LIFE CONGINGENCY MODELS I

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Contents

♡ SECTION 1

Survival Models

The primary responsibility of the life insurance actuary is to maintain the solvency and profitability of the insurer.

Consider, for example, a whole life insurance contract issued to a life aged 50. This contract will pay a fixed sum at the death of the insured individual. This individual will pay monthly premiums, which will be invested by the insurer to earn interest; the accumulated premiums must be sufficient to pay the benefit, on average.

To ensure this goal, the actuary needs to model the survival probabilities of the policyholder, likely investment returns and likely expenses. The actuary may also take into consideration the probability that the policyholder decides to terminate the contract early.

In addition, the actuary must determine how much money the insurer should hold to ensure that future liabilities will be covered with adequately high probability.

To achieve these goals we need to have a mathematical model of a life, a human life or a work-life of an equipment.

section 1.1
Survival Function

Let a positive random variable X represent the duration of a life. We call X the *age-at-death* or *age-at-failure*.

Definition 1.1. The *survival function* of a positive random variable X is

$$S_X(t) = P\left(X > t\right) = 1 - F_X(t),$$

where $F_X\left(t\right)$ is the cumulative distribution function of X.

Intuitively, this is the probability to survive more than t units of time.

From the properties of the cumulative distribution function, it follows that the survival function is non-decreasing, $S\left(\infty\right) = 0$, and $S\left(0\right) = 1$.

Example 1.2. [1] Determine which of the following functions is a survival function of a nonnegative r.v.:

(i)
$$S(x) = \frac{2}{x+2},$$
(ii)
$$S(x) = \frac{1 + \frac{2}{x+2}}{2},$$
(iii)
$$S(x) = (1-x)e^{-x},$$
(iv)
$$S(x) = (1+x)e^{-x},$$
(v)
$$s(x) = \begin{cases} 1 - \frac{x^2}{10{,}000} & \text{for } 0 \le x \le 100, \\ 0 & \text{for } 100 < x. \end{cases}$$

Exercise 1.3. [2] Find the density function for the following survival functions:

(1) $S(x) = (1+x)e^{-x}$, for $x \ge 0$.
(2) $S(x) = \begin{cases} 1 - \frac{x^2}{10{,}000} & \text{for } 0 \le x \le 100, \\ 0 & \text{for } 100 < x. \end{cases}$
(3) $S(x) = \frac{2}{x+2}$, for $x \ge 0$.

section 1.2

Expectation and survival Function

Recall the usual definition of the expectation of a random variable.

Definition 1.4. The expectation of a function $g(X)$ of a random variable X is
$$\mathbb{E}[g(X)] = \sum_x g(x)\mathbb{P}\{X = x\} + \int g(x) f_X(x)\, dx,$$
where $f(x)$ is the density of X.

[1] Biespiel = Example in German
[2] Ubung = Exercise in German

This definition is general in the sense that it applicable for random variables that have both discrete and continuous components.

Often, to find expectations, instead of the density we will use the survival function.

Satz 1.5.

[a] *Let X be a non-negative random variable with survival function S that has a finite or countable number of discontinuities. Suppose that $H : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function which is differentiable everywhere except for a finite number of points. Then,*

$$\mathbb{E}[H(X)] = H(0) + \int_0^\infty S(t)H'(t)\,dt.$$

[a] Satz = Theorem in German

Proof. Given a set $A \subseteq \mathbb{R}$, the **indicator function** of A is the function

$$I_A(t) = I(t \in A) = \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{if } t \notin A \end{cases}$$

Since

$$H(x) - H(0) = \int_0^x H'(t)\,dt = \int_0^\infty I_{[0,x]}(t)H'(t)\,dt,$$

therefore

$$\mathbb{E}[H(X)] - H(0) = \int_0^\infty \mathbb{E}\left[I_{[0,X]}(t)\right] H'(t)\,dt.$$

Note that $\mathbb{E}\left[I_{[0,X]}(t)\right] = \mathbb{P}\{X \geq t\} = S(t) + \mathbb{P}\{X = t\}$. So,

$$\mathbb{E}[H(X)] - H(0) = \int_0^\infty S(t)H'(t)\,dt + \int_0^\infty \mathbb{P}\{X = t\}H'(t).dt$$

The second integral is zero because the function $\mathbb{P}\{X = t\}$ can be different from zero only at a finite or countable number of points. □

Corollary 1.6.

Let X be a nonnegative r.v. with survival function S. Then,

$$\mathbb{E}[X] = \int_0^\infty S(t)\,dt.$$

Corollary 1.7.

Let X be a nonnegative r.v. with survival function S. Then,

$$\mathbb{E}[X^p] = \int_0^\infty S(t)pt^{p-1}\,dt,$$

if $p > 0$.

Question: $E[X^2] = ?$

Corollary 1.8.

Let X be a non-negative r.v. with survival function S. Let $a \geq 0$. Then,

$$\mathbb{E}[\min(X, a)] = \int_0^a S(t)\,dt.$$

Corollary 1.9.

Let X be a non-negative r.v. with survival function S. Let $\delta > 0$. Then, $\mathbb{E}[e^{-\delta X}] = 1 - \int_0^\infty \delta e^{-\delta t} S(t)\,dt$.

Exercise 1.10. Let $S_X(x) = e^{-x}(x+1)$, $x \geq 0$.
$\mathbb{E}(X) = ?$ $\mathbb{E}(X \wedge 10) = ?$

Theorem 1.5 can be written in a different way if X is a discrete random variable.

Satz 1.11.

Let X be a discrete random variable whose possible values are positive integers. Suppose that $H : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function which is differentiable everywhere except for a finite number of points. Assume that $\Pr\{X = 0\} = 0$. *Then,*

$$\mathbb{E}[H(X)] = H(0) + \sum_{k=1}^{\infty} \Pr\{X \geq k\}\big[H(k) - H(k-1)\big].$$

Proof: by Theorem 1.5 on page 5,

$$\begin{aligned}
\mathbb{E}[H(X)] &= H(0) + \int_0^\infty S(t)H'(t)\,dt \\
&= H(0) + \sum_{k=1}^{\infty} \int_{t\in(k-1,k]} S(t)H'(t)\,dt
\end{aligned}$$

Next, we use the definition of $S(t)$ and the fact that $\Pr\{X > t\} = \Pr\{X \geq k\}$ for every $t \in (k-1, k]$.

$$\begin{aligned}
\mathbb{E}[H(X)] &= H(0) + \sum_{k=1}^{\infty} \Pr\{X \geq k\} \int_{k-1}^{k} H'(t)\,dt \\
&= H(0) + \sum_{k=1}^{\infty} \Pr\{X \geq k\}(H(k) - H(k-1)).
\end{aligned}$$

□

By taking $H(x) = x$, we find

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} \Pr\{X \geq k\} = \sum_{k=1}^{\infty} \big(S(k) + \Pr\{X = k\}\big)$$

Since $\sum_{k=1}^{\infty} \Pr\{X = k\} = \Pr\{X > 0\} = S(0)$, we obtain a useful formula:

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} S(k). \tag{1}$$

In a similar fashion we can get two analogous formulae:

$$E[X^2] = \sum_{k=1}^{\infty} \Pr\{X \geq k\}(k^2 - (k-1)^2) = \sum_{k=1}^{\infty} \Pr\{X \geq k\}(2k-1), \tag{2}$$

and

$$E[\min(X, n)] = \sum_{k=1}^{n} \Pr\{X \geq k\},\ n \geq 1. \tag{3}$$

Exercise 1.12. Find $E[X]$ and $E[X^2]$ if

k	0	1	2
$\Pr\{X = k\}$	0.2	0.3	0.5

section 1.3

Actuarial notation

Let (x) denote a life that survives to the age x. The life (x) is called a *life-age-x* or a *life aged x*.

Definition 1.13. The future lifetime of (x) is denoted by T_x. By definition, $T_x = X - x$.

The quantity T_x is especially important for an insurer and is called *time-until-death* of the life-age-x.

Note that T_x is defined only for a life which survives to age x. In particular T_x has the distribution of $X - x$ conditioned on the event $X > x$.

The survival function of T_x is denoted by ${}_tp_x$. It is the probability that a life aged x survives t more years.

$$
{}_tp_x = \mathbb{P}\{X > x + t | X > x\} = \frac{S(x+t)}{S(x)}, \tag{4}
$$

The cumulative distribution function of T_x is

$$
{}_tq_x := 1 - {}_tp_x = 1 - \frac{S(x+t)}{S(x)}.
$$

The density of T_x is

$$
f_{T_x}(t) = -\frac{d}{dt}\frac{S(x+t)}{S(x)} = \frac{f_X(x+t)}{S(x)}.
$$

Exercise 1.14. Consider the survival function $S_X(t) = \frac{90-t}{90}$, for $0 < t < 90$. Find the survival function and the probability density function of T_{30}.

Other notation:

- For simplicity, we denote ${}_1p_x$ and ${}_1q_x$ by p_x and q_x, respectively. For example, p_x is the probability that a life aged x survives one year. The variable q_x is often called *the mortality rate*.
- Given x, s, $t > 0$, ${}_{s|t}q_x$ represents the probability of a life just turning age x will die between ages $x + s$ and $x + s + t$, i.e.,

$$_{s|t}q_x = \mathbb{P}\{s < T_x \le s+t\} = \frac{S(x+s) - S(x+s+t)}{S(x)}.$$

These quantities are called the ***deferred mortality probabilities***. The symbol $s|t$ means deferred for s years and happening within the next t years.

From the definitions, it is clear that

$$_{s|t}q_x = {}_sp_x - {}_{s+t}p_x = {}_{s+t}q_x - {}_sq_x = {}_sp_x \times {}_tq_{x+s}.$$

- For simplicity, $_{s|1}q_x$ is denoted $_{s|}q_x$.

Since $_tp_x$ is the survival function of T_x, we can calculate the density of T_x using $_tp_x$.

$$f_{T(x)}(t) = -\frac{d}{dt}{}_tp_x.$$

Conversely,

$$_tp_x = \int_t^\infty f_{T(x)}(s)ds.$$

Exercise 1.15.
Suppose that $S(t) = \frac{85-t}{85}$, $0 \le t \le 85$.
(i) Calculate $_tp_{40}$, $0 \le t \le 45$.
(ii) Calculate the density function of T_{40}.

Exercise 1.16.
Suppose that $_tp_x = 1 - \frac{t}{90-x}$, $0 \le t \le 90 - x$.
(i) Find the probability that a 25-year-old reaches age 80.
(ii) Find the density of T_x.

Satz 1.17.
For each $t, s \geq 0$,

$$ {}_{t+s}p_x = {}_tp_x \cdot {}_sp_{x+t}. $$

More generally, for each $t_1, \ldots, t_m \geq 0$,

$$ {}_{t_1+\ldots+t_m}p_x = {}_{t_1}p_x \cdot {}_{t_2}p_{x+t_1} \cdot \ldots \cdot {}_{t_m}p_{x+t_1+\ldots+t_{m-1}}. $$

Proof: The first equation easily follows by using (4).

$$ \begin{aligned} {}_{t+s}p_x &= \frac{S(x+t+s)}{S(x)} = \frac{S(x+t+s)}{S(x+t)}\frac{S(x+t)}{S(x)} \\ &= {}_sp_{x+t} \cdot {}_tp_x, \end{aligned} $$

It expresses the fact that to survive for $t+s$ years you need to survive t years and then survive s more years. The second equation easily follow by induction. □

In particular, we have

$$ {}_np_x = p_xp_{x+1}\ldots p_{x+n-1}. $$

Also, we have identities

$$ {}_nq_x = {}_{0|}q_x + {}_{1|}q_x + \ldots + {}_{n-1|}q_x, $$

and

$$ {}_nq_x = {}_0p_xq_x + {}_1p_xq_{x+1} + \ldots + {}_{n-1}p_xq_{x+n-1}, $$

where we set ${}_0p_x = 1$.

The first identity reflects the fact that the event of dying before year $n+1$ is the disjoint union of the events of dying in one of the years $t = 1, \ldots, n$ and the probability that an individual aged-x dies at year t equals ${}_{t-1|}q_x$.

The second identity follows because ${}_{t-1|}q_x = {}_{t-1}p_xq_{x+t-1}$

Exercise 1.18.
(i) Suppose that probability that a 30-year-old reaches age 40 is 0.95, the probability that a 40-year-old reaches age 50 is 0.99, and the probability that a 50-year-old reaches age 60 is 0.95. Find the probability that a 30–year–old reaches age 60.
(ii) Suppose that probability that a 35-year-old reaches age 60 is 0.80,

and the probability that a 50-year-old reaches age 60 is 0.90. Find the probability that a 35-year-old will survive to age 50.

Exercise 1.19. Suppose that the survival function of a person is given by $S_X(x) = \frac{90-x}{90}$, for $0 \leq x \leq 90$. Given a married couple with husband aged 40 and wife aged 35, what is the probability that the husband will die before age 60 and the wife will survive to age 75?

Exercise 1.20. Suppose that

k	29	30	31	32	33	34	35	36
p_k	0.99	0.98	0.97	0.96	0.96	0.95	0.94	0.93

Find the probability that a 30-year old survives to age 35.

Exercise 1.21. Suppose that:
(i) The probability that a 30-year-old will die in less than one year is 0.012
(ii) The probability that a 31-year-old will die in less than one year is 0.013
(iii) The probability that a 32-year-old will die in less than one year is 0.014.
Find the probability that a 30-year-old will die in less than three years.

Exercise 1.22. Suppose that the survival function of a person is given by $S_X(x) = \frac{90-x}{90}$, for $0 \leq x \leq 90$
(i) Find ${}_{s|t}q_x$, for $0 < x, s, t,$ and $x + s + t \leq 90$.
(ii) Find the probability that a 30-year-old dies between ages 55 and 60.

Exercise 1.23. Suppose that the survival function of a person is given by $S_X(x) = \frac{90^6 - x^6}{90^6}$, for $0 \leq x \leq 90$
(i) Find ${}_{s|t}q_x$, for $0 < x, s, t,$ and $x + s + t \leq 90$.
(ii) Find the probability that a 30-year-old dies between ages 55 and 60.

Exercise 1.24. Suppose that:
(i) The probability that a 30-year-old will reach age 60 is 0.90.
(ii) The probability that a 30-year-old will reach age 50 is 0.95.
Find the probability that a 30-year-old will die between age 50 and 60.

Exercise 1.25. Suppose that:
(i) The probability that a 30-year-old will die between ages 40 and 60 is

0.20.
(ii) The probability that a 15-year-old will reach age 30 is 0.95.
Find the probability that a 15-year-old will die between ages 40 and 60.

section 1.4

Force of mortality

Definition 1.26. The **force of mortality** at age x is defined as

$$\mu_x = -\frac{d}{dt}\Big|_{t=0}({}_tp_x) = \frac{f_X(x)}{S_X(x)} = -\frac{d}{dx}\log S_X(x).$$

The above showed that the force of mortality can be computed if we know the survival function $S(x)$. In the other direction, by the Newton-Leibniz formula,

$$S_X(x) = \exp\left(-\int_0^x \mu_t\, dt\right).$$

The force of mortality μ_x gives us the rate at which lives aged x die off. If t is not large, the proportion of people age x who will die within t years is approximately $t\mu_x$.

For example, if $\mu_x = 0.06$, we expect that from each $1,000$ individuals with age x approximately $(1000)(0.06)t = 60t$ individuals will die within t years, where t is small. In particular, we expect that about $60\frac{1}{12} = 5$ individuals will die within a month.

