\delta} \\
&= \frac{1-\overline{A}_{\overline{xy}}}{\delta}.
\end{aligned}$$

□

Example 2.38. For a special continuous joint life annuity on (x) and (y), you are given:

(1) The annuity payments are $25,000$ per year while both are alive and $15,000$ per year when only one is alive.
(2) The annuity also pays a death benefit of $30,000$ upon the first death.
(3) $i = 0.06$
(4) $\overline{a}_{xy} = 8$
(5) $\overline{a}_{\overline{xy}} = 10$

Calculate the actuarial present value of this special annuity.

Solution. The APV of this insurance can be computed as follows:

$$\begin{aligned} APV &= 25,000 \times \overline{a}_{xy} + 15,000 \times \left(\overline{a}_{\overline{xy}} - \overline{a}_{xy}\right) + 30,000 \times \overline{A}_{xy} \\ &= 25,000 \times \overline{a}_{xy} + 15,000 \times \left(\overline{a}_{\overline{xy}} - \overline{a}_{xy}\right) + 30,000 \times \left(1 - \delta\overline{a}_{xy}\right) \\ &= 25,000 \times 8 + 15,000 \times \left(10 - 8\right) + 30,000 \times \left(1 - 8 \times \log(1.06)\right) \\ &= 246,015 \end{aligned}$$

Example 2.39. For two lives, (80) and (90), with independent future lifetimes, you are given:

k	p_{80+k}	p_{90+k}
0	0.9	0.6
1	0.8	0.5
2	0.7	0.4

Calculate the probability that the last survivor will die in the third year.

Solution. This probability can be calculated as follows:

$$\begin{aligned} &{}_2p_{\overline{80:90}} - {}_3p_{\overline{80:90}} \\ &= \left({}_2p_{80} + {}_2p_{90} - {}_2p_{80:90}\right) - \left({}_3p_{80} + {}_3p_{90} - {}_3p_{80:90}\right) \\ &= \left(0.9 \times 0.8 + 0.6 \times 0.5 - 0.9 \times 0.8 \times 0.6 \times 0.5\right) \\ &- \left(0.9 \times 0.8 \times 0.7 + 0.6 \times 0.5 \times 0.4 - 0.9 \times 0.8 \times 0.7 \times 0.6 \times 0.5 \times 0.4\right) \\ &= 0.24048 \end{aligned}$$

Example 2.40. For a special 3-year term life insurance policy on (x) and (y) with dependent future lifetimes, you are given:

(1) A death benefit of $100,000$ is paid at the end of the year of death if both (x) and (y) die within the same year. No death benefits are payable otherwise.
(2) $p_{x+k} = 0.84366$, $k = 0, 1, 2$
(3) $p_{y+k} = 0.86936$, $k = 0, 1, 2$
(4) $p_{x+k:y+k} = 0.77105$, $k = 0, 1, 2$

(5)

Maturity (in years)	Annual Effective Spot Rate
1	3%
2	8%
3	10%

Calculate the expected present value of the death benefit.

Solution. The probability that both (x) and (y) die in the year $k = 0, 1, 2$ is ${}_kp_{xy}q_{\overline{x+k:y+k}}$.

We can calculate,

$$ {}_kp_{xy} = p_{xy}p_{x+1:y+1}\cdots p_{x+k-1:x+k-1} = 0.77105^k. $$

Additionally, for all k,

$$ p_{\overline{x+k:y+k}} = p_{x+k} + p_{y+k} - p_{x+k:y+k} = 0.84366 + 0.86936 - 0.77105 = 0.94197, $$
$$ q_{\overline{x+k:y+k}} = 1 - p_{\overline{x+k:y+k}} = 0.05803 $$

Hence, we have the following calculation:

k	${}_kp_{xy}$	$q_{\overline{x+k:y+k}}$	Discount factor	Product
0	1	0.05803	$1/1.03$ = 0.97087	0.05634
1	0.77105	0.05803	$1/1.08^2$ = 0.85734	0.03836
2	0.59452	0.05803	$1/1.1^3$ = 0.75131	0.02592

Hence

$$ EPV = 100,000 \times (0.05634 + 0.03836 + 0.02592) = 12,062 $$

Example 2.41. For a fully continuous whole life insurance issued on (x) and (y), you are given:

(1) The death benefit of 100 is payable at the second death.
(2) Premiums are payable until the first death.
(3) The future lifetimes of (x) and (y) are dependent.
(4)
$$ {}_tp_{xy} = \frac{1}{4}e^{-0.01t} + \frac{3}{4}e^{-0.03t}, $$
(5) ${}_tp_x = e^{-0.01t}$
(6) ${}_tp_y = e^{-0.02t}$

(7) $\delta = 0.05$

Calculate the annual benefit premium rate for this insurance.

Solution. The equation for the premium is

$$P\overline{a}_{xy} = 100\overline{A}_{\overline{xy}}.$$

We calculate

$$\overline{a}_{xy} = \int_0^\infty e^{-0.05t}\left[\frac{1}{4}e^{-0.01t} + \frac{3}{4}e^{-0.03t}\right] dt = \frac{1}{4}\frac{1}{0.06} + \frac{3}{4}\frac{1}{0.08} = 13.54167$$

$$\overline{A}_{xy} = 1 - \delta\overline{a}_{xy} = 0.3229167$$

$$\overline{A}_x = \int_0^\infty e^{-0.05t} 0.01 e^{-0.01t}\, dt = \frac{0.01}{0.06}$$
$$\overline{A}_y = \int_0^\infty e^{-0.05t} 0.02 e^{-0.02t}\, dt = \frac{0.02}{0.07}$$

Hence,

$$\overline{A}_{\overline{xy}} = \overline{A}_x + \overline{A}_y - \overline{A}_{xy} = 0.1294643.$$

Finally,

$$P = 100\frac{0.1294643}{13.54167} = 0.956044$$

3. Pension Mathematics

3.1. **Introduction.** The two major categories of employer sponsored pension plans are *defined contribution* (DC) and *defined benefit* (DB).

The *defined contribution pension plan* specifies how much the employer will contribute, as a percentage of salary, into a plan. The employee may also contribute, and the employer's contribution may be related to the employee's contribution (for example, the employer may agree to match the employee's contribution up to some maximum).

The contributions are accumulated in an account, which is available to the employee when he or she leaves the company.

The contributions may be set to meet a target benefit level, but the actual retirement income may be well below or above the target, depending on the investment experience.

The *defined benefit plan* specifies a level of benefit, usually in relation to salary near retirement (*final salary plans*), or to salary throughout employment (*career average salary plans*).

The contributions from the employer and, possibly, employee are accumulated to meet the benefit. If the investment or demographic experience is adverse, the contributions can be increased; if experience is favorable, the contributions may be reduced. The pension plan actuary monitors the plan funding on a regular basis to assess whether the contributions need to be changed.

The benefit under a DB plan, and the target under a DC plan, are set by consideration of an appropriate replacement ratio.

The pension plan *replacement ratio* is defined as

$$R = \frac{\text{pension income in the year after retirement}}{\text{salary in the year before retirement}}$$

where it is assumed the plan member survives the year following retirement. The target for the plan replacement ratio depends on other post retirement income, such as government benefits.

A total replacement ratio, including government benefits and personal savings, of around 70% is often assumed to allow retirees to maintain their pre-retirement lifestyle. Employer sponsored plans often target

$50\% - 70\%$ as the replacement ratio for an employee with a full career in the company.

3.2. **The salary scale function.** The contributions and the benefits for most employer sponsored pension plans are related to salaries, so we need to model the progression of salaries through an individual's employment.

We use a deterministic model based on a salary scale, $\{s_y\}_{y \geq x_0}$, where x_0 is some suitable initial age. The value of s_{x_0} can be set arbitrarily, and then for any x, y greater than x_0 we define

$$\frac{s_y}{s_x} = \frac{\text{salary received in year of age } y \text{ to } y+1}{\text{salary received in year of age } x \text{ to } x+1}$$

where it is assumed that the individual remains in employment throughout the period from age x to $y+1$.

The salary scale may be defined as a continuous function of age, or may be summarized in a table of integer age values.

Example 3.1. The final average salary for the pension benefit provided by a pension plan is defined as the average salary in the three years before retirement. Members' salaries are increased each year, six months before the valuation date.

(1) A member aged exactly 35 at the valuation date received $\$75,000$ in salary *in the year to the valuation date*. Calculate his predicted final average salary assuming retirement at age 65.
(2) A member aged exactly 55 at the valuation date was paid salary *at a rate of* $\$100,000$ per year *at that time*. Calculate her predicted final average salary assuming retirement at age 65.

Assume either

(1) a salary scale where $s_y = 1.04^y$, or
(2) the integer age salary scale in the Table.

Solution. (a) The member is aged 35 at the valuation date, so that the salary in the previous year is the salary from age 34 to age 35. The predicted final average salary in the three years to age 65 is then

$$75,000 \frac{s_{62} + s_{63} + s_{64}}{3 s_{34}}$$

x	s_x	x	s_x	x	s_x	x	s_x
30	1.000	40	2.005	50	2.970	60	3.484
31	1.082	41	2.115	51	3.035	61	3.536
32	1.169	42	2.225	52	3.091	62	3.589
33	1.260	43	2.333	53	3.139	63	3.643
34	1.359	44	2.438	54	3.186	64	3.698
35	1.461	45	2.539	55	3.234		
36	1.566	46	2.637	56	3.282		
37	1.674	47	2.730	57	3.332		
38	1.783	48	2.816	58	3.382		
39	1.894	49	2.897	59	3.432		

FIGURE 10. Salary scale table

which gives $\$234,019$ under assumption (i) and $\$201,067$ under assumption (ii).