Example 1.27. Find the force of mortality of the survival function $S_X(x) = \frac{90^6 - x^6}{90^6}$, for $0 < x < 90$.

We have the following characterization of the force of mortality functions.

Satz 1.28.
Let $\mu_t\colon \mathbb{R}^+ \to \mathbb{R}$ be a function which is continuous everywhere except at finitely many points. Then, μ_t is the force of mortality of X if and only if two conditions hold:
(i) $\mu_t \geq 0$ for all $t \geq 0$ except possibly the discontinuity points.
(ii) $\int_0^\infty \mu_t\, dt = \infty$.

Example 1.29. Determine which of the following functions is a legitimate force of mortality of an age-at-death:
(i) $\mu(x) = \frac{1}{(x+1)^2}$.

(ii) $\mu(x) = x \sin x$.
(iii) $\mu(x) = 35$.

Example 1.30. For the force of mortality $\mu(x) = \frac{1}{x+1}$ find S_X, f_X, ${}_tp_x$ and $f_T(x)$.

Satz 1.31.
If X is continuous, then the force of mortality of $T(x)$ is

$$\mu_{T(x)}(t) = \mu_X(x+t).$$

In words, for a life aged x, the force of mortality t years later is the force of mortality for a (x + t)-year old.

Proof: The survival function of $T(x)$ is $\frac{S_X(x+t)}{S_X(x)}$. Hence,

$$\mu_{T(x)}(t) = -\frac{d}{dt} \log \frac{S_X(x+t)}{S_X(x)} = \frac{f_X(x+t)}{S_X(x+t)} = \mu_X(x+t).$$

□

Example 1.32. You are given that ${}_tq_{40} = \frac{40t+t^2}{6000}$ for $0 \le t \le 60$. Find $\mu(60)$.

Example 1.33. Suppose that the survival function of a new born is $S_X(t) = \frac{85^4 - t^4}{85^4}$, for $0 < t < 85$.
(i) Find the force of mortality function of a new born.
(ii) Find the force of mortality function of a life aged 20.

section 1.5

Expectation of Life

Definition 1.34. The **life expectancy** of a newborn is $\mathring{e}_0 = \mathbb{E}[X]$.

The expectation $\mathring{e}_0$ is also called the **complete expectation of life at birth**. The following theorem follows from Corollaries 1.6 and 1.7.

Satz 1.35.
Let X be an age-at-death random variable. Then,

$$\mathring{e}_0 = \int_0^\infty S(x)\,dx$$

and

$$\mathbb{E}X^2 = \int_0^\infty 2xS(x)\,dx.$$

Using the previous theorem, we can find the variance of X, $\mathbb{V}\mathrm{ar}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$.

Example 1.36. An actuary models the lifetime in years of a random selected person as a random variable X with survival function $S_X(x) = \frac{90^6 - x^6}{90^6}$, for $0 < x < 90$. Find $\mathring{e}_x$ and $\mathbb{V}\mathrm{ar}(X)$ using the survival function of X.

Definition 1.37. The **expected future lifetime at age** x is $\mathring{e}_x = \mathbb{E}[T(x)] = E[X - x | X > x]$.

The expectation $\mathring{e}_x$ is also called the **complete expectation** of a life at age x.

Satz 1.38.

$$\mathring{e}_x = \int_0^\infty {}_tp_x\,dt = \int_0^\infty \frac{S(x+t)}{S(x)}\,dt.$$

Proof. It is enough to notice that ${}_tp_x$ is the survival function for the life aged x. □

Example 1.39. An actuary models the lifetime in years of a random selected person as a random variable X with survival function $S(x) = \frac{90^6 - x^6}{90^6}$, for $0 < x < 90$. Find $\mathring{e}_{30}$ using the survival function of $T(30)$.

Definition 1.40. The n-**year temporary complete life expectancy** is the expected number of years lived between age x and age $x+n$ by a survivor aged x. It is denoted by $\mathring{e}_{x:\overline{n}|}$.

Note that the number of years lived between age x and age $x+n$ by a survivor aged x is

$$\begin{Bmatrix} T(x), & \text{if } T(x) < n \\ n, & \text{otherwise.} \end{Bmatrix} = \min\{T(x), n\}.$$

Hence, $\mathring{e}_{x:\overline{n}|} = \mathbb{E}\min\{T(x), n\}$.

Satz 1.41.

$$\mathring{e}_{x:\overline{n}|} = \int_0^n {}_tp_x\,dt.$$

Proof. The function ${}_tp_x$ is the survival function of $T(x)$. Hence, by Corollary 1.8,

$$\mathbb{E}\min\{T(x), n\} = \int_0^n {}_tp_x\,dt.$$

□

Satz 1.42.

$$\mathring{e}_x = \mathring{e}_{x:\overline{n}|} + {}_np_x\mathring{e}_{x+n}$$

Intuitively, this result says that the expected number of years lived after age x equals the (expected) number of years lived from age x to age $x+n$ plus number of years lived after the age $x+n$ by a survivor at age $x+n$ multiplied by the probability that the individual survives to the age $x+n$.

Proof.

$$\begin{aligned}
\mathring{e}_x &= \int_0^\infty {}_tp_x\,dt = \int_0^n {}_tp_x\,dt + \int_n^\infty {}_tp_x\,dt \\
&= \mathring{e}_{x:\overline{n}|} + \int_n^\infty {}_np_x \cdot {}_{t-n}p_{x+n}\,dt \\
&= \mathring{e}_{x:\overline{n}|} + {}_np_x \int_0^\infty {}_tp_{x+n}\,dt = \mathring{e}_{x:\overline{n}|} + {}_np_x\mathring{e}_{x+n}.
\end{aligned}$$

□

This theorem can be generalized as follows.

Satz 1.43.
For $0 < m < n$,

$$\mathring{e}_{x:\overline{n}|} = \mathring{e}_{x:\overline{m}|} + {}_m p_x \mathring{e}_{x+m:\overline{n-m}|}$$

The proof is similar.

Example 1.44. An actuary models the lifetime in years of a random selected person as a random variable X with survival function $S(x) = \frac{90^6 - x^6}{90^6}$, for $0 < x < 90$. Find $\mathring{e}_{30:\overline{10}|}$.

Example 1.45. Assume that
(i) The expected future lifetime of a 40-year old is 45 years.
(ii) The expected future lifetime of a 50-year old is 36 years.
(iii) The probability that a 40-year old survives to age 50 is 0.98.
Find the expected number of years lived between age 40 and age 50 by a 40-year old.

Example 1.46. You are given that:
(i) The expected number of years lived between age 40 and age 50 by a 40-year old is 9.7.
(ii) The probability that a 40-year old survives to age 50 is 0.98.
(iii) The expected number of years lived between age 50 and age 70 by a 50-year old is 19.5.
Find the expected number of years lived between age 40 and age 70 by a 40-year old.

section 1.6
Future curtate lifetime

Definition 1.47. The *future curtate lifetime* of a life age x is the random variable K_x equal to the number of complete future years lived by this life.

Clearly,

$$K_x = \lfloor T_x \rfloor .$$

Definition 1.48. The *curtate life expectation* of a life-age-x is the expectation of the curtate duration of this life, that is, $\mathbb{E}[K(x)]$. We denote the curtate life expectation by e_x.

Since the curtate life-time is less than life-time, $K_x \le T_x$, hence $e_x \le \mathring{e}_x$.

Definition 1.49. The expected number of whole years lived in the interval $(x, x+n]$ by an entity alive at age x is denoted by $e_{x:\overline{n}|}$.

Obviously $e_{x:\overline{n}|} \le \mathring{e}_{x:\overline{n}|}$.

Satz 1.50.

$$e_x = \mathbb{E}[K(x)] = \sum_{k=1}^{\infty} k \cdot {}_{k|}q_x = \sum_{k=1}^{\infty} {}_kp_x,$$

and

$$\mathbb{E}[K(x)^2] = \sum_{k=1}^{\infty} k^2 \cdot {}_{k|}q_x = \sum_{k=1}^{\infty} (2k-1) \cdot {}_kp_x.$$

Proof. Note that $\mathbb{P}(K(x) = k) = {}_{k|}q_x$ and $\mathbb{P}(K(x) > k) = {}_kp_x$. Then the first equalities hold by the definition of the expected values, and the second equalities follow from formulas (1) and (2) on page 7. □

It follows that

$$e_x = \sum_{k=1}^{\infty} \prod_{j=0}^{k-1} p_{x+j}.$$

Example 1.51. Suppose that

x	90	91	92	93	94
p_x	0.2	0.1	0.05	0.01	0

Calculate e_{90}.

Example 1.52. Suppose that $s(t) = \frac{100-t}{100}$, for $0 \le t \le 100$. Find $\mathring{e}_x$ and e_x, where $1 \le x \le 99$ is an integer.

The analogue of Theorem 1.50 for $e_{x:\overline{n}|}$ is as follows.

Satz 1.53.

$$e_{x:\overline{n}|} = \sum_{k=1}^{n-1} k \cdot {}_{k|}q_x + n \cdot {}_n p_x = \sum_{k=1}^{n-1} {}_k p_x.$$

Hence,

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} \prod_{j=0}^{k-1} p_{x+j}.$$

For example,

$$e_{x:\overline{3}|} = p_x + p_x p_{x+1} + p_x p_{x+1} p_{x+2}.$$

Satz 1.54.

$$e_x = e_{x:\overline{n}|} + {}_n p_x e_{x+n}.$$

Example 1.55. Suppose that $e_x = 30$, $p_x = 0.97$ and $p_{x+1} = 0.95$. Find e_{x+2} using Theorem 1.54.

Satz 1.56 —Iterative formulas for e_x and $e_{x:\overline{n}|}$ —.

$$e_x = p_x(1 + e_{x+1})$$
$$e_{x:\overline{n}|} = p_x(1 + e_{x+1:\overline{n-1}|}).$$

Example 1.57. Suppose that $e_x = 30$, $p_x = 0.97$ and $p_{x+1} = 0.95$. Find e_{x+2} using Theorem 1.56.

Satz 1.58.
If $p_{x+k} = p$, for each integer $k \geq 1$. Then,

$$e_x = \frac{p}{1-p},$$
$$e_{x:\overline{n}|} = \frac{p - p^{n+1}}{1-p}.$$

Example 1.59. Let $p_{x+k} = p$, for each integer $k > 0$. Find e_x for $p = 0.92, 0.95, 0.96, 0.98, 0.99$.

Section 1.7
Common analytical survival models

Today, actuaries mostly rely on actuarial tables. Still, one should know the most common analytical survival models. They are De Moivre, generalized De Moivre, exponential, Gompertz, Makeham, and Weibull models.

1.7.1 De Moivre model

Under *De Moivre's model*, deaths happen uniformly on the interval $[0, \omega]$, where ω is the terminal age. The density of the age-at-death is $f_X(x) = \frac{1}{\omega}$, for $0 \leq x \leq \omega$.

It follows that the survival and mortality functions are

$$S_X(x) = 1 - \frac{x}{w}$$
$$\mu(x) = \frac{1}{\omega - x}$$

Example 1.60. Suppose that the De Moivre's law holds and the force of mortality of 70- year old is three times the force of mortality of 35-year old. Calculate the terminal age.

Satz 1.61.
For De Moivre's mortality law, $T(x)$ has a uniform distribution over the interval $[0, \omega - x]$.

Proof.

$$ {}_tp_x = \frac{s(x+t)}{s(x)} = \frac{\omega - x - t}{\omega - x} $$

Hence, $T(x)$ has a uniform distribution over the interval $[0, \omega - x]$. □

Example 1.62. Find the median of the age-at-death random variable subject to De Moivre's law if the probability that a life aged 20 years survives 40 years is $\frac{1}{3}$.

Satz 1.63.
For De Moivre's law,

$$ \overset{\circ}{e}_x = \frac{w - x}{2}, $$

$$ \mathbb{V}\mathrm{ar}(T(x)) = \frac{(w-x)^2}{12}. $$

Proof. It follows noticing that $T(x)$ has a uniform distribution over the interval $[0, \omega - x]$. □

Example 1.64. Suppose that the survival of a cohort follows De Moivre's law. Suppose that the expected age-at-death of a new born is 70 years. Find the mean and the variance of the future lifetime of a 50-year old.

Example 1.65. Suppose that mortality follows de Moivre's law. If $\overset{\circ}{e}_{3:\overline{15|}} = 13.125$, calculate $\overset{\circ}{e}_{30:\overline{30|}}$.

Example 1.66. Suppose that mortality follows Moivre's law with terminal age 110.
(i) Calculate $\overset{\circ}{e}_{x:\overline{5|}}$, for $x = 20$ and $x = 100$.
(ii) Calculate $\overset{\circ}{e}_{50:\overline{n|}}$, for $n = 5$ and $n = 30$.

Satz 1.67.
Let the mortality follow the De Moivre law with terminal age $\omega \in \mathbb{N}$, and suppose that x is an integer between 0 and ω. Then,

(1) $K(x)$ *has a uniform distribution over* $0, 1, 2, \dots, \omega - x - 1$.

(2)
$$ e_x = \frac{\omega - x - 1}{2}. $$

(3)
$$ \mathbb{V}\mathrm{ar}(K(x)) = \frac{(\omega - x)^2 - 1}{12}. $$

Proof.

$$\mathbb{P}\{K(x) = k\} = \mathbb{P}\{k \le T_x < k+1\} = \int_k^{k+1} \frac{1}{\omega - x}\, dx = \frac{1}{\omega - x},$$

which does not depend on k. This proves (1) and (2) and (3) follow from (1) by an easy calculation. □

Note that it follows that for the De Moivre law and integer ω and x, the excess of the future lifetime of (x) over the curtate lifetime of (x) is uniformly distributed on the interval $(0,1)$, Hence, $\overset{\circ}{e}_x - e_x = \frac{1}{2}$.

Example 1.68. Assume the De Moivre law where ω is an integer. The total curtate expected lifetime of a 40-year old and a 60-year old is 46 years. Find ω.

1.7.2 Constant Force of Mortality

The *constant force model* assumes that the mortality rate μ is constant throughout the life.

By integration, the survival function of the constant force model is

$$S(x) = \exp\left(-\int_0^x \mu dt\right) = e^{-\mu x}.$$

For this reason the model is also called *exponential model*.

The constant force model is used in actuarial sciences mostly for interpolation between two known points.

That is, suppose that $S(x)$ and $S(x+n)$ are known and assume that the mortality rate between ages x and $x+n$ is constant. Then the survival function between these ages can be interpolated by

$$S(x+t) = S(x)e^{-\mu t},$$

where $\mu = \log\left(\frac{S(x)}{S(x+n)}\right)$.

Satz 1.69.
Under constant force of mortality, the lifetime X of a new born and the future lifetime $T(x)$ of (x) have the same distribution.

Proof.

$$_tp_x = \frac{S(x+t)}{S(x)} = \frac{e^{-\mu(x+t)}}{e^{-\mu x}} = e^{-\mu t}.$$

That is, the survival function of $T(x)$ is the same as for X. Hence the distributions are the same. $\square$

Example 1.70. Suppose that:
(i) the force of mortality is constant.
(ii) the probability that a 30-year-old will survive to age 40 is 0.95.
Calculate:
(i) the probability that a 40-year-old will survive to age 50.
(ii) the probability that a 30-year-old will survive to age 50.
(iii) the probability that a 30-year-old will die between ages 40 and 50.

Example 1.71. Suppose that:
(i) the force of mortality is constant.
(ii) the probability that a 30-year-old will survive to age 40 is 0.95. Calculate: (i) the expected future lifetime of a 40-year-old. (ii) the curtate life expectation of a 40-year-old.

1.7.3 Mixture distributions

Many distributions are defined by using the conditional probability. Suppose that we know the distribution of $X|Y$ and the distribution of Y. Then we can find the distribution of X. The distribution of X is called a *mixture* distribution.

The conditional expectation $\mathbb{E}(X|Y)$ is a function of the random variable Y and therefore it is itself is a random variable. Hence, we can calculate its expected value.

Satz 1.72 —Iterated Expectations —.

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y)).$$

Satz 1.73 —Iterated Variances —.

$$\mathbb{V}\mathrm{ar}(X) = \mathbb{V}\mathrm{ar}(\mathbb{E}(X|Y)) + \mathbb{E}(\mathbb{V}\mathrm{ar}(X|Y)).$$

Sometimes, these theorems are not sufficient and we need the explicit expression for the density of a mixture distribution.

Satz 1.74.
Suppose that random variable Y takes on the values $1, 2, \ldots, m$, the conditional distribution of X given $Y = j$ has pdf f_j and $\mathbb{P}\{Y = j\} = p_j$. Then, random variable X has pdf $f = \sum_j p_j f_j$.

Example 1.75. You are given that:
(a) Men follow a de Moivre model with terminal age 100.
(b) Women follow a de Moivre model with terminal age 110.
(c) 55% of births are male.
(i) Calculate the expectation life of a randomly chosen life.
(ii) Calculate the probability that a newborn survives 80 years.
(iii) Calculate the density of the future lifetime T of a randomly chosen life.

Another case occurs when the conditioning variable has a continuous distribution.

Satz 1.76.
Suppose that $X|\theta$ has pdf $f(x|\theta)$ and that θ has pdf h. Then, X has pdf

$$f_X(x) = \int_{\mathbb{R}} f(x|\theta)h(\theta)d\theta.$$

Example 1.77. The future lifetime $T(x)$ of a live aged (x) has constant force of mortality μ. Suppose that μ has a uniform distribution on $(0.01, 0.05)$.
(i) Calculate $\overset{\circ}{e}_x$.
(ii) Calculate $\mathbb{V}\text{ar}(T(x))$.

SECTION 2

Using Life Tables

section 2.1

Life Tables

A life table is a bookkeeping system used to keep track of mortality. Suppose that we track a cohort of lives. At the end of each year, we keep track of the number of survivors.

Definition 2.1. The number of individuals alive at age x is called the *number living* or the *number of lives* at age x, and denoted l_x.

Notice that l_0 denotes the initial number of lives, and is called the *radix* of a life table.

There are two types of life tables: the *cohort* (or *generation*) life table and the *period* (or *current*) life table.

The cohort life table presents the mortality experience of a particular birth cohort – all persons born in the year 1900, for example – from the moment of birth through consecutive ages in successive calendar years. It is usually not feasible to construct cohort life tables entirely on

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Table 1. Life table for the total population: United States, 2010

Spreadsheet version available from: ftp://ftp.cdc.gov/pub/Health_Statistics/NCHS/Publications/NVSR/63_07/Table01.xlsx.

Age	Probability of dying between ages x to x+1 q(x)	Number surviving to age x l(x)	Number dying between ages x to x+1 d(x)	Person-years lived between ages x to x+1 L(x)	Total number of person-years lived above age x T(x)	Expectation of life age x e(x)
0-1	0.006123	100,000	612	99,465	7,866,027	78.7
1-2	0.000428	99,388	43	99,366	7,766,561	78.1
2-3	0.000275	99,345	27	99,331	7,667,195	77.2
3-4	0.000211	99,318	21	99,307	7,567,864	76.2
4-5	0.000158	99,297	16	99,289	7,468,556	75.2
5-6	0.000145	99,281	14	99,274	7,369,267	74.2
6-7	0.000128	99,267	13	99,260	7,269,993	73.2
7-8	0.000114	99,254	11	99,249	7,170,733	72.2
8-9	0.000100	99,243	10	99,238	7,071,484	71.3
9-10	0.000087	99,233	9	99,229	6,972,246	70.3
10-11	0.000079	99,224	8	99,220	6,873,017	69.3
11-12	0.000086	99,216	9	99,212	6,773,797	68.3
12-13	0.000116	99,208	12	99,202	6,674,585	67.3

FIGURE 1. A life table for the USA population in 2010

the basis of observed data for real cohorts due to data unavailability or incompleteness. For example, a life table for a cohort of persons born in 1970 would require the use of data projection techniques to estimate deaths into the future.

Unlike the cohort life table, the period life table does not represent the mortality experience of an actual birth cohort. Rather, the period life table presents what would happen to a hypothetical cohort if it experienced throughout its entire life the mortality conditions of a particular period in time.

Figure 1 is an example of a life table from the website of the Centers for the Decease Control and Prevention (www.cdc.gov).

This is a period life table for 2010. It assumes a hypothetical cohort that is subject throughout its lifetime to the age-specific death rates prevailing for the actual population in 2010. The period life table may thus be characterized as rendering a "snapshot" of current mortality experience and shows the long-range implications of a set of age-specific death rates that prevailed in a given year.

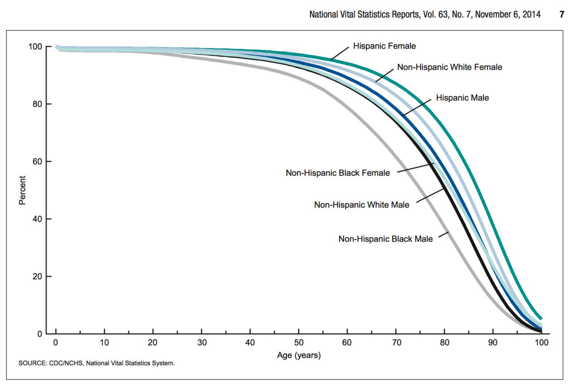


FIGURE 2. Percent surviving by Hispanic origin, race, age and sex: United States, 2010

The life tables can be different for different segments of population. To illustrate this, Figure 2 shows the survival function for the segments of the USA population based on race and gender.

One can clearly see that the survival function for Hispanic Females dominates all other survival functions. That is, the probability to survive to a specific age is highest for Hispanic Females for every age. This probability is lowest for Non-Hispanic Black Males.

Definition 2.2. The number of people which died between ages x and $x+t$ is denoted as ${}_t d_x = l_x - l_{x+t}$. In particular, $d_x \equiv {}_1 d_x$ denotes the number of people which died between ages x and $x+1$, i.e. $d_x = l_x - l_{x+1}$.

From a life table, we can estimate probabilities and expectations related with mortality. For example, we can estimate the survival function of the age-at-death X by $s(x) = l_x/l_0$.

Here are several actuarial variables which can be obtained from a life table:

$$
\begin{aligned}
{}_t p_x &= \frac{l_{x+t}}{l_x}, \\
{}_t q_x &= 1 - \frac{l_{x+t}}{l_x}, \\
p_x &= \frac{l_{x+1}}{l_x}, \\
q_x &= 1 - \frac{l_{x+1}}{l_x} = \frac{d_x}{l_x}, \\
{}_{n|m} q_x &= \frac{l_{x+n} - l_{x+n+m}}{l_x}.
\end{aligned}
$$

Notice that a life table only contains values for non-negative integers. This will make a challenge to estimate quantities which depend on a continuous set of values of the survival function. We will consider this problem later.

Example 2.3. Complete the entries in the following table:

Age	l_x	d_x	p_x	q_x
0	100,000	·	·	·
1	97,523	·	·	·
2	94,123	·	·	·
3	91,174	·	·	·
4	87,234	·	·	·
5	85,938	–	–	–

Satz 2.4.
For $k, n \geq 0$,

$$l_{k+n} = l_k \cdot p_k \cdot p_{k+1} \cdots p_{k+n-1}.$$

Example 2.5. Using a life table, find:

(1) l_{10}.
(2) d_{35}.
(3) ${}_5d_{35}$.
(4) The probability that a newborn will die before reaching 50 years.
(5) The probability that a newborn will live more than 60 years.
(6) The probability that a newborn will die when his age is between 45 years and 65 years old.
(7) The probability that a 25-year old will die before reaching 50 years.
(8) The probability that a 25-year old will live more than 60 years.
(9) The probability that a 25-year old will die when his age is between 50 years and 65 years old.

From a life table, we can find the distribution of the curtate life $K(x)$.

$$\begin{aligned}\mathbb{P}\{K_x = k\} &= \mathbb{P}\{k < T(x) \leq k+1\} \\ &= \frac{l_{x+k} - l_{x+k+1}}{l_x} \\ &= \frac{d_{x+k}}{l_x},\end{aligned}$$

where $k = 0, 1, \ldots$.

From Theorem 1.50 on page 18, it follows that

$$e_x = \sum_{k=1}^{\infty} \frac{l_{x+k}}{l_x}, \tag{5}$$

$$\mathbb{E}[K(x)^2] = \sum_{k=1}^{\infty} (2k-1) \frac{l_{x+k}}{l_x}$$

In addition,

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} \frac{l_{x+k}}{l_x}. \tag{6}$$

Example 2.6. Consider the life table

x	80	81	82	83	84	85	86
l_x	250	217	161	107	62	28	0

(1) Calculate d_x, $x = 80, 81, \ldots, 86$.
(2) Calculate the probability mass function of the curtate life $K(80)$.
(3) Calculate e_{80} and $\mathbb{V}\text{ar}(K(80))$ using the probability mass function of $K(80)$.
(4) Calculate e_{80} and $\mathbb{V}\text{ar}(K(80))$ using (5).
(5) Calculate $e_{80:\overline{3}|}$ using (6).

section 2.2

Continuous Calculations Using Life Tables

The main use of life tables is being an input to mathematical models.

A mathematical model is a sketch that represents a problem in a simpler way. It is a simplified version of a situation in the real world.

Most automobile insurance companies use mathematical models to determine your monthly payment. Information about your age, how far you drive to work or school, what kind of car you have, how old it is, etc is collected. Then, based on this information, you get an insurance rate. This insurance rate depends on how likely you are to file a claim on your insurance. The more likely the computer thinks you are to file a claim, the higher your insurance rate.

Similar models are used for life insurance and home-owners insurance policies.

One problem is that many mathematical models use continuous time and the life tables are able to provide only discrete-time data. A standard approach to this problem is to determine l_x for each real number $x \geq 0$ by interpolation. Namely, we use l_x and l_{x+1}, where x is an integer, and approximate l_{x+t} by interpolating for $t \in (0, 1)$.

Next, knowing l_x for all real positive x, we can find the survival function ${}_tp_x = l_{x+t}/l_x$. Then, all the theory from Section 1 can be applied.

In practice, the most popular is the linear interpolation, which corresponds to the arithmetic average of two numbers. Another method which is used less frequently is the exponential, that corresponds to the geometric average. The differences between the results of these methods is small.

2.2.1 Linear interpolation

First, we consider *linear interpolation* for a function f. Suppose that we know the values of f, for $x_1 < x_2 < \ldots < x_n$.

We estimate the value of f at $x \in [x_{j-1}, x_j]$ by drawing a straight line between points $(x_{j-1}, f(x_{j-1}))$ and $(x_j, f(x_j))$ and choosing $f(x)$ in such a way that the point $(x, f(x))$ is also on this straight line.

This implies that

$$\frac{f(x) - f(x_{j-1})}{x - x_{j-1}} = \frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}},$$

and therefore,

$$f(x) = \frac{x_j - x}{x_j - x_{j-1}} f(x_{j-1}) + \frac{x - x_j}{x_j - x_{j-1}} f(x_j)$$

Example 2.7. Suppose that $f(0) = 10$, $f(2) = 14$, $f(5) = 26$. Using linear interpolation determine $f(x)$, $0 \le x \le 5$.

The linear interpolation model is also called "the uniform distribution of deaths" (UDD) model, and the survival function in this model can be written as

$$l_{x+t} = l_x + t(l_{x+1} - l_x) = l_x - t \cdot d_x,$$

where $0 \le t \le 1$.

Example 2.8. Using a life table and assuming a uniform distribution of deaths, find:
(i) ${}_{0.5}p_{35}$,
(ii) ${}_{1.4}p_{35.3}$.

Satz 2.9.
Given $t \ge 0$, let k be the nonnegative integer such that $k \le t < k+1$. Then, under uniform interpolation,

$$s(t) = \frac{1}{l_0}(l_k - (t-k)d_k).$$

$$f_X(t) = \frac{d_k}{l_0} = {}_{k|}q_0.$$

$$f_{T(x)}(t) = \frac{d_{x+k}}{l_x} = {}_{k|}q_x.$$

Example 2.10. Consider the life table

x	80	81	82	83	84	85	86
l_x	250	217	161	107	62	28	0

(i) Calculate d_x, $x = 81, 82, \ldots, 86$.
By using linear interpolation, calculate:
(ii) l_{80+t}, $0 \le t \le 6$,
(iii) ${}_tp_{80}$, $0 \le t \le 6$.
(iv) the density function of the future life T_{80}.

Satz 2.11.
Under a linear form for the number of living,

(1)
$$\mathring{e}_{x:\overline{1}|} = \frac{1+p_x}{2},$$

(2)
$$\mathring{e}_{x:\overline{n}|} = \frac{{}_nL_k}{l_x} = \frac{1}{l_x}\sum_{k=x}^{x+n-1} L_k = \sum_{k=x}^{x+n-1} \frac{l_k + l_{k+1}}{2l_x}$$
.

(3)
$$\mathring{e}_x = e_x + \frac{1}{2},$$

Example 2.12. Consider the life table

x	80	81	82	83	84	85	86
l_x	250	217	161	107	62	28	0

Assume linear interpolation and calculate:
(i) the complete expected life at 80 using that $\mathring{e}_x = e_x + \frac{1}{2}$;
(ii) $\mathring{e}_{80:\overline{3}|}$;

Recall that $S_x = T(x) - K(x)$, where $K(x)$ is the curtate life duration.

Satz 2.13.
Under UDD, $K(x)$ and S_x are independent random variables for each integer x, and S_x has a distribution uniform on $(0, 1)$.

A useful consequence of this theorem is the following result.

Corollary 2.14.
Under the assumption of uniform distribution of deaths (UDD):
(i)
$$\mathring{e}_x = e_x + \frac{1}{2},$$
(ii)
$$\mathbb{V}\mathrm{ar}(T(x)) = \mathbb{V}\mathrm{ar}(K(x)) + \frac{1}{12}.$$

2.2.2 Exponential interpolation

Under exponential interpolation,

$$\ln l_{x+t} = (1-t)\ln l_x + t\ln l_{x+1},$$

which is the formula for the linear interpolation of the function $\ln l_t$, $t \in [0,1]$.

This expression is equivalent to

$$\ln l_{x+t} = \ln l_x + t\ln \frac{l_{x+1}}{l_x},$$

and therefore

$$l_{x+t} = l_x(p_x)^t.$$

Satz 2.15.
Under an exponential form for the number of living, for each nonnegative integer x *and each* $t \in [0,1]$*:*

(1) ${}_tp_x = (p_x)^t$.
(2) $\mu(x+t) = -\ln p_x$.

By the last point in Theorem 2.15, the force of mortality is a constant in the interval $(x, x+1)$. Hence, the exponential interpolation is also called the *constant force of mortality* interpolation.

Example 2.16. Consider the life table

x	80	81	82	83	84	85	86
l_x	250	217	161	107	62	28	0

Using the exponential interpolation, find ${}_{0.75}p_{80}$ and ${}_{2.25}p_{80}$.