(b) The current annual salary rate of $\$100,000$ is the salary which will be earned in the year of age 54.5 to 55.5, so the final average salary is

$$100,000\frac{s_{62}+s_{63}+s_{64}}{3s_{54.5}}.$$

Under assumption (i) this is $\$139,639$. Under assumption (ii) we need to estimate $s_{54.5}$, which we would normally do using linear interpolation, so that

$$s_{54.5}=\frac{s_{54}+s_{55}}{2}=3.210,$$

giving a final average salary of $\$113,499$.

3.3. **Setting the contribution for a DC plan.** The contribution rate for a DC plan is set to target a fixed replacement ratio for a 'model' employee.

In the case of the life insurance, an individual makes the premium payments until his death and then his family is paid a benefit. The premium is determined by setting equal the expected present values of these two cash flows. It is essential that the time of death is random.

In the case of a DC retirement plan, an individual makes contributions until a known retirement date, and then receives the annuity benefit.

Again, the size of the contribution is determined by setting equal the expected EPVs of these cash flows. Here, the randomness comes as the random time of death of the retired employee.

Let S_t denotes salary rate at time t, F_r denote the value of the DC fund at retirement, and a_r denotes the EPV of the pension benefit that pays $\$1$ annually.

Then, the main equation is

$$F_r = R \times S_{[r-1,r]} \times a_r,$$

where R is the replacement ratio and $S_{[r-1,r]}$ denote the salary obtained in the year before the retirement.

Example 3.2. An employer establishes a DC pension plan. On withdrawal from the plan before retirement age, 65, for any reason, the proceeds of the invested contributions are paid to the employee or the employee's survivors.

The contribution rate is set using the following assumptions.

(1) The employee will use the proceeds at retirement to purchase a pension for his lifetime, plus a reversionary annuity for his wife at 60% of the employee's pension.
(2) At age 65, the employee is married, and the age of his wife is 55.
(3) The target replacement ratio is 65%.
(4) The salary scale is given by $s_y = 1.04^y$ and salaries are assumed to increase continuously.
(5) Contributions are payable monthly in arrear at a fixed percentage of the salary rate at that time.
(6) Contributions are assumed to earn investment returns of 10% per year.
(7) Annuities purchased at retirement are priced assuming an interest rate of 6% per year.
(8) Mortality follows the Illustrative Table and the lives of spouses are assumed independent.

Consider a male new entrant aged 25.

(1) Calculate the contribution rate required to meet the target replacement ratio for this member.

(2) Assume now that the contribution rate will be 5.5% of salary, and that over the member's career, his salary will actually increase by 5% per year, investment returns will be only 8% per year and the interest rate for calculating annuity values at retirement will still be 6% per year.

Calculate the actual replacement ratio for the member.

Solution. (1) First, we calculate the accumulated DC fund at retirement.

(A) The contribution at time t, where $t = \frac{1}{12}, \frac{2}{12}, \ldots, 40$, is

$$\frac{C}{12} S_0 \times 1.04^t,$$

where S_0 is the initial salary and C is the contribution rate per year.

Hence the accumulated amount of contributions at retirement is

$$\begin{aligned} F_{40} &= \frac{CS_0}{12} \sum_{k=1}^{40\times 12} 1.04^{k/12} \times 1.1^{40-k/12} \\ &= \frac{CS_0}{12} \frac{1.1^{40} - 1.04^{40}}{\left(\frac{1.1}{1.04}\right)^{1/12} - 1} = 719.6316 \times CS_0 \end{aligned}$$

(B) The salary in the year before retirement is

$$\begin{aligned} S_{[39,40]} &= S_0 \int_{39}^{40} 1.04^s \, ds \\ &= S_0 \frac{1.04^{40} - 1.04^{39}}{\log 1.04} = S_0 \times 1.04^{39} \frac{0.04}{\log 1.04} \\ &= 4.7081 \times S_0. \end{aligned}$$

Hence, the target pension benefit is

$$R \times S_{[r-1,r]} = 0.65 \times 4.7081 \times S_0 = 3.0603 \times S_0.$$

(C) The EPV of the whole life annuity for husband is

$$\ddot{a}_{65}^{(12)} \approx \ddot{a}_{65} - \frac{11}{24} = 9.8969 - \frac{11}{24} = 9.4386$$

The EPV of the reversionary annuity is

$$\begin{aligned} \ddot{a}_{65|55}^{(12)} &= \ddot{a}_{55}^{(12)} - \ddot{a}_{65:55}^{(12)} \approx \ddot{a}_{55} - \ddot{a}_{65:55} \\ &= 12.2758 - 8.8966 = 3.3792. \end{aligned}$$

Hence, the cost of buying the pension for the target replacement ratio is

$$3.0603 \times S_0(9.4386 + 0.6 \times 3.3792) = 35.0898$$

(D) Equating the accumulation of contributions to age 65 with the expected cost of the benefits at age 65 gives

$$C = \frac{35.0898}{719.6316} = 4.88\%.$$

(2) We repeat the calculation, using the actual experience rather than estimates. We use an annual contribution rate of 5%, and solve for the amount of benefit funded by the accumulated contributions, as a proportion of the final year's salary.

The values of the annuities will not change as we still use 6% interest rate for their pricing.

For the accumulated contributions we have

$$\begin{aligned} F_{40} &= \frac{0.05S_0}{12} \sum_{k=1}^{40\times 12} 1.05^{k/12} \times 1.08^{40-k/12} \\ &= \frac{CS_0}{12} \frac{1.08^{40} - 1.05^{40}}{\left(\frac{1.08}{1.05}\right)^{1/12} - 1} = 520.6548 \times 0.05S_0 \\ &= 26.0327. \end{aligned}$$

It follows that the annuity will pay

$$S_0 \frac{26.0327}{9.4386 + 0.6 \times 3.3792} = 2.2704 \times S_0$$

The salary in the last year before the retirement is

$$S_0 \times 1.05^{39} \frac{0.05}{\log 1.05} = 6.8710 \times S_0.$$

Hence, the actual replacement ratio is

$$R = \frac{2.2704}{6.8710} = 33\%$$

Note that relatively small changes in the actual experience relative to the assumptions lead to dramatic change in the replacement ratio.

This sensitivity to assumptions is true for both DC and DB benefits. In the DC case, the risk is taken by the member, who takes a lower benefit, relative to salary, than the target. In the DB case, the risk is usually taken by the employer, whose contributions are adjusted when the difference becomes apparent.

3.4. **Service table.** Often, an employee has several ways to exit employment. Besides retirement by age, the employee can withdraw to enter a different employment, retire because of a disability, or die in service. For each of these possible exits, different benefits can be assigned. Hence the value of a retirement plan often can be obtained by using a multiple decrement model (see Figure 11).

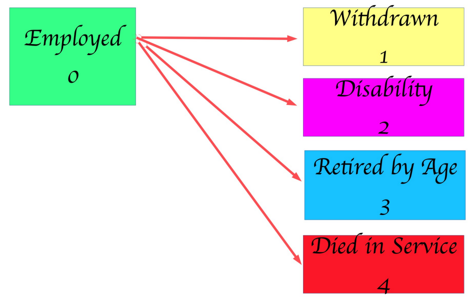

FIGURE 11. Retirement Model

Definition 3.3. A *service table* is a table that summarizes the retirement multiple decrement model at integer ages.

Namely, at some minimum integer age x_0, an initial cohort size (radix) is defined. Then, for each integer age $x = x_0 + k$, we define number of people who withdraw in the age interval $[x, x+1]$,

$$w_x = l_{x_0} \times {}_kp_{x_0}^{00} \times p_x^{01}.$$

Similarly, we define the number of people who were injured and retired by disability, i_x, the number of people who retired by age, r_x, and the number of people who died in service, d_x. We also write l_x to denote the number of people who are still employed at age x,

$$l_x = l_{x_0} \times {}_kp_{x_0}^{00}$$

A sample service table can be seen in Figure 12.

The row 60^- shows the number of people that retired exactly at age 60 and the row 60^+ shows the appropriate numbers for exits in the age interval $(60, 61)$. The age 65 is the age of compulsory retirement.

x	l_x	w_x	i_x	r_x	d_x	x	l_x	w_x	i_x	r_x	d_x
20	1 000 000	95 104	951	0	237	44	137 656	6 708	134	0	95
21	903 707	85 946	859	0	218	45	130 719	2 586	129	0	100
22	816 684	77 670	777	0	200	46	127 904	2 530	127	0	106
23	738 038	70 190	702	0	184	47	125 140	2 476	124	0	113
24	666 962	63 430	634	0	170	48	122 428	2 422	121	0	121
25	602 728	57 321	573	0	157	49	119 763	2 369	118	0	130
26	544 677	51 800	518	0	145	50	117 145	2 317	116	0	140
27	492 213	46 811	468	0	134	51	114 572	2 266	113	0	151
28	444 800	42 301	423	0	125	52	112 042	2 216	111	0	163
29	401 951	38 226	382	0	117	53	109 553	2 166	108	0	176
30	363 226	34 543	345	0	109	54	107 102	2 118	106	0	190
31	328 228	31 215	312	0	102	55	104 688	2 070	103	0	206
32	296 599	28 207	282	0	96	56	102 308	2 023	101	0	224
33	268 014	25 488	255	0	91	57	99 960	1 976	99	0	243
34	242 181	23 031	230	0	86	58	97 642	1 930	96	0	264
35	218 834	10 665	213	0	83	59	95 351	1 884	94	0	288
36	207 872	10 131	203	0	84	60^-	93 085	0	0	27 926	0
37	197 455	9 623	192	0	84	60^+	65 160	0	62	6 188	210
38	187 555	9 141	183	0	85	61	58 700	0	56	5 573	212
39	178 147	8 682	174	0	86	62	52 860	0	50	5 018	213
40	169 206	8 246	165	0	87	63	47 579	0	45	4 515	214
41	160 708	7 832	157	0	89	64	42 805	0	41	4 061	215
42	152 631	7 438	149	0	90	65^-	38 488	0	0	38 488	0
43	144 954	7 064	141	0	93						

FIGURE 12. Pension plan service table

Example 3.4. You are calculating the expected present value (EPV) of group insurance benefits for Gary, age 35, an active employee of BMK. You are given:

(i) The benefit for death of an active employee is $100,000$. The benefit for disability of an active employee is $50,000$. There are no other benefits.