Satz 2.17.
Under the exponential interpolation for l_{x+t},

(1)
$$\mathring{e}_{x:\overline{1}|} = \frac{1}{l_x}\frac{d_x}{-\ln p_x}.$$

(2)
$$\mathring{e}_x = \frac{1}{l_x}\sum_{k=x}^{\infty}\frac{d_k}{-\ln p_k}.$$

(3)
$$\mathring{e}_{x:\overline{n}|} = \frac{1}{l_x}\sum_{k=x}^{x+n-1}\frac{d_k}{-\ln p_k}.$$

Example 2.18. Consider the life table

x	80	81	82	83	84	85	86
l_x	250	217	161	107	62	28	0

Using exponential interpolation, calculate $\mathring{e}_{80}$ and $\mathring{e}_{80:\overline{3}|}$.

section 2.3

Select and ultimate tables

A *select table* is a mortality table for a group of people subject to a special circumstance (disability, retirement, etc.).

For the term insurance, an important circumstance is the age at which an individual entered the group of policyholders, that is, at which the insurance was purchases.

Indeed, passing the health checks at age x indicates that the individual is in a good health and so has lower mortality rate than someone who has passed health checks several years ago.

However, the initial selection effect is assumed to have worn off after a period of m years which is called the *select period*. After this period the mortality follows the general pattern and recorded in the *ultimate table*.

Here is another example of when the select table of mortality is appropriate.

Example 2.19. Consider men who need to undergo a complicated surgery. The probability that they will survive for a year following surgery is only 50%. If they do survive for a year, then they are fully cured and their future mortality follows the pattern of general population.

Suppose that for the general population, we have $l_{60} = 89,777$, $l_{61} = 89,015$, and $l_{70} = 77,946$.

Calculate probabilities that

(1) a man aged 60 who is just about to have surgery will be alive at age 70,
(2) a man aged 60 who had surgery at age 59 will be alive at age 70, and
(3) a man aged 60 who had surgery at age 58 will be alive at age 70.

Now let us look at the notation which is used for select and ultimate tables. Suppose that we start with $l_{[x]}$ lives that entered the select period at age x. Then the number of survivors at time t is denoted by $l_{[x]+t}$.

The probabilities of survival from age $[x]+t$ to age $[x]+t+n$ are denoted

$$_np_{[x]+t} = \frac{l_{[x]+t+n}}{l_{[x]+t}}.$$

As usual, $p_{[x]+t}$ is a shorthand for $_1p_{[x]+t}$.

The rows in the select and ultimate table show the age x at which an individual joined the group. The columns correspond to the time t elapsed from the moment an individual joined the group. Its header is $l_{[x]+t}$ for the select table and l_{x+t} for the ultimate table.

Example 2.20. You are given the following extract from a 2-year select-and-ultimate mortality table:

$[x]$	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
45	1235	1124	1039	47
46	1135	1025	978	48
47	1012	996	965	49

(i) Complete the table

$[x]$	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
45				47
46				48
47			-	49

(ii) Find $_2p_{[47]}$, $_2p_{[46]+1}$ and $_2p_{47}$.

Example 2.21. You are given the following extract from a 2-year select-and-ultimate mortality table:

$[x]$	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
45	0.009	0.008	0.007	47
46	0.008	0.006	0.005	48
47	0.004	0.003	-	49

Complete the table

$[x]$	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
45	10,000			47
46				48
47				49

Example 2.22. You are given the following extract from a 3-year select mortality table:

$[x]$	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	l_{x+3}	$x+2$
40	96489	96319	96084	95906	43
41	96312	96164	95998	95667	44
42	96157	95954	95265	95406	45
43	95895	95480	95243	95122	46
44	98743	96812	95012	94813	47
45	97239	95123	94753	94479	48

Compute $\mathring{e}_{[44]:\overline{4}|}$, $\mathring{e}_{[42]+2:\overline{4}|}$ and $\mathring{e}_{44:\overline{4}|}$.

SECTION 3

Life Insurance

section 3.1

Introduction

In this section, we develop formulae for the valuation of traditional insurance benefits. In particular, we consider whole life, term and endowment insurance.

Because of the dependence on death or survival, the timing and possibly the amount of the benefit are uncertain. So, the present value of the benefit can be modeled as a random variable.

We develop valuation functions for benefits based on the future lifetime $T_x = T(x)$ and the curtate future lifetime $K_x = K(x) - 1$, which we introduced in previous sections.

The valuation of benefits involve some interest theory functions. For convenience, we review some of them here.

Given the *effective annual rate of interest* i, we use $v = 1/(1+i)$, so that the present value of a payment of S which is to be paid in t years' time is Sv^t.

The *force of interest* per year is denoted δ where

$$\delta = \log(1+i),$$

$$1+i = e^{\delta},$$

and
$$v = e^{-\delta}.$$
The variable δ is also known as ***continuously compounded rate of interest***.

The ***nominal rate of interest*** compounded p times per year is denoted $i^{(p)}$,
$$i^{(p)} = p\left((1+i)^{1/p} - 1\right).$$
The ***effective rate of discount*** per year is d, where
$$d = 1 - v = 1 - e^{-\delta}.$$

section 3.2
Whole life insurance

Definition 3.1. A policy is called a *whole life policy* if it pays a fixed amount, called the *face value* after the death of the policyholder.

The face value paid in a life insurance is also called the *death benefit*.

The payment in a whole life insurance can be paid at different times. In this section, we consider the situation when the face value is paid at the end of the year of death.

"... So, how much would I get, if he was accidentally pushed down the stairs?"

An insurer offering life insurance takes a liability. It is of interest to know the present value of this liability. We will use the notation Z_x to denote the present value of the death benefit payment of a unit whole life insurance.

Example 3.2. On January 1, 2000, John entered a whole insurance contract. This contract pays a death benefit of $\$50,000$ at the end of year of death. On June 13, 2009, John died. The annual effective rate of interest is 6%. Calculate the present value of the benefit payment at the time of the issue of this contract.

Solution: The insurer makes a payment of $50,000$ at the end of the year 2009 (on January 1, 2010). This date is in ten years after the issue of the contract. The present value on January 1, 2000, of the paid benefit payment is $(50,000)(1.06)^{-10} = 27,919.73$.

Definition 3.3. The *actuarial present value* (APV) of a whole life insurance with a unit payment made at the end of the year of the death is the expectation of the present value Z_x and denoted by A_x,

$$A_x = \mathbb{E}Z_x.$$

Notice that $v^{\omega-x} \leq A_x \leq v$.

Once we know how to deal with insurances paying a unit, we can deal with insurances making a general payment. The actuarial present value of a whole life insurance with payment b is bA_x.

The quantity A_x is also called the ***net single premium*** of a whole life insurance with a unit payment made at the end of the year of the death.

In this case,

$$A_x = \sum_{k=1}^{\infty} v^k \mathbb{P}\{K_x = k-1\} = \sum_{k=1}^{\infty} e^{-\delta k} \mathbb{P}\{K_x = k-1\}.$$

The m-th moment of the random variable Z_x is denoted ${}^m A_x$.

$${}^m A_x = \sum_{k=1}^{\infty} v^{mk} \mathbb{P}\{K_x = k-1\} = \sum_{k=1}^{\infty} e^{-\delta mk} \mathbb{P}\{K_x = k-1\}.$$

The variance can be calculated by using the usual formula

$$\mathbb{V}\text{ar}(Z_x) = \mathbb{E}[Z_x^2] - (\mathbb{E}Z_x)^2 = {}^2A_x - (A_x)^2.$$

Finally, note that

$$\mathbb{P}\{K_x = k-1\} = {}_{k-1|}q_x = \frac{l_{x+k-1} - l_{x+k}}{l_x},$$

and so these probabilities can be easily obtained from life tables.

Example 3.4. Suppose $i = 5\%$ and K_x has probability mass function:

k	1	2	3
$\mathbb{P}\{K_x = k-1\}$	0.2	0.3	0.5

Find A_x and $\mathbb{V}\text{ar}(Z_x)$.

Suppose that an insurer offers a whole life insurance to n lives aged x with a benefit payment of b paid at the end of the year of death. Let $Z_{x,1}, \ldots, Z_{x,n}$ be the present values per unit of the benefit payments for these n insurees. The aggregate present value of these benefit payments is

$$Z^{ag} = b \sum_{j=1}^{n} Z_{x,j}.$$

Suppose that the insurer sets up a fund with an amount F to pay future benefits. The insurer would like that with probability p close to one, the fund is enough to pay future death benefits, $p = \mathbb{P}\{Z^{ag} \le F\}$. By the central limit theorem, Z^{ag} has approximately a normal distribution,

$$Z^{ag} \sim \mathcal{N}(nbA_x, nb^2({}^2A_x - (A_x)^2).$$

Hence we can find the required fund size from the formula

$$F = nbA_x + \Phi^{(-1)}(p)\sqrt{nb^2({}^2A_x - (A_x)^2)},$$

where $\Phi^{(-1)}(p)$ is the inverse distribution function. Its values can be found in the standard statistical tables.

For some values of p, $\Phi^{-1}(p)$ are recorded in the following table:

$\Phi(x)$	0.800	0.850	0.900	0.950	0.975	0.990	0.995
x	0.8416	1.0364	1.2815	1.6448	1.9599	2.3263	2.5758

Suppose now that the insurer collects a premium of P per insuree so that the probability that the collected premiums suffices to cover for the aggregate benefit payments is p, where p is close to one. In this case, $F = nP$. Hence,

$$P = b\left(A_x + \Phi^{(-1)}(p)\sqrt{\frac{{}^2A_x - (A_x)^2}{n}}\right).$$

Example 3.5. Consider the life table

x	80	81	82	83	84	85	86
l_x	250	217	161	107	62	28	0

An 80-year old buys a whole life policy insurance which will pay $50,000$ at the end of the year of his death. Suppose that $i = 6.5\%$.

(1) Find the actuarial present value of this life insurance.
(2) Find the probability that the APV of this life insurance is adequate to cover this insurance.
(3) Find the variance of the present value random variable of this life insurance.

(4) An insurance company offers this life insurance to 250 80-year old individuals. How much should each policyholder pay so that the insurer has a probability of 1% that the present value of these 250 policies exceed the total premiums received?

Solution:

(1) First we compute the relevant probabilities.

k	1	2	3	4	5	6
${}_{k-1\|}q_{80}$	0.132	0.224	0.216	0.18	0.136	0.112

Then we find the APV:

$$50,000 \times A_{80} = 50,000 \left(\frac{0.132}{1.065} + \ldots + \frac{0.112}{(1.065)^6} \right) = 40,809.5$$

(2) We have a cash inflow of 40,809.5 and a cash outflow of 50,000.

If the fund grows at 6.5%, it will require 4 years to grow the fund above $50,000$. (We assume that payments of interest to the fund occur only at the end of the year.)

Hence the probability in question is ${}_3p_{80} = 107/250 = 42\%$

(3) First we calculate ${}^2A_{80}$.

Note that $(1.065)^2 = 1.1342$. Hence,

$${}^2A_{80} = \left(\frac{0.132}{1.1342} + \ldots + \frac{0.112}{(1.1342)^6} \right) = 0.6724$$

and

$$\mathbb{V}\mathrm{ar}(Z_{80}) = {}^2A_{80} - (A_{80})^2 = 0.0062339.$$

(4) From the tables, $\Phi^{(-1)}(0.99) = 2.33$.

$$\begin{aligned} P &= 40,809.5 + 2.33 \times 50,000 \times \sqrt{0.0062339/250} \\ &= 40,809.5 + 581.75 = 41,391.25 \end{aligned}$$

□

Table 1 contains the values of A_x based on the life table for the USA population in 2004.

Table C.2: Single benefit premiums using $i = 6\%$ and the life table for the USA population in 2004.

Age	$1000A_x$	$1000 \cdot {}^2A_x$	Age	$1000A_x$	$1000 \cdot {}^2A_x$	Age	$1000A_x$	$1000 \cdot {}^2A_x$
0	25.56509	8.80028	40	132.64232	36.48695	80	602.29000	398.78023
1	20.43797	3.10914	41	138.92362	39.12511	81	617.59267	416.26584
2	21.19120	3.01160	42	145.45820	41.94200	82	632.69815	433.88508
3	22.17703	3.09261	43	152.24547	44.94022	83	647.60043	451.63017
4	23.28129	3.24385	44	159.28512	48.12273	84	662.27452	469.46404
5	24.50120	3.46401	45	166.56845	51.48294	85	676.61946	487.22737
6	25.80432	3.72143	46	174.10407	55.03330	86	690.64692	504.92794
7	27.19565	4.02073	47	181.89272	58.77789	87	704.32128	522.49937
8	28.68047	4.36710	48	189.95271	62.74114	88	717.61936	539.89113
9	30.26438	4.76635	49	198.30360	66.95055	89	730.53409	557.07628
10	31.97284	5.24510	50	206.95786	71.42730	90	743.03797	573.99323
11	33.78399	5.78307	51	215.92905	76.19525	91	755.11854	590.60171
12	35.70401	6.38759	52	225.19854	81.24183	92	766.76587	606.86359
13	37.70061	7.02680	53	234.79707	86.61209	93	777.97875	622.75341
14	39.74945	7.67501	54	244.74096	92.33720	94	788.75951	638.25042
15	41.80558	8.28329	55	255.04797	98.45246	95	799.13939	653.38239
16	43.87947	8.85675	56	265.72936	104.98798	96	809.10797	668.10721
17	45.95325	9.37102	57	276.78968	111.96812	97	818.76523	682.57177
18	48.06577	9.85873	58	288.21074	119.39098	98	828.16570	696.85524
19	50.21961	10.31648	59	299.97432	127.25443	99	837.42633	711.15592
20	52.45587	10.78051	60	312.03085	135.51662	100	846.20037	724.86583
21	54.77970	11.25154	61	324.34442	144.14912	101	854.64503	738.23257
22	57.19655	11.73035	62	336.86984	153.10886	102	862.68274	751.09985
23	59.72177	12.22786	63	349.64322	162.45322	103	870.49210	763.76315
24	62.39068	12.77643	64	362.70266	172.24665	104	877.99753	776.07824
25	65.23111	13.40252	65	376.09614	182.57183	105	885.29025	788.19794
26	68.25350	14.11580	66	389.83025	193.46383	106	891.94945	799.34341
27	71.45945	14.91710	67	403.86070	204.89280	107	898.47471	810.37122
28	74.86961	15.82745	68	418.15497	216.84194	108	904.76640	821.06620
29	78.47756	16.84035	69	432.67785	229.28906	109	912.52099	834.53860
30	82.29543	17.96859	70	447.40501	242.22463	110	919.99884	847.68072
31	86.32673	19.21651	71	462.36634	255.71311	111	925.59630	857.36218
32	90.57530	20.58899	72	477.55684	269.77592	112	943.39623	889.99644
33	95.04531	22.09153	73	492.91905	284.36250			
34	99.74133	23.73035	74	508.36536	299.37481			
35	104.65912	25.50242	75	523.83660	314.74666			
36	109.80361	27.41441	76	539.41014	330.60416			
37	115.18019	29.47383	77	555.14598	347.05975			
38	120.77661	31.66919	78	570.97004	364.03834			
39	126.59832	34.00756	79	586.73082	381.34315			

Table 1. Table of APVs

Example 3.6. An insurer issues a whole life insurance to a hundred lives of age 40 which pays $20,000$ at the end of the year of her death. Assume $i = 0.06$. The insurer has a fund with amount of $300,000$ of dollars to pay for these hundred life insurances. Calculate the probability that this fund is not enough to cover the payments of these hundred life insurances.

Since there are several different ways to calculate ${}_{k-1}q_x$, we have several ways to calculate A_x. Here is one of them.

$$A_x = vq_x + v^2 p_x q_{x+1} + v^3 p_x p_{x+1} q_{x+2} + \dots.$$

Example 3.7. Suppose that mortality of (x) is given by the table

k	0	1	2	3	4
p_{x+k}	0.05	0.01	0.005	0.001	0

Calculate A_x if $i = 7.5\%$.

Satz 3.8 —Recursion formula for A_x —.
For each $x > 0$, $A_x = vq_x + vp_x A_{x+1}$.

Example 3.9.

Jess and Jane buy a whole life policy insurance on the day of their birthdays. Both policies will pay $\$50,000$ at the end of the year of death. Jess is 45 years old and the net single premium of her insurance is $\$25,000$. Jane is 44 years old and the net single premium of her insurance is $\$23,702$. Suppose that $i = 0.06$.

Find the probability that a 44-year old will die within one year.

"Your life insurance policy only pays if you're completely dead. No partial benefits for zombies."

Next theorem deals with the case when the curtate lifetime K_x follows a geometric distribution.

Satz 3.10.
Suppose that for each $k = 1, 2, \ldots$, we have $p_{x+k} = p_x$. Then,

$$A_x = \frac{q_x}{q_x + i},$$
$$^m A_x = \frac{q_x}{q_x + (1+i)^m - 1},$$

Proof. Since K_x and K_{x+1} have the same distribution, $A_x = A_{x+1}$. Hence,

$$(1+i)A_x = q_x + p_x A_{x+1} = q_x + (1 - q_x)A_x.$$

and $A_x = q_x/(q_x + i)$.

The second equality holds because we can calculate $^m A_x$ as if we were calculating A_x but with the interest rate set to $(1+i)^m - 1$. □

Corollary 3.11.
Under a constant force of mortality μ,

$$A_x = \frac{q_x}{q_x + i},$$

where $q_x = 1 - e^{-\mu}$.

Example 3.12. Jane is 30 years old. She buys a whole life policy insurance which will pay $\$20000$ at the end of the year of her death. Suppose that $p_x = 0.9$, for each $x > 0$, and $i = 5\%$. Find the actuarial present value of this life insurance.

Example 3.13. A benefit of $\$500$ is paid at the end of the year of failure of a home electronic product. Let K be the end of the year of failure. Suppose that $\mathbb{P}\{K = k\} = \frac{(0.95)^k}{19}$, $k = 1, 2, \ldots$. The annual effective interest rate is $i = 6\%$. Calculate the actuarial present value of this benefit.

Example 3.14. Mariah is 40 years old. She buys a whole life policy insurance which will pay $\$150,000$ at the end of the year of her death. Suppose that the force of mortality is 0.01 and the force of interest is 0.07. Find the mean and the standard deviation of the present value random variable of this life insurance.

Example 3.15. An actuary models the future lifetime of (30) as follows. The actuary classifies lives according with health into three groups: good health, average health, and poor health. The probabilities of belonging to a given group are given by the following table:

Group	Good Health	Average Health	Poor Health
Probability	0.1	0.3	0.6

Individuals for the same group have the same constant force of mortality. The force of mortality for each group is given in the following table:

Group	Good Health	Average Health	Poor Health
Force of Mortality	0.01	0.05	0.1

The annual effective rate of interest is $i = 7.5\%$.

Calculate A_x and $\mathbb{V}\mathrm{ar}(Z_x)$.

section 3.3
Term Life Insurance

Definition 3.16. An n-th *term life insurance* policy is an insurance policy that pays a face value at the end of the year of death, if the insured dies within n years of the issue of the policy.

The present value of an n-th term life insurance policy with unit payment is denoted $Z^1_{x:\overline{n}|}$.

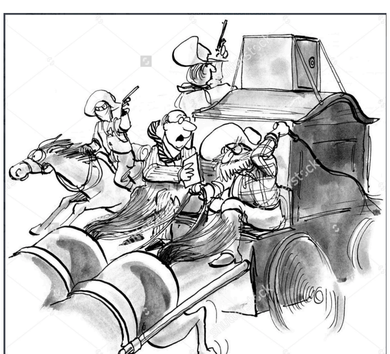
By definition,

$$Z^1_{x:\overline{n}|} = v^{K_x+1}\mathbb{1}_{K_x<n} = \begin{cases} v^{K_x+1}, & \text{if } K_x < n, \\ 0, & \text{if } K_x \geq n. \end{cases}$$

(The upper index 1 in $Z^1_{x:\overline{n}|}$ means that (x) must fail before the n-th term.)

Definition 3.17. The actuarial present value of an n-th term life insurance policy, $A^1_{x:\overline{n}|}$, is the expectation of $Z^1_{x:\overline{n}|}$,

$$A^1_{x:\overline{n}|} = \mathbb{E}Z^1_{x:\overline{n}|} = \sum_{k=1}^{n} v^k \mathbb{P}\{K_x = k-1\}.$$



"Do you have a moment to talk about life insurance?"

We use notation ${}^mA^1_{x:\overline{n}|}$ to denote the m-th moment of the actuarial present value, ${}^mA^1_{x:\overline{n}|} = \mathbb{E}\left(Z^1_{x:\overline{n}|}\right)^m$, that is, ${}^mA^1_{x:\overline{n}|} = \sum_{k=1}^{n} v^{mk}\mathbb{P}\{K_x = k-1\}$.

Most of the formulae valid for the whole life insurance are also valid for term life insurance, except that summation are terminated at the n-th step. For example, $\mathbb{V}\mathrm{ar}\left(Z^1_{x:\overline{n}|}\right) = {}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2$, or

$A^1_{x:\overline{3}|} = vq_x + v^2 p_x q_{x+1} + v^3 p_x p_{x+1} q_{x+2}$.

The recursion formula can also be adapted in a straightforward manner. We simply condition on whether the insuree survived the first year.

Satz 3.18 —Recursion —.

$$A^{1}_{x:\overline{n}|} = vq_x + vp_x A^{1}_{x+1:\overline{n-1}|}.$$

We can calculate the value of the term life insurance from the table by the following formula:

$$A^{1}_{x:\overline{n}|} = A_x - v^n \times {}_np_x \times A_{x+n}.$$

This formula can be verified directly from definitions.

The factor $v^n \times {}_np_x$ is often denoted as ${}_nE_x$, and for $n = 5$, 10, 20, it is often in the tables.

Example 3.19. Suppose that $i = 0.05$, $q_x = 0.05$ and $q_{x+1} = 0.02$. Find $A^{1}_{x:\overline{2}|}$ and $\mathbb{V}\mathrm{ar}(Z^{1}_{x:\overline{2}|})$.

Example 3.20. Suppose that $\delta = 0.04$ and (x) has force of mortality $\mu = 0.03$. Find $A^{1}_{x:\overline{10}|}$ and $\mathbb{V}\mathrm{ar}(Z^{1}_{x:\overline{2}|})$.

Example 3.21. Using $i = 0.05$ and a certain life table, $A^{1}_{37:\overline{10}|} = 0.52$. Suppose that an actuary revises this life table and changes p_{37} from 0.95 to 0.96. Other values in the life table are unchanged.

Find $A^{1}_{37:\overline{10}|}$ using the revised life table.

Example 3.22. Consider the life table

x	80	81	82	83	84	85	86
l_x	250	217	161	107	62	28	0

An 80-year old buys a three-year term life policy insurance which will pay $\$50000$ at the end of the year of his death. Suppose that $i = 6.5\%$.

(i) Find the actuarial present value of this life insurance.

(ii) Find the standard deviation of the present value of this life insurance.

(iii) Find the probability that the APV of this life insurance is adequate to cover this insurance.

(iv) Find the probability that the present value of this life insurance exceeds one standard deviation to its APV.

ſection 3.4

Deferred Life Inſurance

Definition 3.23. An n-year *deferred life insurance* is a life insurance policy that pays a face value if the insured dies at least n-years after the issue of the policy.

The present value of an n-year deferred life insurance with unit payment at the end of the year of death is denoted by ${}_{n|}Z_x$

$$ {}_{n|}Z_x = v^{K_x+1}\mathbb{1}_{K_x \geq n} = \begin{cases} 0, & \text{if } K_x < n, \\ v^{K_x+1}, & \text{if } K_x \geq n. \end{cases} $$

The actuarial present value of an n-year deferred life insurance with unit payment at the end of the year of death is denoted by ${}_{n|}A_x$,

$$ {}_{n|}A_x = \mathbb{E}[{}_{n|}Z_x] = \sum_{k=n+1}^{\infty} v^k \mathbb{P}\{K_x = k-1\}. $$

The m-th moment of ${}_{n|}Z_x$ is denoted ${}^{m}_{n|}A_x$.

$$ {}^{m}_{n|}A_x = \mathbb{E}({}_{n|}Z_x)^m = \sum_{k=n+1}^{\infty} v^{mk} \mathbb{P}\{K_x = k-1\}. $$

From the tables, the deferred insurance can be calculated as follows.

$$ {}_{n|}A_x = v^n {}_n p_x A_x = {}_n E_x A_x. $$

This formula immediately implies the following result for the case of constant force of mortality.

Satz 3.24 —Constant Force of Mortality —.
Under constant force of mortality μ,

$$ {}_{n|}A_x = e^{-n(\mu+\delta)} \frac{q_x}{q_x + i}. $$

We have also two recursion formulas.

Satz 3.25 —Recursion I —.
For each $x > 0$,

$$ {}_{n|}A_x = v p_x \times {}_{n-1|}A_{x+1}. $$

Proof. Intuitively, there are two possibilities. Either the insuree dies in the first year, and then the expected PV of payment is 0. Or he will survive and the payoff structure is the same as for the deferred insurance with age $x+1$ and deferral term $n-1$. The expected PV of this payment is $vp_x \times {}_{n-1|}A_{x+1}$. Formally,

$$
\begin{aligned}
{}_{n|}A_x &= \mathbb{E}\big(v^{K_x+1}\mathbb{1}_{\{K_x \geq n\}}\big) \\
&= \mathbb{E}\big(v^{K_x+1}\mathbb{1}_{\{K_x \geq n\}} | K_x = 0\big)\mathbb{P}(K_x = 0) \\
&+ \mathbb{E}\big(v^{K_x+1}\mathbb{1}_{\{K_x \geq n\}} | K_x > 0\big)\mathbb{P}(K_x > 0) \\
&= 0 + vp_x \mathbb{E}\big(v^{K_{x+1}+1}\mathbb{1}_{\{K_{x+1} \geq n-1\}}\big) \\
&= vp_x \times {}_{n-1|}A_{x+1}
\end{aligned}
$$

□

Satz 3.26 —Recursion II —.
For each $x > 0$,

$$ {}_{n|}A_x = v^{n+1} {}_{n|}q_x + {}_{n+1|}A_x. $$

Proof.

$$\begin{aligned}
{}_{n|}A_x &= \sum_{k=n+1}^{\infty} v^k \mathbb{P}(K_x = k-1) \\
&= v^{n+1}{}_{n|}q_x + \sum_{k=n+2}^{\infty} v^k \mathbb{P}(K_x = k-1) \\
&= v^{n+1}{}_{n|}q_x + {}_{n+1|}A_x
\end{aligned}$$

□

Example 3.27. Consider the life table

x	80	81	82	83	84	85	86
l_x	250	217	161	107	62	28	0

An 80-year old buys a three-year deferred policy insurance which will pay $\$50,000$ at the end of the year of his death. Suppose that $i = 6.5\%$.

(i) Find the actuarial present value of this life insurance.

(ii) Find the probability that APV of this life insurance is adequate to cover this insurance.

Example 3.28. An insurance company offers a 10-year deferred life insurance for individuals aged 25, which will pay $\$250,000$ at the end of the year of his death.

Suppose that $p_x = 0.95$, for each $x \geq 0$, and $\delta = 0.065$.

(i) Find the expected value and the standard deviation of the present value of this life insurance.

(ii) Fifty lives enter this insurance contract. Using the normal approximation, calculate the amount F such that the probability that the aggregate PV of these 50 lives is less than F equals 0.95.

Example 3.29. Suppose that $i = 0.10$, $q_x = 0.05$ for all x.

Find ${}_{2|}A_x$ and $\mathbb{V}\text{ar}({}_{2|}Z_x)$.

Solution.

$${}_{2|}A_x = \frac{q}{q+i} e^{-n(\mu+\delta)} = \frac{q}{q+i}\left(\frac{p}{1+i}\right)^2 = 0.249.$$

Example 3.30. Suppose that ${}_{14|}A_{35} = 0.24$, $i = 8\%$, $p_{35} = 0.96$.

Find ${}_{13|}A_{36}$.

Solution.

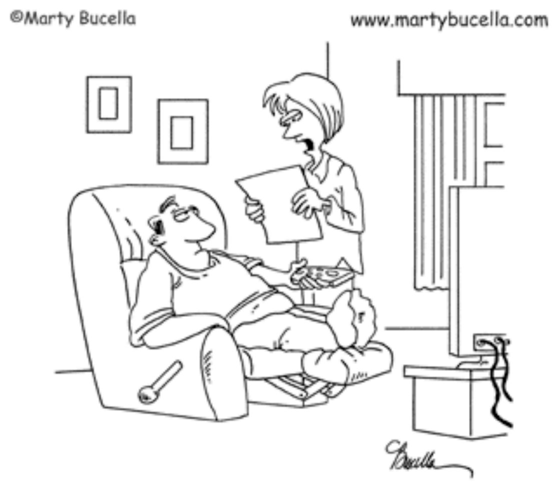
$$ {}_{13|}A_{36} = \frac{{}_{14|}A_{35}(1+i)}{p_{35}} = 0.27. $$

Example 3.31. Suppose that ${}_{14|}A_{35} = 0.24$, $i = 8\%$, ${}_{14}p_{35} = 0.7$, $q_{49} = 0.03$.

Find ${}_{15|}A_{35}$.

Solution.

$$ {}_{15|}A_{35} = {}_{14|}A_{35} - (1+i)^{-15}\,{}_{14}p_{35}\, q_{49} = 0.233. $$



"Just as I thought....you can't collect on the life insurance by boring me to death."

section 3.5

Pure Endowment Life Insurance

Definition 3.32. An n-year *pure endowment* life insurance is a life insurance policy that pays its face value in n years provided the insuree is alive at that time.

We denote the present value of this insurance as $Z_{x:\overset{1}{\overline{n}|}}$,

$$Z_{x:\overset{1}{\overline{n}|}} = v^n \mathbb{1}_{\{K_x \geq n\}} = \begin{cases} 0, & \text{if } K_x < n, \\ v^n & \text{if } K_x \geq n. \end{cases}$$

The upper index 1 over the term subscript means that the term must expire before the life does.

Similarly, the actuarial present value of an n-year pure endowment life insurance with unit payment is denoted by $A_{x:\overset{1}{\overline{n}|}}$. Unfortunately, this notation leads to a possibility of confusion between the term and pure endowment insurance policies. For this reason, the APV of the pure endowment insurance is often denoted by ${}_nE_x$.