(ii) Benefits are payable as a lump sum at the end of the year of the event.

(iii) Decrements follow the Illustrative Service Table, with d, w, and i representing death, withdrawal, and disability, respectively.

(iv) $i = 0.08$

Calculate the EPV of benefits for Gary for the first three years.

Solution.

$$EPV = 100,000 \times \frac{d_{35}^{(d)}v + d_{36}^{(d)}v^2 + d_{37}^{(d)}v^3}{l_{35}^{(\tau)}} + 50,000 \times \frac{d_{35}^{(i)}v + d_{36}^{(i)}v^2 + d_{37}^{(i)}v^3}{l_{35}^{(\tau)}}$$
$$= 100,000 \times \frac{64v + 64v^2 + 65v^3}{45,730} + 50,000 \times \frac{46v + 43v^2 + 45v^3}{45,730}$$
$$= 362.40 + 125.94 = 488.34$$

Example 3.5. BMK amends its benefit plan so that the lump sum is payable at the moment of death or disability.

You assume decrements are uniformly distributed over each year of age in the multiple decrement table.

Calculate the EPV of benefits for Gary for the first three years.

Solution. The EPV is $\frac{i}{\delta}488.34 = 507.62$.

Example 3.6. You revise your assumption about the timing of decrements to assume that the force of decrement for each decrement is constant over each year of age.

(i) Calculate $\mu_{35}^{(\tau)}$.

(ii) Calculate $\mu_{35}^{(d)}$ and $\mu_{35}^{(i)}$.

(iii) Calculate the EPV of benefits for Gary in the first year.

Solution. (i)

$$\mu_{35}^{(\tau)} = -\log\left(\frac{42,927}{45,730}\right) = 0.06325$$

(ii)

$$\mu_{35}^{(d)} = -\log\left(\frac{45,730 - 64}{45,730}\right) = 0.001400$$

$$\mu_{35}^{(i)} = -\log\left(\frac{45,730 - 46}{45,730}\right) = 0.001006$$

(iii) EPV of the death benefit is

$$\begin{aligned}
&100,000 \int_0^1 e^{-\delta t} {}_t p_{35}^{(\tau)} \mu_{35+t}^{(d)} \, dt \\
&= 100,000 \int_0^1 e^{-\left(\delta + \mu_{35+t}^{(\tau)}\right) t} \mu_{35}^{(d)} \, dt \\
&= 100,000 \times 0.001400 \times \frac{1 - e^{-(0.06325+0.07691)}}{0.06325 + 0.07691} \\
&= 134.76
\end{aligned}$$

The EPV for the disability benefit is calculated similarly and equals 48.43. The EPV of the total benefit is 183.18.

3.5. **Valuation of benefits in DB plans.** Our next task is to determine how the employer's contributions are determined in Defined Benefit plans. For this we need to study the structure and valuation of typical pension benefits.

3.5.1. *Final salary plans.* In a DB *final salary* pension plan, the basic annual pension benefit is equal to

$$n\alpha S_f,$$

where n is the total number of years of service, S_f is the average salary in a specified period before retirement, and α is the accrual rate, typically between 0.01 and 0.02.

For an employee who has been a member of the plan all her working life, say $n = 40$ years, this typically gives a replacement ratio in the range 40%–80%.

We can interpret this benefit formula to mean that the employee earns a pension of 100α% of final average salary for each year of employment.

Since the benefit is annual, the expected value of the benefit at the retirement date is

$$n\alpha S_f \ddot{a}_z^{(m)},$$

where z is the age at retirement and m is the number of times per year the benefit is paid. For example, if it is paid monthly, then $m = 12$.

If we want to calculate the present value of the benefit at some other date, then we need to discount the benefit value at retirement. Suppose we want to calculate the EPV of the benefit at age $y < z$. Then,

$$\begin{aligned}\text{EPV(benefit)} &= v^{z-y}{}_{z-y}p_y^{00} \times n\alpha S_f \ddot{a}_z^{(m)} \\ &= v^{z-y}\frac{l_z^{(\tau)}}{l_y^{(\tau)}} n\alpha S_f \ddot{a}_z^{(m)}\end{aligned}$$

Example 3.7. A pension plan provides an annual retirement benefit of 2% of final year's salary for each year of service, payable at the start of each month, upon retirement at age 65. The annual retirement benefit cannot exceed 60% of final year's salary.

A member, now age 45, joined the plan at age 30. Her current salary is $50,000$ and will increase at the rate of 3% per year at the start of each year in the future.

You are given:
(i) $l_{45}^{(\tau)} = 5000$ and $l_{65}^{(\tau)} = 3000$,
(ii) $i = 0.05$,
(iii) $\ddot{a}_{65}^{(12)} = 7.80$.

Calculate the expected present value of this member's retirement benefit.

Solution. By age 65, member would have served total of 35 years, in which case benefit would be $35 \times 0.02 = 70\%$. Thus set it at 60%. Then,

$$\begin{aligned}\text{EPV(benefits)} &= 0.60 \times 50,000 \times \frac{1.03^{19}}{1.05^{20}} \times \frac{l_{65}^{(\tau)}}{l_{45}^{(\tau)}}\ddot{a}_{65}^{(12)} \\ &= 92,787.29\end{aligned}$$

For regulatory purposes, the current value of the pension benefit sometimes calculated differently.

Consider a member who is currently aged y, who joined the pension plan at age x and for whom the normal retirement age is 60. Our estimate of her annual pension at retirement is

$$(60 - x)\alpha S_f = (y - x)\alpha S_f + (60 - y)\alpha S_f.$$

The first part is related to the employee past service, and is called the *accrued benefit*. The second part is related to future service.

The employer who sponsors the pension plan retains the right to stop offering pension benefits in the future. If this were to happen, the second part in the estimate is foregone. For this reason, the regulatory value of the pension benefit is often based only on the accrued benefits.

3.5.2. *Career average earnings plans.* Under a *career average earnings* (CAE) defined benefit pension plan, the benefit formula is based on the average salary, rather than the final average salary. Suppose a plan member retires at age z with n years of service and *total pensionable earnings* during their service of $(TPE)_z$. Then the annual pension benefit is

$$\alpha n \frac{(TPE)_z}{n} = \alpha (TPE)_z.$$

Under a career average earnings plan, the *accrued benefit* at age x is $\alpha(TPE)_x$, where $(TPE)_x$ denotes the total pensionable earnings up to age x.

The methods available for valuing such benefits are the same as for a final salary benefit.

A popular variation of the career average earnings plan is the *career average revalued earnings* plan, in which an inflation adjustment of the salary is made before averaging. The accrual principle is the same. The accrued benefit is based on the total past earnings after the revaluation calculation.

Example 3.8. Phillip, who is age 40, joins XYZ company which offers him a choice of two pension plans:

Plan 1 pays an annual pension of 1250 for each year of service.

Plan 2 pays an annual pension of 2% of his career average salary for each year of service.

You are given:

(i) His starting salary is S_0 and he will receive a 4% salary increase at the beginning of each year starting at age 41.

(ii) He will retire at age 65.

(iii) Plan 1 and Plan 2 both pay benefits at the beginning of each year.

(iv) Plan 1 and Plan 2 yield the same replacement ratio for him.

Calculate S_0.

Solution. We can derive expression for the replacement ratio in both cases. For Plan 1, we have

$$R_1 = \frac{1250 \times 25}{S_0 \times 1.04^{24}}.$$

For Plan 2, we use the formula for the sum of a geometric progression and find that

$$R_2 = \frac{0.02 \times S_0 \times \frac{1.04^{25}-1}{1.04-1}}{S_0 \times 1.04^{24}}$$

Since these replacement ratios must be equal, we get the equation

$$1250 \times 25 = 0.02 \times S_0 \times \frac{1.04^{25}-1}{0.04},$$

from which we can obtain

$$S_0 = 37,518.69$$

3.6. **Funding for DB plans.** The *reserve* is the assets set aside to meet the accrued liabilities as they fall due in the future.

Definition 3.9. The *reserve* at time t is the sum of the EPVs of all the accrued benefits at that time, taking into consideration all the appropriate benefits.

The reserve is denoted ${}_tV$. It is also called the *actuarial liability*. The task of the employer is to maintain the reserve from year to year.

Assume for simplicity that

(1) all employer contributions are paid at the start of the year,
(2) there are no employee contributions, and
(3) any benefits payable during the year are paid exactly half-way through the year.

These assumptions make the development of the principles and formulae clearer, but they can be relaxed quite easily.

With these assumptions, we define the employer's *normal contribution* due at the start of the year t to $t+1$ for a member aged x at time t, and denote it C_t.

Definition 3.10.

$$C_t = v \times {}_1p_x^{00} \times {}_{t+1}V + \text{EPV of benefits for mid-year exits} - {}_tV.$$

Here

$$\begin{aligned}&\text{EPV of benefits for mid-year exits}\\ =&\text{EPV of benefits that are payable given that the life exits during the year}\\ &\times \left(1 - {}_1p_x^{00}\right),\end{aligned}$$

and the EPV is evaluated at the start of year t.

Example 3.11. A member aged 50 has 20 years past service. His salary in the year to valuation was $\$50,000$.

You are given the following pension plan information:

- Accrual rate: 1.5%
- Final salary plan
- Pension based on earnings in the year before age retirement

- Normal retirement age 65
- The pension benefit is a life annuity payable monthly in advance
- There is no benefit due on death in service

The valuation assumptions are as follows:

- No exits other than by death before normal retirement age.
- Interest rate: 5% per year effective.
- Salaries increase at 4% per year (projected unit credit).
- Mortality before and after retirement follows the Illustrative Mortality Table.

Calculate the value of his accrued pension benefit and the normal contribution due at the start of the year assuming

(a) projected unit credit (PUC) funding, and

(b) traditional unit credit (TUC) funding, assuming valuation uses "final pensionable earnings" at the valuation date.

Solution. (a) Using the *projected unit credit* approach, the funding and valuation are based on projected final average earnings, so

$$S_f = 50,000 \times 1.04^{14} = 86,583.8$$

The reserve at the start of the year is

$$\begin{aligned} {}_0V &= n\alpha S_f \times {}_{15}p_{50} \times v^{15}\ddot{a}_{65}^{(12)} \\ &= 20 \times 0.015 \times 86,583.8 \times \frac{7,533,964}{8,950,901} \times 1.05^{-15}\Big(9.8969 - \frac{11}{24}\Big) \\ &= 99,262 \end{aligned}$$

Then,

$$\begin{aligned} vp_{50} \times {}_1V &= vp_{50} \times (n+1)\alpha S_f \times {}_{14}p_{51} \times v^{14}\ddot{a}_{65}^{(12)} \\ &= (n+1)\alpha S_f \times {}_{15}p_{50} \times v^{15}\ddot{a}_{65}^{(12)} \\ &= \frac{n+1}{n} \times {}_0V = \frac{21}{20} \times {}_0V \end{aligned}$$

In this example there are no benefits for mid-year exits. Hence, the normal contribution is

$$C = vp_{50} \times {}_1V - {}_0V = \frac{1}{20} \times {}_0V = 4,963.1$$

(b) At the start of the year, the salary for valuation is $\$50,000$; at the year end the projected salary is $\$50,000 \times 1.04 = \$52,000$.

Buy using the traditional unit credit approach, we have

$$\begin{aligned} {}_0V &= n\alpha S_{50} \times {}_{15}p_{50} \times v^{15}\ddot{a}_{65}^{(12)} \\ &= 57,321 \end{aligned}$$

and

$$\begin{aligned} vp_{50} \times {}_1V &= (n+1)\alpha S_{51} \times {}_{15}p_{50} \times v^{15}\ddot{a}_{65}^{(12)} \\ &= 62,595. \end{aligned}$$

Hence,

$$C = 62,595 - 57,321 = 5,274$$

4. Profit testing

4.1. Calculation of the profit vector. The purpose of a *profit test* is to identify the profit which the insurer can claim from the contract at the end of each time period.

The set of assumptions used in the profit test is called the *profit test basis*. Here is an example of a profit test basis.

Interest:	5.5% per year effective on all cash flows.
Initial expenses:	$\$400$ plus 20% of the first premium.
Renewal expenses:	3.5% of premiums.
Survival model:	$q_{60+t} = 0.01 + 0.001t$ for $t = 0, 1, \ldots, 9$.

In addition, we need to determine how the reserves are calculated. This set of assumptions is called the *reserve basis*. Often the reserves are calculated as net premium policy values under a more conservative assumptions on the interest rates and mortality. Here is an example of a reserve basis:

Interest:	4% per year effective on all cash flows.
Survival model:	$q_{60+t} = 0.011 + 0.001t$ for $t = 0, 1, \ldots, 9$.

Let Pr_t denote the expected profit at the end of year $t = 1, \ldots, n$ given that the policy is still in force at the start of the year. For convenience let us also define Pr_0, which is negative and equals to the initial expenses paid just before the contract started. Then the vector

$$\big(Pr_0, Pr_1, \ldots, Pr_n\big)',$$

where n is the length of the contract, is called the *profit vector* for the contract.

Let

$$\Pi_0 = Pr_0, \text{ and } \Pi_t = {}_{t-1}p_x Pr_t \text{ for } t = 1, \ldots, n$$

This is the expected profit at the end of year t given only that the policy is in effect at age x.

The vector

$$\Pi = \big(\Pi_0, \Pi_1, \ldots, \Pi_n\big)$$

is called the ***profit signature*** for the contract. The profit signature is the key to assessing the profitability of the contract.

How do we calculate these quantities?

Let B denote the sum insured which is payable at the end of the year of death. The level annual premiums are denoted P. Expenses, incurred in the year t are denoted E_t for $t = 1, 2, \ldots$, it is assumed that they are incurred at the start of the year from time $t-1$ to time t. The initial expenses are denoted E_0 and it is assumed that they were incurred before the start of the contract at time $t = 0$.

The reserve required at time t is denoted ${}_tV$.

Then we have the following equations for the profits Pr_t.

$$\begin{aligned} Pr_0 &= -E_0, \\ Pr_t &= \Big({}_{t-1}V + P - E_t\Big)(1+i) - \Big(q_{x+t-1} \times B + p_{x+t-1} \times {}_tV\Big). \end{aligned}$$

Sometimes the latter equation is written differently, as

$$Pr_t = \Big(P - E_t\Big)(1+i) + \Delta({}_tV) - q_{x+t-1} \times B,$$

where $\Delta({}_tV)$ is called the ***change in reserve*** in year t and defined as

$$\Delta({}_tV) = {}_{t-1}V \times (1+i) + p_{x+t-1} \times {}_tV.$$

Example 4.1. An insurer issues fully discrete whole life policies with a death benefit of $100,000$ to independent lives age 50. You are given the following premium assumptions:

(1) Mortality follows the Illustrative Life Table.
(2) $i = 6\%$
(3) Pre-contract expenses are 1000.
(4) Maintenance expenses of 50 are incurred at the start of each year including the first.
(5) The gross premium is computed using the equivalence principle.

Calculate the gross premium.

Solution. We have the equation:

$$P\ddot{a}_{50} = 100,000A_{50} + 1000 + 50\ddot{a}_{50}.$$

From the table we have $\ddot{a}_{50} = 13.2668$ and $A_{50} = 0.24905$. By using this information we can calculate that

$$P = 2002.6.$$

Example 4.2. The insurer performs a profit test of these policies using the same expenses and mortality assumptions as in the premium basis. You are given the following additional information for the profit test:

(1) Reserves for the policy at $t = 0, 1, 2, 3$ are:

$${}_0V = 0, \quad {}_1V = 750, \quad {}_2V = 1500, \quad {}_3V_0 = 2250.$$

(2) Cash Values for policyholders surrendering at the end of the t-th year, $t = 1, 2, 3$, are:

$$CV_1 = 0, \quad CV_2 = 600, \quad CV_3 = 1400.$$

(3) At the end of each year, 5% of policyholders still in force surrender their policies.

(4) The insurer earns interest of 5% in the first year and 7% in each subsequent year.

Calculate components of the profit vector and profit signature for $t = 0, 1, 2, 3$.

Solution. It is convenient to compile the following table:

t	${}_{t-1}V$	P_t	E_t	I_t	EDB_t	ECV_t	$E({}_tV)$	Pr_t	Π_t
0^-		1000						- 1000	- 1000
1	0	2002.6	50	97.6	592	0.0	708.3	749.9	749.9
2	750	2002.6	50	189.2	642	29.8	1415.9	804.1	799.3
3	1500	2002.6	50	241.7	697	69.5	2122.6	805.2	795.3

Here I_t is the interest profit, EDB_t is the expected death benefit payment, and ECV_t is the expected cash value payment to the customers who withdraw.

This table is compiled in the following way. The interest profit is

$$I_t = i_t({}_{t-1}V + P_t - E_t),$$

where $i_t = 0.05$ for $t = 1$, and $i_t = 0.07$ for $t = 2, 3$.

From the Illustrative table, we can obtain

$$q_{50} = 0.00592, \quad q_{51} = 0.00642, \quad q_{52} = 0.00697$$

and then we calculate the expected death benefit as

$$EDB_t = q_{50+t-1} 100,000,$$

and the expected cash value benefit as

$$ECV_t = p_{50+t-1} \times 0.05 \times CV_t.$$

Finally, the expected reserve $\mathbb{E}_t V$ is

$$\mathbb{E}({}_tV) = p_{50+t-1} \times 0.95 \times {}_tV.$$

As an example of the profit calculation:

$$Pr_2 = 750 + 2002.6 - 50 + 189.2 - 642 - 29.8 - 1415.9 = 804.1$$
$$\Pi_2 = p_{50} \times Pr_2 = 799.34$$

4.2. **Profit Measures.** Once we have the profit signature, we need to assess whether the emerging profit is adequate by a suitable measure of profit.

Definition 4.3. The *internal rate of return* (IRR) is the interest rate j such that the present value of the expected cash flows is zero. Given a profit signature $(\Pi_0, \Pi_1, \ldots, \Pi_n)'$ for an n-year contract, the internal rate of return is j, where

$$\sum_{t=0}^{n} \Pi_t (1+j)^{-t} = 0. \tag{2}$$

The insurer may set a minimum *hurdle rate* or *risk discount rate* for the internal rate of return, so that the contract is deemed adequately profitable if the IRR exceeds the hurdle rate.

One problem with the internal rate of return is that there may be no real solution to equation (2), or there may be several.

As an alternative, we can use the *net present value* (NPV) of the contract.

Definition 4.4. The net present value of a contract is

$$NPV = \sum_{t=0}^{n} \Pi_t (1+r)^{-t},$$

where r is the risk discount rate.