Satz 3.33.
We have

$$
\begin{aligned}
{}_nE_x &= v^n \cdot {}_np_x, \\
{}^2_nE_x &= v^{2n} \cdot {}_np_x, \\
\mathbb{V}\mathrm{ar}(Z_{n:\overline{x}|}^{\ 1}) &= v^{2n} \cdot {}_np_x \cdot {}_nq_x.
\end{aligned}
$$

The value of ${}_nE_x$ can also be interpreted as the n-year discount factor that takes in account both interest and mortality. Namely, ${}_nE_x$ is the expected present value of a unit payment made to a still living insuree at time n.

section 3.6
Endowment Life Insurance

Definition 3.34. An n-year *endowment* life insurance is a life insurance policy that makes a payment when either death happens before n years, or at the end of the n years if the insuree survived til that time.

Under an n-year endowment life insurance, every insuree receives a payments. The deceased within n years receive a death benefit. The n-year survivors receive a payment at time n.

The present value of an n-year endowment life insurance is denoted by $Z_{x:\overline{n}|}$, and the corresponding actuarial present value by $A_{x:\overline{n}|}$. (Note the absence of the superscript.)

$$
Z_{x:\overline{n}|} = v^{\min(K_x+1,n)} = \begin{cases} v^{K_x+1}, & \text{if } K_x < n, \\ v^n, & \text{if } K_x \geq n. \end{cases}
$$

$$
A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + {}_nE_x = \sum_{k=1}^{n} v^k \cdot {}_{k-1|}q_x + v^n \cdot {}_np_x.
$$

Similarly, the m-th moment of the present value is

$$
{}^mA_{x:\overline{n}|} = \sum_{k=1}^{n} v^{mk} \cdot {}_{k-1|}q_x + v^{mn} \cdot {}_np_x.
$$

For the endowment insurance the following recursion formula holds.

Satz 3.35.

$$A_{x:\overline{n}|} = vq_x + vp_x A_{x+1:\overline{n-1}|}.$$

Example 3.36. Consider the life table

x	80	81	82	83	84	85	86
l_x	250	217	161	107	62	28	0

An 80-year old buys a three-year endowment life policy insurance which will pay $\$50000$ at the end of the year of his death. Suppose that $i = 6.5\%$.

(i) Find the actuarial present value of this life insurance.

(ii) Find the probability that APV of this life insurance is adequate to cover this insurance.

section 3.7

Deferred Term Life Insurance

Definition 3.37. An m-year deferred n-year term life insurance is an insurance policy that makes a payment if death happens during the period of n years that starts m years from now.

The present value of an m-year deferred n-year term life insurance with unit payment paid at end of year of death is denoted by ${}_{m|n}Z_x^1$, and the corresponding APV by ${}_{m|n}A_x^1$.

section 3.8

Non-level payments paid at the end of the year

In this section we consider the case when payments change over time. Suppose that a life insurance provides a benefit of b_k paid at the end of the k-th year if death happens in this year. The present value of this benefit is $B_x = b_{K_x} v^{K_x}$. The actuarial present value is

$$\mathbb{E}[B_x] = \sum_{k=1}^{\infty} b_k v^k \mathbb{P}\{K_x = k\}.$$

Example 3.38. A whole life insurance on (50) pays a death benefit at the end of the year of death. The death benefit is $\$50,000$ for the first year and it increases at annual rate of 3% per year. The annual effective rate of interest is 6.5%. We have that $A_{50} = 0.47$ when the annual effective rate of interest is $\frac{1.065}{1.03} - 1$. Calculate the net single premium for this insurance.

Definition 3.39. An *increasing-by-one* whole life insurance pays k at time k, for each $k > 1$, if the failure happens in the k-th year.

This insurance is also called an *annually increasing* whole life insurance.

The present value of an increasing by one unit whole life insurance is denoted by $(IZ)_x$,

$$(IZ)_x = K_x v^{K_x}.$$

The actuarial present value of an increasing by one unit whole life insurance is denoted by $(IA)_x$,

$$(IA)_x = \mathbb{E}[(IZ)_x] = \sum_{k=1}^{\infty} k v^k \mathbb{P}\{K_x = k\} = \sum_{k=1}^{\infty} k v^k \cdot {}_{k-1|}q_x.$$

Satz 3.40.

$$(IA)_x = A_x + {}_1E_x (IA)_{x+1}.$$

Proof. We calculate

$$\begin{aligned}
(IA)_x &= \sum_{k=1}^{\infty} kv^k \mathbb{P}\{K_x = k\} \\
&= \sum_{k=1}^{\infty} v^k \mathbb{P}\{K_x = k\} + \sum_{k=2}^{\infty} (k-1)v^k \mathbb{P}\{K_x = k\} \\
&= A_x + vp_x \sum_{k-1=1}^{\infty} (k-1)v^{k-1} \mathbb{P}\{K_{x+1} = k-1\} \\
&= A_x + {}_1E_x (IA)_{x+1}
\end{aligned}$$

□

Example 3.41. Suppose that $A_{30} = 0.13$, $(IA)_{30} = 0.45$, $v = 0.94$ and $p_30 = 0.99$. Find $(IA)_{31}$.

Definition 3.42. An *increasing by one n-year term* life insurance pays an annually increasing-by-one payment if insured dies within n years of the issue of the policy.

The present value of an increasing by one n-year term life insurance is denoted by $(IZ)^1_{x:\overline{n}|}$,

$$(IZ)^1_{x:\overline{n}|} = K_x v^{K_x} \mathbb{1}_{(K_x leqn)}.$$

The actuarial present value of a unit increasing n-year term life insurance is denoted by $(IA)^1_{x:\overline{n}|}$,

$$(IA)^1_{x:\overline{n}|} = \mathbb{E}[(IZ)^1_{x:\overline{n}|}] = \sum_{k=1}^{n} kv^k \mathbb{P}\{K_x = k\} = \sum_{k=1}^{n} kv^k \cdot {}_{k-1|}q_x.$$

Definition 3.43. An *increasing by one n-year endowment* insurance pays an annually increasing-by-one payment when either death happens before n-years, or at the end of the n years if death happens after n years.

The actuarial present value of this insurance is

$$\begin{aligned}
IA_{x:\overline{n}|} &= \sum_{k=1} nkv^k \cdot {}_{k-1|}q_x + nv^n \cdot {}_np_x \\
&= (IA)^1_{x:\overline{n}|} + n \cdot {}_nE_x.
\end{aligned}$$

Definition 3.44. A *decreasing-by-one n-year term* life insurance pays $n+1-k$ at time k if the failure happens in the k-th interval, where $1 \le k \le n$.

The actuarial present value of this insurance is

$$(DA)^{1}_{x:\overline{n}|} = \sum_{k=1}^{n} (n+1-k) v^k \mathbb{P}\{K_x = k\}.$$

Section 3.9
Life insurance paid m times a year

It is not realistic that claims are paid only at the end of the year. A more realistic model assumes that claims are paid at the end of each month, or other period.

Suppose claim payments are made at m equally spaced times in a year. To indicate this, a superscript (m) is added to the actuarial notation of insurance variables.

For example, $A_x^{(12)}$ is the APV of the whole life insurance where payments are made at the end of the month of death, and $A_x^{(4)}$ means that claims are paid quarterly, that is, every three months.

Let $J_x^{(m)} = j$ if $T_x \in \left(\frac{j-1}{m} \frac{j}{m}\right]$ for some positive integer $j \ge 1$.

Then the present value of a whole life insurance paid m times a year is

$$Z_x^{(m)} = v^{J_x^{(m)}/m},$$

and the actuarial present value is

$$\begin{aligned} A_x^{(m)} = \mathbb{E} Z_x^{(m)} &= \sum_{j=1}^{\infty} v^{j/m} \mathbb{P}(J_x^{(m)} = j) \\ &= \sum_{j=1}^{\infty} v^{j/m} \cdot {}_{\frac{j-1}{m}|\frac{1}{m}} q_x. \end{aligned}$$

If the force of mortality is constant, then

$$\begin{aligned} A_x^{(m)} &= \frac{{}_{1/m}q_x}{{}_{1/m}q_x + (1+i)^{1/m} - 1} \\ &= \frac{1 - e^{-\mu/m}}{e^{\delta/m} - e^{-\mu/m}}. \end{aligned}$$

Example 3.45. Suppose that $\mu_{T_x}(t) = 0.03$ and $\delta = 0.06$. Calculate $A_x^{(3)}$ and $\mathbb{V}\mathrm{ar}(Z_x^{(3)})$.

Solution.

$$A_x^{(3)} = \frac{1 - e^{-0.01}}{e^{0.02} - e^{-0.01}} = 0.330,$$

$${}^2A_x^{(3)} = \frac{1 - e^{-0.01}}{e^{0.04} - e^{-0.01}} = 0.1960,$$

and

$$\mathbb{V}\mathrm{ar}(Z_x^{(3)}) = 0.1871.$$

section 3.10

Level benefit insurance in the continuous case

In this section, we consider the case of benefits paid *at the moment of of death*. This is also called *immediate payment of a claim*.

The actuarial notation in the continuous case is similar to the discrete case. A bar is added to each actuarial symbol to denote that payments are made continuously.

For example, the present value of a unit payment whole life insurance paid at the moment of death is denoted by $\overline{Z}_x$,

$$\overline{Z}_x = v^{T_x} = e^{-\delta T_x}.$$

The actuarial present value of this insurance is denoted by $\overline{A}_x$,

$$\begin{aligned} \overline{A}_x &= \int_0^\infty v^t f_{T_x}(t)\, dt \\ &= \int_0^\infty v^t {}_tp_x \mu_{x+t}\, dt \end{aligned}$$

An interpretation of $v^t {}_tp_x \mu_{x+t}\, dt$ is that it is the actuarial present value of the unit payment made in the interval $(t, t + dt)$.

It is easy to calculate $\overline{A}_x$ in the case of the constant force of mortality.

Satz 3.46.
In the model with constant force of mortality μ,

$$\overline{A}_x = \frac{\mu}{\mu + \delta}.$$

Section 3.11
Calculating APV from life tables

If the mortality is given by life tables and the claim payments happens more frequently than once a year, then we need to approximate mortality at non-integer times.

In this case the easiest situation is when we assume the uniform distribution of deaths over each year.

The following result is very handy when we need to evaluate the value of the insurance policy that pays at the moment of death.

Satz 3.47.
Assume a uniform distribution of deaths over each year of death. Suppose that the benefit b_t *is constant in each interval* $(k-1, k]$, $k = 1, 2, \ldots$. *Then,*

$$\mathbb{E}[b_{T_x} v^{T_x}] = \frac{i}{\delta} \mathbb{E}[b_{K_x} v^{K^x}].$$

Proof. Recall the decomposition $T_x = K_x - 1 + S_x$. We use the fact that K_x and S_x are independent and that S_x has a uniform distribution on the interval $[0, 1]$. Then,

$$\begin{aligned} \mathbb{E}[b_{T_x} v^{T_x}] &= \mathbb{E}[b_{K_x} v^{K_x - 1 + S_x}] \\ &= \mathbb{E}[b_{K_x} v^{K_x}] \mathbb{E}[v^{S_x - 1}], \end{aligned}$$

and we can calculate

$$\mathbb{E}[v^{S_x - 1}] = \int_0^1 (1+i)^{1-t} dt = \frac{i}{\ln(1+i)} = \frac{i}{\delta}.$$

□

Hence the APV of the insurance with immediate payment equals the APV of the insurance that pays at the end of the year multiplied by the factor of i/δ.

Note that $1 + i = e^\delta > 1 + \delta$. Hence the factor $i/\delta > 1$, and the value of the insurance with immediate payment is always greater than the value of insurance that pays at the year end.

A similar result holds for the insurance that pays m times a year.

Satz 3.48.
Assume a uniform distribution of deaths over each year of death. Suppose that the benefit b_t is constant in each interval $(k-1, k]$, $k = 1, 2, \ldots$. Then,

$$\mathbb{E}[b_{J_x^m/m} v^{J_x^m/m}] = \frac{i}{i^{(m)}} \mathbb{E}[b_{K_x} v^{K_x}],$$

where

$$i^{(m)} = m\big((1+i)^{1/m} - 1\big).$$

So for example,

$$\overline{A}^{1}_{x:\overline{n}|} = \frac{i}{\delta} A^{1}_{x:\overline{n}|}$$

and

$$A^{1}_{x:\overline{n}|}{}^{(m)} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{n}|}$$

For endowment insurance, these formulas a slightly different

$$\overline{A}_{x:\overline{n}|} = \frac{i}{\delta} A^{1}_{x:\overline{n}|} + {}_nE_x$$

and

$$A_{x:\overline{n}|}{}^{(m)} = \frac{i}{i^{(m)}} A^{1}_{x:\overline{n}|} + {}_nE_x.$$

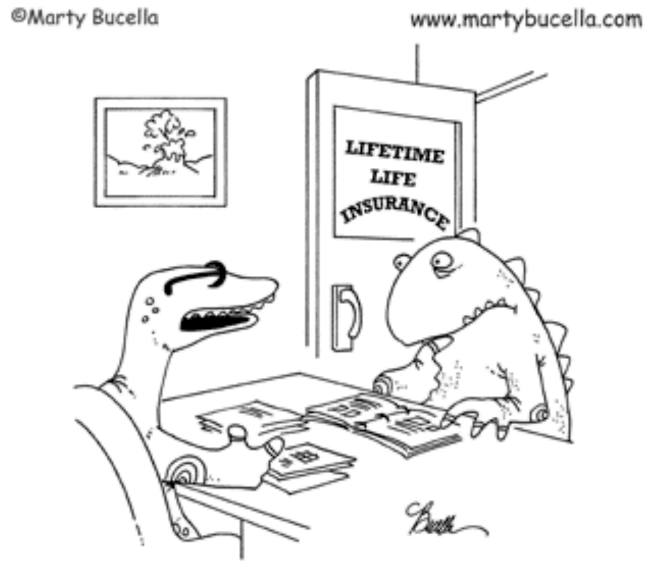
This is because the pure endowment portion of the endowment insurance is always paid at the end of the year n.

Example 3.49. Consider the life table

x	80	81	82	83	84	85	86
l_x	250	217	161	107	62	28	0

Suppose that $i = 6.5\%$. Assume a uniform distribution of deaths. Calculate: (i) A_{80}, (ii) $A^{1}_{80:\overline{3}|}$, (iii) $A_{80:\overline{3}|}$,

(iv) $\overline{A}_{80}$, (v) $\overline{A}^{\,1}_{80:\overline{3}|}$, (vi) $\overline{A}_{80:\overline{3}|}$,
(vii) $A_{80}^{(12)}$ (viii) ${A^{1}_{80:\overline{3}|}}^{(12)}$ (ix) $A_{80:\overline{3}|}^{(12)}$.



"Okay, say you go extinct before the rest of your family..."

section 3.12

Insurance for joint life and last survivor

Suppose that the future lifetimes of lives (x) and (y) are denoted by T_x and T_y. We will assume here that these random variables are independent but may have different survival functions.

Then we define ***time to first death***, $T_{xy} = \min(T_x, T_y)$, and ***time to last death***, $T_{\overline{xy}} = \max(T_x, T_y)$.

The subscript in these expressions is called a ***status***; the subscript xy is the ***joint life status***, and $\overline{xy}$ is the ***last survivor status***. So, the random variables T_{xy} and $T_{\overline{xy}}$ represent the time until the failure of joint life status and the last survivor status, respectively.

In cases where numbers are used instead of symbols x and y, it is usual to put colon between these numbers. For example, we write $T_{\overline{34:41}}$ instead of $T_{\overline{34\,41}}$.

It is very useful to note that the realized value of T_x matches one of T_{xy} and $T_{\overline{xy}}$, and the realized value of T_y matches the other.