Definition 4.5. The *profit margin* is the NPV expressed as a proportion of the EPV of the premiums, evaluated at the risk discount rate.

For a contract with level premiums of P per year payable m-thly throughout an n-year contract issued to a life aged x, the profit margin is

$$\text{Profit Margin} = \frac{NPV}{P\ddot{a}^{(m)}_{x:\overline{n}|}},$$

using the same risk discount rate for all calculations.

Another profit measure is the *discounted payback period* (DPP), also known as the *break-even* period. This is calculated using the risk discount rate, r, and is the smallest value of m such that

$$NPV(m) = \sum_{t=0}^{m} \Pi_t (1+r)^{-t} \geq 0.$$

The DPP represents the time until the insurer starts to make a profit on the contract.

Example 4.6. For a 5-year term insurance policy on (x), you are given:
(i) The profit signature is $\Pi = [-550, 300, 275, 75, 150, 100]$.
(ii) The risk discount rate is 12%.
Calculate the discounted payback period (DPP) for this policy.

Solution. We calculate:

$$\begin{aligned}
NPV(0) &= \Pi_0 = -550 \\
NPV(1) &= NPV(0) + v\Pi_1 = -550 + \frac{300}{1.12} = -282 \\
NPV(2) &= NPV(1) + v^2\Pi_2 = -282 + \frac{275}{1.12^2} = -62.91 \\
NPV(3) &= NPV(2) + v^3\Pi_3 = -62.91 + \frac{75}{1.12^3} = -9.53 \\
NPV(4) &= NPV(3) + v^4\Pi_4 = -9.53 + \frac{150}{1.12^4} = 82.50
\end{aligned}$$

Hence DPP of the contract equals 4.

5. Universal Life Insurance

The *Universal Life* (UL) insurance contract is a mixture of life insurance and an investment product. The policyholder may vary the amount and timing of premiums, within some constraints. The premium is deposited into a *notional* account, which is used to determine the death and survival benefits.

The account is *'notional'* because there is no separate account for the policyholder, and the premiums are actually merged within the insurer's general funds. The insurer will share the investment proceeds of the general funds, through the credited interest rate which is declared by the insurer at regular intervals (typically monthly). The UL policy terms will set a minimum value for the credited interest rate, regardless of the investment performance of the insurer's assets.

The notional account, made up of the premiums and credited interest, is subject to monthly deductions (also notional) for management expenses and for the cost of life insurance cover.

The *account balance* or *account value* (AV) is the balance of funds in the notional account. Note that the cost of insurance and expense charge deductions are set by the insurer, and need not be the best estimate of the anticipated expenses or insurance costs. In the profit test examples that follow in this section, the best estimate assumptions for expenses and the cost of death benefits are quite different to the charges set by the insurer for expenses and the cost of insurance.

The account value represents the the insurer's liability, analogously to the reserve under a traditional contract. UL contracts are generally sold as 'permanent' or whole life products. For a whole life UL, the account value also represents the cash value for a surrendering policyholder, after an initial period (typically 7-10 years).

In the basic UL contract, the insurer expects to earn more interest than will be credited to the policyholder account value. The difference between the interest earned and the interest credited is the *interest spread*, and this is the major source of profit for the insurer.

This is, in fact, no different to any traditional whole life or endowment insurance. For a traditional insurance, the premium might be set assuming interest of 5% per year, even though the insurer expects to earn 7% per year. The difference generates profit for the insurer.

The difference for UL, perhaps, is that the interest spread is more transparent.

It is useful to analyze the profitability of UL policies, using the profit test techniques. However, one have to perform an additional step before the usual profit test. The benefits depend on the account value, so the first step in the UL profit test is a projection of the account value and the cash value of the contract, assuming the policyholder survives to the final projection date.

5.1. Key design features.

1. Death Benefit

On the policyholder's death the total benefit paid is the account value of the policy, plus an *additional death benefit* (ADB).

The ADB is required to be a significant proportion of the total death benefit, except at very advanced ages, to justify the policy being considered an insurance contract. The proportions are set through the *corridor factor requirement* which sets the minimum value for the ratio of the total death benefit (i.e. Account Value + ADB) to the account value at death. In the US the corridor factor is around 2.5 up to age 40 , decreasing to 1.05 at age 90, and to 1.0 at age 95 and above.

There are two types of death benefit, Type A and Type B.

The *Type A* universal life insurance offers a level total death benefit, which comprises the account value plus the additional death benefit. As the account value increases, the ADB decreases. However the ADB cannot decline to zero, except at very old ages, because of the corridor factor requirement.

For a Type A UL policy the level death benefit is the *Face Amount* of the policy.

The *Type B* universal life insurance offers a level additional death benefit (ADB). The amount paid on death would be the Account Value plus the level Additional Death Benefit selected by the policyholder, provided this satisfies the corridor factor requirement.

The policyholder may have the option to adjust the ADB to allow for inflation. Other death benefit increases may require evidence of health to avoid adverse selection.

2. Premiums

The premiums may be subject to some minimum level, but otherwise are highly flexible.

3. Expense Charges

These are deducted from the account value. The rates will be variable at the insurer's discretion, subject to a maximum specified in the original contract.

4. Credited Interest

Usually the credited interest rate will be decided at the insurer's discretion, but it may be based on a published exogenous rate, such as yields on government bonds. A minimum guaranteed annual credited interest rate will be specified in the policy document.

5. Cost of Insurance

Each year the UL account value is subject to a charge to cover the cost of the selected death benefit cover. The charge is called the *Cost of Insurance*, or CoI. Usually, the CoI is calculated using an estimate (perhaps conservative) of the mortality rate for that period, so that, as the policyholder ages, the mortality charge (per $1 of ADB) increases. The CoI is then the single premium for a 1-year term insurance with sum insured equal to the ADB.

6. Surrender Charge

If the policyholder chooses to surrender the policy early, the surrender value paid will be the policyholder's account balance reduced by a surrender charge. The main purpose of the charge is to ensure that the insurer receives enough to pay its acquisition expenses. The total cash available to the policyholder on surrender is the account value minus the surrender charge (or zero if greater), and is referred to as the *Cash Value* of the contract at each duration.

7. Secondary Guarantees

There may be additional benefits or guarantees attached to the policy. A common feature is the *no lapse guarantee* under which the death benefit cover continues even if the Account Value declines to zero, provided that the policyholder pays a pre-specified minimum premium at each premium date.

8. Policy Loans

A common feature of UL policies is the option for the policyholder to take out a loan using the policy account or cash value as collateral. The interest rate on the loan could be fixed in the policy document, or could depend on prevailing rates at the time the loan is taken, or might be variable. Pre-specified fixed interest rates add substantial risk to the contract; if interest rates rise, it could benefit the policyholder to take out the maximum loan at the fixed rate, and re-invest at the prevailing, higher rate.

5.2. **Universal Life Type B.** In this section we illustrate the Type B UL policy design through several simple examples. We start with Type B as it is a simpler policy to analyze. The Type B policy total death benefit is the account value plus some fixed additional death benefit (ADB).

Example 5.1. A universal life policy is sold to a 45 year old man. The initial premium is $\$2250$ and the ADB is a fixed $\$100,000$. The policy charges are:

Cost of Insurance: Mortality in the Illustrative Life Table. 6% per year interest.

Expense Charges: $\$48 + 1\%$ of premium at the start of each year.

Surrender penalties at each year end are the lesser of the full account value and the following surrender penalty schedule:

Year of surrender	1	2	3-4	5-7	8-10	> 10
Penalty	\$4500	\$4100	\$3500	\$2500	\$1200	\$0

Assume (i) the policy remains in force for 20-years, (ii) interest is credited to the account at 5% per year, (iii) a no lapse guarantee applies to all policies provided full premiums are paid for at least 6 years, (iv) all cash flows occur at policy anniversaries and (v) there are no corridor factors requirements for the policy.

Project the account value and the cash value at each year end for the 20-year projected term, given

(a) the policyholder pays the full premium of $\$2250$ each year;

(b) the policyholder pays the full premium of $\$2250$ for 6 years, and then pays no further premiums.

Solution. Projecting the account value shows how the policy works under the idealized assumptions – level premiums, level credited interest. It also gives us the total death benefit and surrender benefit values which we need to profit test the contract.

Each year, the insurer deducts from the account value the expense charge and the cost of insurance (which is the price for a 1-year term insurance with sum insured equal to the Additional Death Benefit), and adds to the account value any new premiums paid, and the credited interest for the year.

Here are the calculations for the first two years.

First Year	
Premium	2250
Expense Charge	$48 + 0.01 \times 2250 = 70.50$
CoI	$100,000 \times 0.004 \times v_{6\%} = 377.36$
Interest Credited	$0.05 \times (2250 - 70.50 - 377.36) = 90.11$
Account value	$2250 - 70.50 - 377.36 + 90.11 = 1892.25$
Cash Value	$\max(1892.25 - 4500, 0) = 0$

Second Year	
Premium	2250
Expense Charge	$48 + 0.01 \times 2250 = 70.50$
CoI	$100,000 \times 0.00431 \times v_{6\%} = 406.60$
Interest Credited	$0.05 \times (1892.25 + 2250 - 70.50 - 406.60) = 183.26$
Account value	$1892.25 + 2250 - 70.50 - 406.60 + 183.26 = 3848.41$
Cash Value	$\max(3848.41 - 4100, 0) = 0$

Example 5.2. Calculate the profit signature for the previous example if it is assumed that

(1) Policies remain in force for a maximum of 20 years.

(2) Premiums of $\$2250$ are paid for six years, and no premiums are paid thereafter.

(3) The insurer does not change the CoI rates, or expense charges from the initial values given in the previous example.
(4) Interest is credited to the policyholder account value in the t-th year using a 2% interest spread, with a minimum credited interest rate of 2%. In other words, if the insurer earns more than 4%, the credited interest will be the earned interest rate less 2%. If the insurer earns less than 4% the credited interest rate will be 2%.
(5) The ADB remains at $100,000$ throughout.
(6) Interest earned on all insurer's funds at 7% per year.
(7) Mortality experience is as assumed.
(8) Incurred expenses are $\$2000$ at inception, $\$45$ plus 1% of premium at renewal, $\$50$ on surrender (even if no cash value is paid), $\$100$ on death.
(9) Surrenders occur at year ends. The surrender rate given in the following table is the proportion of in-force policyholders surrendering at each year end.

Duration	Surrender Rate at year end
1	5%
2-5	2%
6-10	3%
11	10%
12-19	15%
20	100%

(10) The insurer holds the full account value as reserve for this contract.

Solution.

Example 5.3. For a Type B Universal Life policy issued to (55) you are given:

(1) $AV_4 = 500$
(2) The additional death benefit is $1,000,000$.
(3) The COI rate in year 5 is 20 per 1000 of insurance.
(4) Expense charges are 500 per year plus 5% of premium
(5) $i^c = 0.045$.
(6) The policy does not have a no-lapse guarantee.

Calculate the minimum premium to be paid at the beginning of year 5 so that the policy does not lapse before the next premium is paid at the beginning of year 6.

Solution. Recall that AV_t denotes the account value at the end of year t.

The minimum premium to prevent lapse will be the premium such that $AV_5 = 0$. Let P be this premium.

WE have the following equation that relates AV_5 and AV_4.

$$AV_5 = \left(AV_4 + 0.95P - 500 - \frac{20,000}{1.045} \right) \times 1.045.$$

From the condition $AV_5 = 0$, we find that $P = 20,146$.

Example 5.4. For a universal life insurance policy with death benefit of 100,000 plus the account value, you are given:

(i)

Component	Policy Year	
	1	2
Percent of Premium Charge	25%	$R\%$
Cost of Insurance Rate per Month	0.002	0.003
Monthly Expense Charge	6	6
Surrender Charge	3000	1000

(ii) The credited interest rate is $i^{(12)} = 0.06$.

(iii) The actual cash surrender value at the end of month 11 is $10,000$ and at the end of month 13 is $13,330$.

(iv) Premiums of 1000 are paid at the beginning of months 12 and 13.

(v) The policy is in force at the end of month 13.

Calculate $R\%$.

Solution. Since $i^{(12)} = 0.06$, the interest rate per month is $0.06/12 = 0.005$.

The account value at the end of month 11 is $10,000 + 3,000 = 13,000$.

The Cost of Insurance charge at the beginning of month 12 is $100,000 \times 0.002/1.005 = 199.00$.

From this data we can calculate the account value at the end of month 12,

$$\begin{aligned} AV_{12} &= \Big[13,000 + 1000 \times (1 - 0.25) - 6 - 199.00\Big] \times 1.005 \\ &= 13,612.75 \end{aligned}$$

Next, we do similar calculations for the month 13. The Cost of Insurance for that month is $100,000 \times 0.003/1.005 = 298.51$. Hence, the account value at the end of month 13 is

$$AV_{12} = \Big[13,612.75 + 1000 \times (1 - R\%) - 6 - 298.51\Big] \times 1.005.$$

By condition, this equals $13,330 + 1,000 = 14,330$. By solving for R we find that

$$R = 0.0495$$

Example 5.5. Dana buys a Type B universal life contract of $100,000$. You are given:

(i)

Policy Year k	Annual Premium	Annual COI Rate per 1000 of Insurance	Percent of Premium Charge	Annual Expense Charge	Surrender Charge
1	1000	–	60%	–	–
2	P_2	2	10%	10	200
3	P_3	3	10%	10	100
$k \geq 4$	P_4	k	5%	10	0

(ii) The credited interest rate is $i = 0.06$

(iii) Dana's account value at the end of the year 1 is 165.

(iv) Except as indicated, there are no deaths or surrenders.

(a) Calculate Dana's account value at the end of year 2 assuming that P_2 were 1000.

Solution.

$$\begin{aligned} AV_2 &= (AV_1 + 0.9P_2 - 10 - 200v) \times 1.06 \\ &= (165 + 900 - 10) \times 1.06 - 200 \\ &= 918.3 \end{aligned}$$

(b) Dana's account value at the end of year 3 can be expressed as $aP_2 + bP_3 + c$. Calculate a, b, and c.

Solution.

$$\begin{aligned} AV_2 &= (165 + 0.9P_2 - 10) \times 1.06 - 200 \\ &= 0.954P_2 - 35.7 \\ AV_3 &= (AV_2 + 0.9P_3 - 10) \times 1.06 - 300 \\ &= (0.954P_2 - 35.7 + 0.9P_3 - 10) \times 1.06 - 300 \\ &= 1.0112P_2 + 0.954P_3 - 348.4. \end{aligned}$$

Hence,

$$a = 1.0112 \quad b = 0.954 \quad c = -348.4$$

(c) In year 2, Dana pays a premium of 1000 with probability 0.6, or 200 with probability 0.4.

If he paid 1000 in year 2, then in year 3 he will pay either 1000 with probability 0.6, or 200 with probability 0.4.

If he paid 200 in year 2, then in year 3 he will pay either 1000 with probability 0.2, or 200 with probability 0.8

(1) Calculate the expected death benefit payable at the end of year 3, if Dana dies then.
(2) Calculate the expected surrender benefit payable at the end of year 3, if Dana surrenders the contract then.

Solution. Here the result from (b) is very helpful, since it gives an easy way to calculate the account value at the end of the year 3 in various scenarios.

P_2	P_3	Prob.	AV_3
1000	1000	0.36	1616.8
1000	200	0.24	853.6
200	1000	0.08	807.8
200	200	0.32	44.6

Hence, the death benefit is

$$\begin{aligned} \mathbb{E}(DB_3) &= 100,000 + 0.36 \times 1.616.8 + 0.24 \times 853.6 + 0.08 \times 807.8 \\ &\quad + 0.32 \times 44.6 = 100,865.8 \end{aligned}$$

For the second part of this problem, it is not appropriate to simply to subtract $\$100$ from the expected account value calculated in the third part, since in the last case the account value is strictly less than 100. This happens with probability 0.32. Hence

$$\begin{aligned}\mathbb{E}(CV_3) &= (865.8 - 100) + 0.32 \times (100 - 44.6) \\ &= 765.8 + 0.32 \times 55.4 = 783.53\end{aligned}$$

(d) Dana's identical twin, Mark, buys a contract identical to Dana's. If Mark pays 1000 every year, Mark's account value at the end of year 10 will be 5114. Mark will pay premiums of 1000 in 9 of the first 10 years. Mark will pay no premium in one year, with the year of no premium equally likely to be year 3 or year 10.

Calculate Mark's expected surrender value at the end of year 10.

Solution. Let $AV_{10} = 5114$ denote the AV assuming all premiums paid; $AV_{10}^{(3)}$ and $AV_{10}^{(10)}$ denote the AV at time 10 assuming all premiums are paid except the third or last, respectively. Then

$$\begin{aligned}AV_{10}^{(3)} &= AV_{10} - 1000 \times 0.9 \times 1.06^8 = 3679.5, \\ AV_{10}^{(10)} &= AV_{10} - 1000 \times 0.9 \times 1.06 = 4107.0\end{aligned}$$

Then,

$$\mathbb{E}(AV) = 0.5AV_{10}^{(3)} + 0.5AV_{10}^{(10)} = 3893.3$$

5.3. **Universal Life Type A.** The Type A contract is a little more complicated. The total death benefit is set at the Face Amount (FA), so the additional death benefit (ADB) is the excess of the Face Amount over the Account Value (AV); however, there is also, generally, a corridor factor requirement. This requirement ensures that the death benefit cannot be smaller than a certain multiple of the Account Value.

Hence if the corridor requirement does not apply, then we can find the Additional Death Benefit and the Account Value by solving the system

$$\begin{aligned}AV_t &= (AV_{t-1} + P(1-\alpha) - E)(1+i) - rADB_t, \\ ADB_t &= FA - AV_t,\end{aligned}$$

where α is the portion of the premium charged by the insurance company, r is the Cost of Insurance as a portion of Additional Death Benefit, and FA is the Face Amount (i.e., total death benefit).

If the corridor factor of γ_t applies, then the total death benefit is $\gamma_t AV_t$, and the second equation in the above system should be changed to

$$ADB_t = (\gamma_t - 1)AV_t.$$

In order to decide which of the systems gives the correct solution one have to solve both of them and check if the corridor factor requirement applies. Or, one can simply use the largest of the additional death benefits.

Example 5.6. Suppose a Type A UL contract was issued some time ago to a life now age 50 and it has face amount FA = $\$100,000$. The CoI mortality rate for the year is $q_{50} = 0.004$. The account value at the start of the year is $AV_{t-1} = \$50,000$. The corridor factor for the year is $\gamma_t = 2.2$. There is no premium paid and no expense deduction from the account value in the year. The credited interest rate is 5%.