In particular, this implies that

$$T_x + T_y = T_{xy} + T_{\overline{xy}}, \text{ and}$$
$$v^{T_x} + v^{T_y} = v^{T_{xy}} + v^{T_{\overline{xy}}}.$$

We can extend our notation in a standard way, which is illustrated by the following examples.

$$\begin{aligned} {}_{u|t}q_{xy} &= \mathbb{P}[\text{(x) and (y) are both alive in } u \text{ years but not in } u+t \text{ years}] \\ &= \mathbb{P}[u < T_{xy} \leq u+t]. \end{aligned}$$

$$\begin{aligned} {}_{u|t}q_{\overline{xy}} &= \mathbb{P}[\text{at least one (x) and (y) is alive in } u \text{ years but both are} \\ &\qquad \text{deceased after } u+t \text{ years}] \\ &= \mathbb{P}[u < T_{\overline{xy}} \leq u+t]. \end{aligned}$$

In addition, some notation is specific to policies on multiple lives. For example,

$$\begin{aligned} {}_tq^1_{xy} &= \mathbb{P}[\text{(x) dies before (y) and within } t \text{ years}] \\ &= \mathbb{P}[T_x \leq t \text{ and } T_x < T_y]. \end{aligned}$$

$$\begin{aligned} {}_tq^2_{xy} &= \mathbb{P}[\text{(x) dies after (y) and within } t \text{ years}] \\ &= \mathbb{P}[T_y < T_x \leq t]. \end{aligned}$$

By using the independence assumption, we can derive formulas for the probabilities related to the joint life status and the last survival status.

$${}_tp_{xy} = \mathbb{P}[T_x > t \text{ and } T_y > t] = {}_tp_x \cdot {}_tp_y,$$

and

$${}_tp_{\overline{xy}} = \mathbb{P}[T_x > t \text{ or } T_y > t] = 1 - {}_tq_x \cdot {}_tq_y.$$

The notation here is a bit ambiguous and might lead to confusion if the survival functions of these two lives are different. So, in some cases we might need to write the formulas more carefully. For example,

$${}_tp^{m\ f}_{45:40} = {}_tp^m_{45} \cdot {}_tp^f_{40}.$$

Using the joint life and last survivor probabilities, we can develop formulas for the actuarial present value of an insurance in the usual manner.

For example, A_{xy} denotes the APV of the whole life insurance on the joint life. It pays one unit at the end of the year of the first death. We can calculate its value as follows.

$$\begin{aligned} A_{xy} &= \mathbb{E}[v^{T_{xy}}] \\ &= \sum_{k=1}^{\infty} v^k \cdot {}_{k-1|}q_{xy}, \end{aligned}$$

where

$$\begin{aligned} {}_{k-1|}q_{xy} &= {}_{k-1}p_{xy} - {}_k p_{xy} \\ &= {}_{k-1}p_x \cdot {}_{k-1}p_y - {}_k p_x \cdot {}_k p_y. \end{aligned}$$

We can calculate $A_{\overline{xy}}$ in a similar fashion,

$$\begin{aligned} A_{\overline{xy}} &= \mathbb{E}[v^{T_{\overline{xy}}}] \\ &= \sum_{k=1}^{\infty} v^k \cdot {}_{k-1|}q_{\overline{xy}}, \end{aligned}$$

where

$$\begin{aligned} {}_{k-1|}q_{\overline{xy}} &= {}_k q_{\overline{xy}} - {}_{k-1}q_{\overline{xy}} \\ &= {}_k q_x \cdot {}_k q_y - {}_{k-1}q_x \cdot {}_{k-1}q_y. \end{aligned}$$

It is also worth noting, that

$$A_{\overline{xy}} + A_{xy} = A_x + A_y,$$

because two individual insurances on (x) and (y) have the same total payoff as the insurances on joint life and on last survivor life, bought together.

Example 3.50. Consider the following life tables appropriate for a husband and wife.

x	l_x	y	l_y
65	43,302	60	47,260
66	42,854	61	47,040
67	42,081	62	46,755
68	41,351	63	46,500
69	40,050	64	46,227

(1) Calculate ${}_3p_{xy}$ and ${}_3p_{\overline{xy}}$ for a husband aged $x = 65$ and a wife aged $y = 60$.
(2) Calculate $A^{\ 1}_{xy:\overline{2}|}$ and $A^{\ 1}_{\overline{xy}:\overline{2}|}$ if interest rate is 5% per year.

Solution. (1)

$$ {}_3p_{xy} = {}_3p_x {}_3q_y = \frac{41,351}{43,302} \times \frac{46,500}{47,260} = 0.9396 $$

$$ \begin{aligned} {}_3p_{\overline{xy}} &= 1 - {}_3q_x {}_3q_y = {}_3p_x + {}_3p_y - {}_3p_x {}_3p_y \\ &= \frac{41,351}{43,302} + \frac{46,500}{47,260} - \frac{41,351}{43,302} \times \frac{46,500}{47,260} = 0.9993 \end{aligned} $$

(2) We calculate

$$ \begin{aligned} A^{\ 1}_{xy:\overline{2}|} &= \sum_{k=1}^{2} (1+i)^{-k} {}_{k-1|}q_{xy} \\ &= 1.05^{-1} \times \left(1 - \frac{42,854}{43,302} \times \frac{47,040}{47,260}\right) \\ &+ 1.05^{-2} \times \left(\frac{42,854}{43,302} \times \frac{47,040}{47,260} - \frac{42,081}{43,302} \times \frac{46,755}{47,260}\right) \\ &= 1.05^{-1} \times 0.0150 + 1.05^{-2} \times 0.0236 = 0.0357 \end{aligned} $$

$$
\begin{aligned}
A^{1}_{\overline{xy}:\overline{2}|} &= \sum_{k=1}^{2}(1+i)^{-k}{}_{k-1|}q_{\overline{xy}} \\
&= 1.05^{-1} \times \left(\frac{43,302-42,854}{43,302} \times \frac{47,260-47,040}{47,260}\right) \\
&+ 1.05^{-2} \times \left(\frac{43,302-42,081}{43,302} \times \frac{47,260-46,755}{47,260}\right. \\
&- \left.\frac{43,302-42,854}{43,302} \times \frac{47,260-47,040}{47,260}\right) \\
&= 1.05^{-1} \times 0.00004816 + 1.05^{-2} \times 0.000253 = 0.0003
\end{aligned}
$$

SECTION 4

Annuities

section 4.1

Introduction

A *life annuity* is a financial contract according to which a seller (issuer) makes periodic payments in the future to the buyer (annuitant). Life annuities are one of the most often used plans to fund retirement. The payment for a life annuity can be made at the time of issue. But, in the case of retirement, contributions are made to the retirement fund while the annuitant works. Common retirement plans are $401(k)$ plans and (individual retirement accounts) IRAâs. At the time of retirement, the insurance company uses the accumulated deposit to issue a life annuity.

Contributions to this retirement fund can be made by either the employer and/or the employee. Contributions made by the employee can be tax free.

Another way to get retirement funds is done by the Social Security. So, Social Security is some how similar to an insurance company issuing life annuities being funded while an individual works.

section 4.2
Whole life annuities

A *whole life annuity* is a series of payments made while an individual is alive.

4.2.1 Whole life due annuity

Definition 4.1. A *whole life due annuity* is a series payments made at the beginning of the year while an individual is alive.

The present value of a whole life due annuity for (x) with unit payment is denoted by $\ddot{Y}_x$. It is the present value of the cashflow of unit payments made at times $0, 1, \ldots, K_x - 1$,

$$\ddot{Y}_x = \sum_{k=0}^{K_x-1} v^k.$$

The standard notation for the present value of a due annuity that makes n payments at the beginning of the first n years is $\ddot{a}_{\overline{n}|}$. So, $\ddot{Y}_x = \ddot{a}_{\overline{K_x}|}$.

The actuarial present value of this annuity is denoted by $\ddot{a}_x$.

$$\ddot{a}_x = \mathbb{E}\ddot{Y}_x = \sum_{k=1}^{\infty} \ddot{a}_{\overline{k}|} \cdot {}_{k-1|}q_x$$

Satz 4.2 —Relation between annuity and insurance —.

(i) If $i > 0$,

$$\ddot{Y}_x = \frac{1 - v^{K_x}}{d} = \frac{1 - Z_x}{d}$$
$$\ddot{a}_x = \frac{1 - A_x}{d}$$
$$\mathbb{V}\text{ar}(\ddot{Y}_x) = \frac{\mathbb{V}\text{ar}(Z_x)}{d^2} = \frac{{}^2A_x - A_x^2}{d^2},$$

where $d = 1 - v = i/(1 + i)$.

(ii) If $i = 0$, $\ddot{Y}_x = K_x$, $\ddot{a}_x = e_x + 1$, *and* $\mathbb{V}\text{ar}(\ddot{Y}_x) = \mathbb{V}\text{ar}(K_x)$.

Proof.

$$\ddot{a}_{\overline{K_x}|} = 1 + v + v^2 + \ldots + v^{K_x - 1} = \frac{1 - v^{K_x}}{1 - v} = \frac{1 - Z_x}{d}.$$

All other equalities in (i) directly follow. Equalities in (ii) follow from definitions. □

The APV of the life annuity due can also be found by adding the present value of yearly payments.

Satz 4.3.

$$\ddot{Y}_x = \sum_{k=0}^{\infty} Z_{x:\frac{1}{k|}}$$

and

$$\ddot{a}_x = \sum_{k=0}^{\infty} {}_kE_x = \sum_{k=0}^{\infty} v^k \cdot {}_kp_x.$$

Proof. The payment at time k is made if and only if $k < T_x$. Hence,

$$\ddot{Y}_x = \sum_{k=0}^{\infty} v^k \mathbb{1}_{(k<T_x)} = \sum_{k=0}^{\infty} Z_{x:\frac{1}{k|}}.$$

The second identity is obtained by taking the expectation on both sides of the first identity. □

Satz 4.4 —Recursion formula for $\ddot{a}_x$ —.

$$\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}.$$

Satz 4.5 —Constant force of mortality —.
For the constant force of mortality model,

$$\ddot{a}_x = \frac{1+i}{q_x + i} = \frac{1}{1 - e^{-(\delta+\mu)}} = \frac{1}{1 - vp_x},$$

where $q_x = 1 - e^{-\mu}$.

Proof. Recall that under the constant mortality assumption,

$$A_x = \frac{q_x}{i + q_x}.$$

Hence,

$$\ddot{a}_x = \frac{1 - A_x}{d} = \frac{1+i}{i}\frac{i}{i+q_x} = \frac{1+i}{i+q_x}$$
$$= \frac{e^{\delta}}{e^{\delta} + e^{-\mu}} = \frac{1}{1 - e^{-(\delta+\mu)}}.$$

□

Example 4.6. Suppose that $p_{x+k} = 0.97$, for each integer $k \geq 0$ and $i = 6.5\%$. Find $\ddot{a}_x$ and $\mathbb{V}\text{ar}(\ddot{Y}_x)$.

Example 4.7. Assume $i = 6\%$ and the De Moivre model with terminal age 100. Find $\ddot{a}_{30}$.

Solution. In general, for the De Moivre model, we have

$$_{k-1|}q_x = \frac{1}{\omega - x},$$

therefore,

$$A_x = \sum_{k=1}^{\omega-x} v^k \frac{1}{\omega - x} = \frac{1}{\omega - x} v \frac{1 - v^{\omega - x}}{1 - v} = \frac{1}{\omega - x}\frac{1 - v^{\omega-x}}{i},$$

which is, incidentally, equals the value of annuity-immediate $a_{\overline{\omega - x|}}$ divided by $\omega - x$. In our case,

$$A_{30} = \frac{a_{\overline{70|}}}{70} = 0.2341.$$

Hence

$$\ddot{a}_{30} = \frac{1 - A_{30}}{d} = 13.5315.$$

Example 4.8. John, age 65, has $\$750,000$ in his retirement account. An insurance company offers a whole life due annuity to John which pays $\$P$ at the beginning of the year while (65) is alive, for $\$750,000$. The annuity is priced assuming that $i = 6\%$ and the life table for the USA population in 2004 (see pages 628 - 631). The insurance company charges John 30% more of the APV of the annuity. Calculate P.

Solution. From the life table, we get that $\ddot{a}_{65} = 11.022302$. The APV of this annuity is $P\ddot{a}_{65} = (11.022)P$. We have that $750,000 = (1.3)(11.022302)P$ and $P = 52341.43257$.

Example 4.9. Consider the life table

x	80	81	82	83	84	85	86
l_x	250	217	161	107	62	28	0

An 80-year old buys a due life annuity which will pay $\$50,000$ at the beginning of the year. Suppose that $i = 6.5\%$. Calculate the single benefit premium for this annuity.

Answer: $\ddot{a}_{80} = 3.0116$, and $(50,000)\ddot{a}_{80} = 150,582.71$.

Example 4.10. Suppose that $\ddot{a}_x = \ddot{a}_{x+1} = 10$ and $q_x = 0.01$. Find i.

Answer: $i = 10\%$.

Example 4.11. An insurance company issues 800 identical due annuities to independent lives aged 65. Each of of this annuities provides an annual payment of $30,000$. Suppose that $p_{x+k} = 0.95$ for each integer $k > 0$, and $i = 7.5\%$.
(i) Find $\ddot{a}_x$ and $\mathbb{V}\text{ar}(\ddot{Y}_x)$.
(ii) Using the central limit theorem, estimate the initial fund needed at time zero in order that the probability that the present value of the random loss for this block of policies exceeds this fund is 1%.

Solution. (i) We have that

$$\begin{aligned}
\ddot{a}_x &= \frac{1+i}{q_x + i} = 8.6, \\
A_x &= \frac{q_x}{q_x + i} = 0.4, \\
{}^2A_x &= \frac{q_x}{q_x + i(2+i)} = 0.2445, \\
\mathbb{V}\text{ar}(\ddot{Y}_x) &= \frac{{}^2A_x - A_x^2}{d^2} = 17.3598.
\end{aligned}$$

Let Q be the fund needed. Then Q is $30,000\times$ the 99-th percentile of a normal r. v. with mean 800×8.6 and variance 800×17.3598. Hence,

$$Q = 30,000 \times \left[800 \times 8.6 + (2.326)\sqrt{800 \times 17.3598}\right] = 214,623,343.70.$$

4.2.2 Whole life immediate and other annuities

Here are the definitions of other types of annuities. For insurances in the table below, the payments are made only while the insured individual is alive.