Let ADB^f denote the additional death benefit based on the excess of face value over account value, that is

$$ADB_t^f = FA - AV_t$$

and let ADB^c denote the additional death benefit based on the corridor factor, so that

$$ADB_t^c = \gamma_t AV_t - AV_t = (\gamma_t - 1)AV_t.$$

Then

$$ADB_t = \max(ADB_t^f, ADB_t^c).$$

We have the following results, where we use the superscript f to refer to the CoI based on ADB^f and c for the CoI based on ADB_c. The final CoI value is the maximum of these two values.

$$\begin{aligned}
AV_t &= AV_{t-1} \times 1.05 - CoI \\
&= AV_{t-1} \times 1.05 - 0.004ADB_t^f \\
ADB_t^f &= 100,000 - AV_t \\
&= 100,000 - (1.05 \times 50,000 - 0.004ADB_t^f)
\end{aligned}$$

Hence,

$$ADB_t^f = \frac{100,000 - 1.05 \times 50,000}{0.996} = 47,691$$

$$AV_t = 50,000 \times 1.05 - 0.004 \times 47,691 = 52,309.$$

The corridor factor requirement is not satisfied:

$$\frac{ADB_t^f}{AV_t} = \frac{47,691}{52,309} < \gamma_t - 1 = 1.2.$$

Now let us calculate the additional death benefit under the corridor factor requirement. In this case we have

$$\begin{aligned} ADB_t^c &= 1.2AV_t \\ &= 1.2 \times (1.05 \times 50,000 - 0.004ADB_t^c), \\ ADB_t^c &= \frac{1.2 \times 1.05 \times 50,000}{1 + 1.2 \times 0.004} = 62,699, \end{aligned}$$

and

$$AV_t = 1.05 \times 50,000 - 0.004 \times 62,699 = 52,249.$$

Then,

$$\frac{ADB_t^c}{AV_t} = 1.2,$$

as expected.

So the total death benefit is $52,249 + 62,699 = 114948$, which is greater than the Face Amount.

Example 5.7. For a Type A universal life insurance policy issued to (50), you are given:

(1) The total death benefit is $500,000$.
(2) Level annual premiums are 5000 payable at the beginning of each year.
(3) Death benefits are paid at the end of the year of death.
(4) The cost of insurance rates are 120% of the mortality from the Illustrative Life Table.
(5) $i^q = 0.03$ and $i^c = 0.045$ for all years.
(6) Expense charges payable at the start of each year: 75 plus 3.5% of each annual premium.
(7) Account values are calculated annually.
(8) The account value at the end of the first year is 1369.90.

Calculate the account value at the end of the second year.

Solution. From the table we get

$$q_{51} = 0.00642.$$

Then,

$$AV_2 = \left(AV_1 + 5000(1 - 0.035) - 75 - (500,000 - AV_2)\frac{1.2 \times 0.00642}{1.03} \right) \times 1.045,$$

$$AV_2 = \frac{(1369.90 + 5000 \times 0.965 - 75 - 500,000 \times 0.0075) \times 1.045}{1 - 0.0075 \times 1.045}$$
$$= 2496.1$$

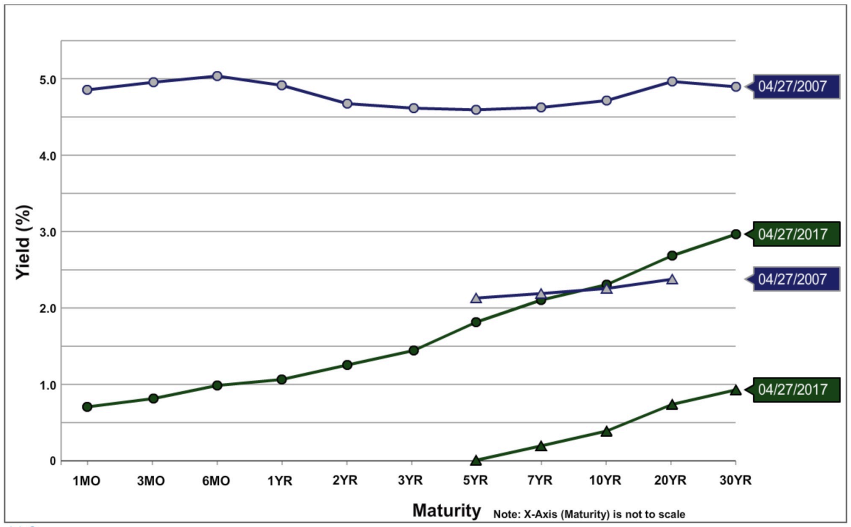

FIGURE 13. Yield curve for Treasury bonds, comparison of 4/27/2017 and 4/27/07. Circles are for nominal interest rate and triangles are for real interest rate (that is, the nominal interest rate minus inflation).

6. Interest rate risk

6.1. **The yield curve.** In practice, at any given time interest rates vary with the duration of the investment; that is, a sum invested for a period of, say, 5 years, would typically earn a different rate of interest than a sum invested for a period of 15 years or a sum invested for a period of six months.

Let $v(t)$ denote the current market price of a t-year zero-coupon bond; that is, the current market price of an investment which pays a unit amount with certainty t years from now. Note that, at least in principle, there is no uncertainty over the value of $v(t)$ although this value can change at any time as a result of trading in the market.

Definition 6.1. The t-year *spot rate of interest*, denoted y_t , is the yield per year on this zero-coupon bond, so that

$$v(t)(1+y_t)^t = 1, \text{ and}$$
$$v(t) = (1+y_t)^{-t}.$$

The *term structure* of interest rates describes the relationship between the term of the investment and the interest rate on the investment, and it is expressed graphically by the *yield curve*, which is a plot of the spot rate y_t against t.

The yield curve reflects investors' expectations about future interest rates and can have various shapes. The most common shape is a rising yield curve, flattening out after 10–15 years, with spot rates increasing at a decreasing rate. See Figure 13.

When we have a term structure, we should discount each future payment using the spot interest rate appropriate to the term until that payment is due. This is a replication argument: the present value of any cash flow is the cost of purchasing a portfolio which exactly replicates the cash flow.

Hence, $v(t)$ can be interpreted as a discount function which generalizes v^t.

Definition 6.2. *The forward rate of interest* $f(t, t+k)$ is the annualized interest rate of an investment contracted at time 0, made at time t, and maturing at $t+k$.

The forward rates of interest can be determined from the yield curve. The relevant equation is

$$(1 + y_{t+k})^{t+k} = (1 + y_t)^t f(t, t+k)^k,$$

which represents the intuition that the expected gain from the investment in t-year bonds followed by a contracted investment in k-year bonds should be equal to the direct investment in the $(t+k)$ - year bonds.

6.2. **Valuation of insurances and life annuities.** The valuation of insurances and life annuities with varying interest rate is similar to the case when the interest rate is fixed. The only difference is that we have to use the discount function $v(t)$ instead of v^t.

Here are two examples of this valuation.

The present value random variable for a life annuity-due with annual payments, issued to a life aged x, given a yield curve $\{y_t\}$, is

$$Y = \sum_{k=0}^{K_x} v(k) = \sum_{k=0}^{\infty} \mathbb{1}_{\{k \leq K_x\}} v(k)$$

By taking the expectation, we find the expected present value of the annuity, denoted $\ddot{a}_x(y)$,

$$\ddot{a}_x(y) = \sum_{k=0}^{\infty} {}_kp_x \, v(k)$$

Similarly, the present value random variable for a whole life insurance for (x), payable immediately on death, is

$$Z = v(T_x).$$

Since the density of the random variable T_x is ${}_tp_x\mu_{x+t}$, we find the expression for the expected present value,

$$\overline{A}_x(y) = \int_0^{\infty} v(t)_t p_x \mu_{x+t} \, dt$$

For a non-flat yield curve, many relationships that we have developed for flat interest rates, are lost, as for example, the formula linking $\ddot{a}_x$ and A_x.

6.3. **Diversifiable and non-diversifiable risk.** The yield curve represent expectations of future interest rates. In reality, these rates can deviate from the expectations. As a result the present value of the insurance contract can fluctuate.

Fluctuations of the value can also occur because of mortality variability. However these two sources of value fluctuations represent different types of risk: ***non-diversifiable*** and ***diversifiable***, respectively.

0.4 Consider a portfolio consisting of N life insurance policies. We can model as a random variable, X_i, $i = 1, \ldots, N$, many quantities of interest for the ith policy in this portfolio. For example, X_i could take the value 1 if the policyholder is still alive, say, 10-years after the policy was issued and the value zero otherwise. In this case, $S_n = \sum_{i=1}^{N} X_i$ represents the number of survivors after 10 years.

Alternatively, X_i could represent the present value of the loss on the i- th policy so that S_N represents the present value of the loss on the whole portfolio. Suppose that the X_is are independent, identically distributed with common mean μ and standard deviation σ. Then we have the central limit theorem that says that

$$\frac{S_n - \mu}{\sigma\sqrt{(N)}} \to \mathcal{N}(0,1),$$

where convergence is in distribution.

So, as N increases, the variation of the mean of the X_i from their expected value will tend to zero. In this case we can reduce the risk measured by X_i, relative to its mean value, by increasing the size of the portfolio.

This result relies on the fact that we have assumed that the X_i are independent; it is not generally true if $\rho \neq 0$, as in that case $\mathbb{V}\mathrm{ar}[S_n]$ is of order N_2, which means that increasing the number of policies increases the risk relative to the mean value.

Definition 6.3. the risk within portfolio, as measured by the random variable X_i, is called *diversifiable* if the following condition holds,

$$\lim_{N\to\infty} \frac{\sqrt{\mathbb{V}\mathrm{ar}(S_N}}{N} = 0.$$

A risk is *non-diversifiable* if this condition does not hold.

In simple terms, a risk is diversifiable if we can eliminate it (relative to its expectation) by increasing the number of policies in the portfolio. An important aspect of financial risk management is to identify those risks which can be regarded as diversifiable and those which cannot. Diversifiable risks are generally easier to deal with than those which are not.

Example 6.4. SoA Life simultaneously sells N two-year joint life term insurance policies, each with a death benefit of $10,000$ payable at the end of the year of death of the first death. The ages at issue are 65 and 70. The premiums are payable annually during the term of the policy.

You are given:

(i) The future lifetimes are independent.

(ii) $q_{65+t} = 0.05 + 0.01t$, for $t = 0, 1, 2, \ldots, 10$.

(iii) The spot rates are stochastic with the following distribution:

Scenario Number	Probability	One Year Spot Rate	Two Year Spot Rate
1	0.40	0.05	0.06
2	0.60	0.05	0.07

(iv) The annual premium per policy, P, is calculated using the equivalence principle.

(a) Calculate P.

Solution. Let $v(t)$ denote the discount function. Then, $v(1) = 1.05^{-1}$ under both scenarios.

$$v(2) = \begin{cases} 1.06^{-2} & \text{under Scenario 1,} \\ 1.07^{-2} & \text{under Scenario 2.} \end{cases}$$

Then we calculate the probabilities.

$$\begin{aligned} p_{65:70} &= p_{65} \times p_{70} = (1 - 0.05)(1 - 0.1) = 0.855, \\ q_{65:70} &= 0.145, \\ {}_2p_{65:70} &= (1 - 0.05)(1 - 0.06)(1 - 0.1)(1 - 0.11) = 0.7153, \\ {}_2q_{65:70} &= 0.2847, \\ {}_{1|}q_{65:70} &= {}_2q_{65:70} - q_{65:70} = 0.2847 - 0.145 = 0.1397, \end{aligned}$$

The EPV of premiums is the same for both scenarios and equals

$$P(1 + p_{65:70} \times 1.05^{-1}) = 1.81429P$$

For the first scenario, the EPV of benefits equals

$$10,000(q_{65:70} \times 1.05^{-1} + {}_{1|}q_{65:70} \times 1.06^{-2}) = 2624.3.$$

For the second scenario, it is

$$10,000(q_{65:70} \times 1.05^{-1} + {}_{1|}q_{65:70} \times 1.07^{-2}) = 2601.2.$$

Hence, the expected present value of the benefit is

$$0.4 \times 2624.3 + 0.6 \times 2601.2 = 2610.5,$$

and

$$P = \frac{2610.5}{1.81429} = 1438.82$$

$\square$

Let $L_{0,j}$ denote the loss-at-issue random variable for the j-th insurance policy, and let $L = \sum_{j=1} N L_{0,j}$.

(b)

(1) Calculate the expected value and variance of $L_{0,j}$ given that the spot rates follow scenario 1.
(2) Calculate the expected value and variance of $L_{0,j}$ given that the spot rates follow scenario 2.

Solution. In terms of the discount function $v(t)$, we have

$$L_{0,j} = \begin{cases} 10,000v(1) - P & \text{with probability } q_{65:70}, \\ 10,000v(2) - P(1 + v(1)) & \text{w. p. } {}_{1|}q_{65:70}, \\ -P(1 + v(1)) & \text{w. p. } {}_{2}p_{65:70}. \end{cases}$$

After calculation, we find that under scenario 1, we have

$$L_{0,j} = \begin{cases} 8085.0 & \text{with probability } 0.145, \\ 6090.1 & \text{w. p. } 0.13971, \\ -2809.2 & \text{w. p. } 0.71529. \end{cases}$$

Then,

$$\begin{aligned} \mathbb{E}[L_{0,j}|Sc.1] &= 8085.0 \times 0.145 + 6090.1 \times 0.13971 \\ &\quad - 2809.2 \times 0.71529 = 13.88 \\ \mathbb{V}\text{ar}[L_{0,j}|Sc.1] &= 8085.0^2 \times 0.145 + 6090.1^2 \times 0.13971 \\ &\quad + 2809.2^2 \times 0.71529\text{-}13.882^2 \\ &= 4506.2^2 \end{aligned}$$

For Scenario 2, we have

$$L_{0,j} = \begin{cases} 8085.0 & \text{with probability } 0.145, \\ 5925.3 & \text{w. p. } 0.13971, \\ -2809.2 & \text{w. p. } 0.71529, \end{cases}$$

which gives,

$$\mathbb{E}[L_{0,j}|Sc.2] = -9.25$$
$$\mathbb{V}\text{ar}[L_{0,j}|Sc.2] = 4475.2^2$$

□

(c)
Express $\mathbb{V}\text{ar}(L)$ in terms of N.

Solution. We use the following formula:

$$\mathbb{V}\text{ar}(L) = \mathbb{E}\big[\mathbb{V}\text{ar}(L|Sc)\big] + \mathbb{V}\text{ar}\big[\mathbb{E}(L|Sc)\big]$$

Note that given a scenario, the random variables $L_{0,j}$ are independent so we can write:

$$\mathbb{V}\text{ar}(L|Sc) = \mathbb{V}\text{ar}\Big(\sum_{j=1} NL_{0,j}|Sc\Big) = N\mathbb{V}\text{ar}(L_{0,j}|Sc).$$

We calculate

$$\mathbb{E}\big[\mathbb{V}\text{ar}(L|Sc)\big] = N(0.4 \times 4506.2^2 + 0.6 \times 4475.2^2) = 4487.6^2 N,$$

and

$$\mathbb{E}\big[\mathbb{E}(L|Sc)\big] = N(0.4 \times 13.88 + 0.6 \times (-9.25)) = 0,$$
$$\mathbb{V}\text{ar}\big[\mathbb{E}(L|Sc)\big] = N^2(0.4 \times 13.88^2 + 0.6 \times 9.25^2) = 128.4N^2.$$

Hence,

$$\mathbb{V}\text{ar}(L) = 4487.6^2 N + 128.4N^2.$$

□

(d) Demonstrate mathematically that the interest rate risk in these policies is not diversifiable.

Solution. We calculate:

$$\lim_{N\to\infty} \frac{\sqrt{\mathbb{V}\mathrm{ar}(L)}}{N} = \lim_{N\to\infty} \frac{\sqrt{4487.6^2 N + 128.4 N^2}}{N} = \sqrt{128.4} > 0.$$

By definition, this means that the risk is not diversifiable.

In this example, the risk is not diversifiable because the interest scenario applies to all policies simultaneously. It does not become negligible as the number of policies grows.

Example 6.5. ABC Life Insurance Company sells 5-year pure endowment policies of 1 to 100 independent lives age 85. You are given:

(i) Mortality follows the Illustrative Life Table.

(ii) There is a 60% chance that the force of interest over the next five years will be $\delta_t = 0.03\sqrt{t}$, for $0 \le t \le 5$, and there is a 40% chance that the force of interest over the next five years will be $\delta_t = 0.02$, for $0 \le t \le 5$.

(iii) The present value of benefits random variable for the portfolio of policies is denoted by Y.

(a) Calculate the mean of Y.

Solution. Let Y_j denote the PV of benefits for the j-th life,

$$Y = \sum_{j=1}^{100} Y_j,$$

and let $v(5)$ denote the discount factor for time 5.

Then,

$$\mathbb{E}(Y|v(5)) = 100\mathbb{E}(Y_j|v(5)) = 100 v(5)\,{}_5p_{85}.$$

From the table data, we calculate

$$100\,{}_5p_{85} = 100\frac{1,058,491}{2,358,246} = 44.885.$$

As for $v(5)$, we have

$$v(5) = \begin{cases} \exp\left(-\int_0^5 0.03 t^{1/2}\,dt\right) = 0.79963, & \text{with prob. } 0.6, \\ \exp\left(-\int_0^5 0.02\,dt\right) = 0.90484, & \text{with prob. } 0.4. \end{cases}$$

Hence

$$\begin{aligned}\mathbb{E}(Y) &= \mathbb{E}\big(\mathbb{E}(Y|v(5))\big)\\ &= 44.885 \times (0.79963 \times 0.6 + 0.90484 \times 0.4)\\ &= 37.780\end{aligned}$$

(b) Calculate the probability that Y is less than 30 using the normal approximation without continuity correction.

Solution. First, we compute the variance of Y by using the formula:

$$\mathbb{V}\text{ar}(Y) = \mathbb{E}\big(\mathbb{V}\text{ar}(Y|v(5))\big) + \mathbb{V}\text{ar}\big(\mathbb{E}(Y|v(5))\big).$$

We have

$$\mathbb{E}(Y|v(5) = 100v(5)_5p_{85}.$$

So,

$$\begin{aligned}\mathbb{V}\text{ar}\big(\mathbb{E}(Y|v(5))\big) &= 44.885^2 \times (0.79963^2 \times 0.6 + 0.90484^2 \times 0.4) - 37.78^2\\ &= 5.352,\end{aligned}$$

Since Y are independent given $v(5)$, we have

$$\begin{aligned}\mathbb{V}\text{ar}(Y|v(5)) &= 100\mathbb{V}\text{ar}(Y_j|v(5))\\ &= 100 \times v(5)^2 \times {}_5p_{85} \times (1 - {}_5p_{85})\end{aligned}$$

After some calculation, we find that

$$\mathbb{V}\text{ar}(Y|v(5)) = \begin{cases} 15.818, & \text{with prob. } 0.6,\\ 20.254, & \text{with prob. } 0.4.\end{cases}$$

Hence,

$$\mathbb{E}\big(\mathbb{V}\text{ar}(Y|v(5))\big) = 17.592,$$

and

$$\mathbb{V}\text{ar}(Y) = 5.352 + 17.592 = 22.944 = 4.790^2.$$

So, the required probability is

$$\begin{aligned}
\mathbb{P}(Y \leq 30) &= \Phi\left(\frac{30 - 37.78}{4.790}\right) \\
&= \Phi(-1.62) = 1 - \Phi(1.62) = 1 - 0.974 \\
&= 0.0526
\end{aligned}$$

(c) A colleague of yours claims that because the lives are independent, the risk for this portfolio of policies as measured by Y is diversifiable. State with reasons whether your colleague is correct.

Solution. The risk is not diversifiable, because the uncertainty about the discount factor cannot be eliminated by increasing the number of policies.