Name of the annuity	Definition	Symbols for PV and APV
Whole life discrete due	Pays at the beginning of the year.	$\ddot{Y}_x$, $\ddot{a}_x$
Whole life discrete immediate	Pays at the end of the year.	Y_x, a_x
Whole life continuous	A continuous flow of payments with constant rate.	$\overline{Y}_x$, $\overline{a}_x$
Due n-year deferred	Pays at the beginning of the year starting in n years.	${}_{n\|}\ddot{Y}_x$, ${}_{n\|}\ddot{a}_x$
Immediate n-year deferred	Pays at the end of the year starting in n years.	${}_{n\|}Y_x$, ${}_{n\|}a_x$
An n-year deferred continuous	A continuous flow of payments starting in n years	${}_{n\|}\overline{Y}_x$, ${}_{n\|}\overline{a}_x$
Due n-year temporary	Pays at the beginning of the year for at most n years	$\ddot{Y}_{x:\overline{n}\|}$, $\ddot{a}_{x:\overline{n}\|}$
Immediate n-year temporary	Pays at the end of the year for at most n years	$Y_{x:\overline{n}\|}$, $a_{x:\overline{n}\|}$
An n-year temporary continuous	A continuous flow of payments for at most n years	$\overline{Y}_{x:\overline{n}\|}$, $\overline{a}_{x:\overline{n}\|}$

Annuity	Formula 1	Formula 2
$\ddot{a}_x$	$\frac{1-A_x}{d}$	$\sum_{k=0}^{\infty} {}_kE_x = \sum_{k=0}^{\infty} v^k \cdot {}_kp_x$
a_x	$\ddot{a}_x - 1 = \frac{v-A_x}{d}$	$\sum_{k=1}^{\infty} {}_kE_x = \sum_{k=1}^{\infty} v^k \cdot {}_kp_x$
$\overline{a}_x$	$\frac{1-\overline{A}_x}{\delta}$	$\int_0^{\infty} v^t \cdot {}_tp_x \, dt$
$\ddot{a}_{x:\overline{n}\rceil}$	$\frac{1-A_{x:\overline{n}\rceil}}{d}$	$\sum_{k=0}^{n-1} {}_kE_x = \sum_{k=0}^{n-1} v^k \cdot {}_kp_x$
$a_{x:\overline{n}\rceil}$	$\ddot{a}_{x:\overline{n+1}\rceil} - 1$	$\sum_{k=1}^{n} {}_kE_x = \sum_{k=1}^{n} v^k \cdot {}_kp_x$
$\overline{a}_{x:\overline{n}\rceil}$	$\frac{1-\overline{A}_{x:\overline{n}\rceil}}{\delta}$	$\int_0^{n} v^t \cdot {}_tp_x \, dt$
${}_{n\vert}\ddot{a}_x$	${}_nE_x \frac{1-A_{x+n}}{d} = \frac{{}_nE_x - {}_{n\vert}A_x}{d}$	$\sum_{k=n}^{\infty} {}_kE_x = \sum_{k=n}^{\infty} v^k \cdot {}_kp_x$
${}_{n\vert}a_x$	${}_{n\vert}\ddot{a}_x - {}_nE_x = {}_{n+1\vert}\ddot{a}_x$	$\sum_{k=n+1}^{\infty} {}_kE_x = \sum_{k=n+1}^{\infty} v^k \cdot {}_kp_x$
${}_{n\vert}\overline{a}_x$	${}_nE_x \frac{1-\overline{A}_{x+n}}{\delta} = \frac{{}_nE_x - {}_{n\vert}\overline{A}_x}{\delta}$	$\int_n^{\infty} v^t \cdot {}_tp_x \, dt$

For example,

$$\ddot{a}_{x:\overline{n}\rceil} = \mathbb{E}\big(1 + v + \ldots + v^{\min(K_x,n)-1}\big) = \mathbb{E}\frac{1 - v^{\min(K_x,n)}}{1-v} = \frac{1 - A_{x:\overline{n}\rceil}}{d}.$$

Satz 4.12.

$$\ddot{a}_{x:\overline{n}\rceil} = 1 + vp_x\ddot{a}_{x+1:\overline{n-1}\rceil}.$$

Definition 4.13. The *actuarial accumulated value* at time n of n-year temporary due annuity is defined by

$$\ddot{s}_{x:\overline{n}\rceil} = \frac{\ddot{a}_{x:\overline{n}\rceil}}{{}_nE_x}.$$

We have that $\ddot{a}_{x:\overline{n}\rceil} = {}_nE_x\ddot{s}_{x:\overline{n}\rceil}$. To take care that the number of living decreases over time, in actuarial computations, the n-year discount factor is ${}_nE_x = v^n \, {}_np_x$. So the actuarial accumulated value can be thought of as the actuarial future value of an n-year due life annuity to (x).

In other words, an insurer offers an n-year due life annuity to lives aged (x). The insurer can fund this annuity by either:

1. charging $\ddot{a}_{x:\overline{n}|}$ at time zero to each of the living.
2. charging $\ddot{s}_{x:\overline{n}|}$ at time n to each of the living at that time.

One other type of annuity is *certain due life annuity*.

Definition 4.14. An *n-year certain due life annuity* pays at the beginning of the year while either the individual is alive or the number of payments does not exceed n.

So if the individual dies before n-th year, then the annuity will still pay until n payments are made. If the individual dies after n-th year then the annuity pays while he or she is alive.

The PV and APV of this insurance are denoted $\ddot{Y}_{\overline{x:\overline{n}|}}$ and $\ddot{a}_{\overline{x:\overline{n}|}}$, respectively.

Satz 4.15.

$$\ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + {}_{n|}\ddot{a}_x = \ddot{a}_{\overline{n}|} + \sum_{k=n}^{\infty} v^k \cdot {}_kp_x$$

Example 4.16. An insurer offers a 20-year temporary life annuity due to lives age (60) with an annual payment of $\$40,000$. Mortality follows the life table for the US population in 2004 (see pages 628-631). The annual effective rate of interest is 6%. Calculate the APV of this life annuity.

Example 4.17. Suppose that $\overline{a}_x = 10$, $q_x = 0.02$ and $\delta = 0.07$. Deaths are uniformly distributed within each year of age. Find $\overline{a}_{x+1}$.

Example 4.18. You are given:
(i) $v = 0.94$.
(ii) $p_x = 0.99$.
(iii) $p_{x+1} = 0.95$.
(iv) $\ddot{a}_x = 5.6$.

Calculate $\ddot{a}_{\overline{x:\overline{3}|}}$.

4.2.3 Annuities paid m times a year

A whole life due annuity with payments paid m times a year is a series payments made at the beginning m-thly time interval while an individual is alive.

If a whole life due annuity for (x) pays the amount $\$\frac{1}{m}$ and does it m times a year, then its present value is denoted by $\ddot{Y}_x^{(m)}$ and the expectation of the present value (APV) is denoted by $\ddot{a}_x^{(m)}$.

Satz 4.19.
If $v \neq 1$, then

$$\ddot{Y}_x^{(m)} = \frac{1 - Z_x^{(m)}}{d^{(m)}},$$

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}},$$

$$\mathbb{V}\mathrm{ar}\big(\ddot{Y}_x^{(m)}\big) = \frac{{}^2A_x^{(m)} - \big(A_x^{(m)}\big)^2}{\big(d^{(m)}\big)^2}.$$

where

$$d^{(m)} = m\Big((1+i)^{1/m} - 1\Big).$$

Example 4.20. Suppose that $\mu(t) = 0.03$, and $\delta = 0.06$. Calculate $\ddot{a}_x^{(12)}$ and $\mathbb{V}\mathrm{ar}(\ddot{Y}_x^{(12)})$.

Section 4.3

Annuities – Woolhouse approximation

We studied previously that for evaluation of annuities paid monthly, a very convenient method is available if we assume the uniform of distribution of deaths (UDD). In this case, we have formula

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x,$$

where

$$i^{(m)} = m\Big((1+i)^{\frac{1}{m}} - 1\Big).$$

Then, for the life annuity we get

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}},$$

where

$$d^{(m)} = m\Big(1 - (1+i)^{-\frac{1}{m}}\Big).$$

Another method to approximate the value of these annuities is called the *Woolhouse method*. Recently, it has become quite popular in MLC exams, although it gives almost the same values as the UDD method. The method is based on the Euler-Maclaurin formula for numerical integration, which is an improvement of the trapezoidal rule.

$$\int_0^\infty g(t)\,dt = h\Big(\sum_{k=0}^{\infty} g(kh) - \frac{1}{2}g(0)\Big) + \frac{h^2}{12}g'(0) - \frac{h^4}{720}g''(0) + \ldots$$

for a positive constant h.

This formula is applied to the function $g(t) = v^t {}_tp_x$. By taking the derivative we find,

$$\begin{aligned} g'(t) &= (\log v)v^t {}_tp_x + v^t(\log {}_tp_x)' {}_tp_x, \\ g'(0) &= \log v - \mu_x = -\delta - \mu_x. \end{aligned}$$

Now, by taking either $h = 1$ or $h = 1/m$, we obtain two formulas for $\overline{a}_x$, one in terms of $\ddot{a}_x$ and another in terms of $\ddot{a}_x^{(m)}$. This allows us to express $\ddot{a}_x^{(m)}$ in terms of $\ddot{a}_x$.

We skip the details of the calculation and present the result:

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x).$$

This is the Woolhouse formula for the whole life annuity paid m times a year.

For the temporary life annuities we can use the formula

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_x^{(m)} - {}_nE_x \ddot{a}_{x+n}^{(m)}.$$

By using this formula and keeping only two terms in the Woolhouse approximation, we get:

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x).$$

This is the *Woolhouse formula with two terms*.

If we keep all three terms, we obtain the *Woolhouse formula with three terms,*

$$\begin{aligned}\ddot{a}^{(m)}_{x:\overline{n}|} =& \ddot{a}_{x:\overline{n}|} - \frac{m-1}{2m}(1 - {}_nE_x) \\ &- \frac{m^2-1}{12m^2}\big[\delta + \mu_x - {}_nE_x(\delta + \mu_{x+n})\big].\end{aligned}$$

In order to apply this formula, we need an estimate for mortality force function. Here is one possibility,

$$\mu_x \approx -\frac{1}{2}(\log p_{x-1} + \log p_x).$$

Example 4.21. You are given:

(1) $A_{35} = 0.188$
(2) $A_{65} = 0.498$
(3) ${}_{30}p_{35} = 0.883$
(4) $i = 0.04$

Calculate $1000\ddot{a}^{(2)}_{35:\overline{30}|}$ using the two-term Woolhouse approximation.

Solution. We have

$$\ddot{a}^{(2)}_{35:\overline{30}|} = \ddot{a}^{(2)}_{35} - {}_{30}E_{35}\ddot{a}^{(2)}_{65}.$$

Since $m = 2$, therefore,

$$\begin{aligned}\ddot{a}^{(2)}_{35} &= \ddot{a}_{35} - \frac{1}{4} = \frac{1 - A_{35}}{d} - \frac{1}{4} \\ &= \frac{1 - 0.188}{1 - 1.04^{-1}} - \frac{1}{4} = 20.8620\end{aligned}$$

Similarly,

$$\begin{aligned}\ddot{a}^{(2)}_{65} &= \frac{1 - A_{65}}{d} - \frac{1}{4} \\ &= \frac{1 - 0.498}{1 - 1.04^{-1}} - \frac{1}{4} = 12.8020\end{aligned}$$

Next,

$${}_{30}E_{35} = v^{30}{}_{30}p_{35} = 1.04^{-30} \times 0.883 = 0.2722$$

Hence,

$$1000\ddot{a}^{(2)}_{35:\overline{30}|} = 1000 \times (20.8620 - 0.2722 \times 12.8020) = 17,377$$

Section 4.4
Annuities and UDD assumption

Recall that the value of an annuity which is paid m-times a year can be approximated by the value of the corresponding annuity which is paid annually plus a correction term.

One method is the Woolhouse approximation given by the formula

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}(\delta + \mu_x).$$

Often, the third term is omitted.

Another method of approximation based on the assumption that deaths are distributed uniformly during each of the years when the contract is in force. This is known as the ***uniform distribution of deaths*** (UDD) assumption.

Satz 4.22.
Under the UDD assumption,

$$\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x - \beta(m),$$

where

$$\alpha(m) = \frac{i}{i^{(m)}} \times \frac{d}{d^{(m)}} \textit{ and } \beta(m) = \frac{i - i^{(m)}}{d^{(m)}}.$$

Proof. We know that the EPVs of the corresponding whole life insurances are related by the following exact formula:

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x.$$

At the same time, we have these relations:

$$A_x = 1 - d\ddot{a}_x \text{ and } A_x^{(m)} = 1 - d^{(m)}\ddot{a}_x^{(m)}.$$

It follows that

$$i^{(m)}\Big(1 - d^{(m)}\ddot{a}_x^{(m)}\Big) = i\Big(1 - d\ddot{a}_x\Big),$$

which gives

$$\ddot{a}_x^{(m)} = -\frac{i - i^{(m)}}{d^{(m)}} + \Big(\frac{i}{i^{(m)}} \times \frac{d}{d^{(m)}}\Big)\ddot{a}_x.$$

□

The formulas and selected values for $\alpha(m)$ and $\beta(m)$ can be found in the Illustrative Table.

Note that it can be shown that $\alpha(m) = 1 + O(1/m^2)$ and $\beta(m) = (m-1)/(2m)$ so that this approximation is close to the approximation by the Woolhouse formula with two terms.

By taking the limit $m \to \infty$, we get a similar formula for continuous annuities:

$$\overline{a}_x^{(m)} = \Big(\frac{i}{\delta} \times \frac{d}{\delta}\Big)\overline{a}_x - \frac{i - \delta}{\delta^2}.$$

Corollary 4.23.

For the temporary annuities, we have

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \alpha(m)\ddot{a}_{x:\overline{n}|} - \beta(m)\Big(1 - v^n \times {}_n p_x\Big)$$

Proof. This result immediately follows from the equation:

$$\ddot{a}_{x:\overline{n}|}^{(m)} = \ddot{a}_x^{(m)} - v^n \times {}_n p_x \times \ddot{a}_{x+n}^{(m)}.$$

□

♡ SECTION 5

Benefit Premiums

section 5.1

Funding a liability

When an insurance takes an insuree it assumes a liability. The insurance company will make one or several payments in the future. The present value of this liability is contingent on the death of the insuree. In this section, we study the benefit premiums needed to fund insurance liabilities.

Previously, we considered the value of an insurance product at issue time. The *net single premium* of an insurance product is the (actuarial present value) APV of the benefit payments for this insurance product.

However, usually insurance products are funded periodically while the contract is in hold. These payments are made while the individual is alive and the obligations of the contract are not expired. Payments made to fund an insurance contract are called *benefit premiums*, and they are usually made annually. The *annual premium* (also called the *net annual premium* and the *benefit annual premium*) is the amount which an insurance company allocates to fund an insurance product.

An insurance product is funded according with the *equivalence principle* if the actuarial present values of the funding scheme and of the contingent benefits agree.

The annual premium found under the equivalence principle is the basis to the payment charged to the insuree. Costs and commissions are added to this basis. The final value of each payment in an insurance contract is called a *contract premium*.

The *loss* of an insurance product is a random variable equal to the excess of the present value of benefit payments over the present value of funding.

section 5.2

Fully discrete benefit premiums: Pricing by equivalence principle

"If I let myself get bitten by a vampire so I become immortal and only a wooden stake can kill me, can I get a better rate on my life insurance?"

In this section, we will consider the funding of insurance products paid at the end of the year of death with annual benefits premiums made at the beginning of the year. The funding is made as far as the individual is alive and the term of the insurance has not expired.

This type of the insurance contract is called a *fully discrete life insurance*.

Let us denote the loss for a unit fully discrete whole life insurance by L_x.

Satz 5.1.

For a fully discrete whole insurance,
(i) The loss random variable is

$$L_x = Z_x - P\ddot{Y}_x = Z_x\Big(1 + \frac{P}{d}\Big) - \frac{P}{d}.$$

(ii) The actuarial present value of the loss L_x is

$$\mathbb{E}L_x = A_x - P\ddot{a}_x = A_x\Big(1 + \frac{P}{d}\Big) - \frac{P}{d}.$$

(iii) The variance of the loss L_x is

$$\mathbb{V}\mathrm{ar}(L_x) = \Big(1 + \frac{P}{d}\Big)^2 \mathbb{V}\mathrm{ar}(Z_x) = \Big(1 + \frac{P}{d}\Big)^2 \Big({}^2A_x - (A_x)^2\Big)$$

The benefit premium of a fully discrete whole insurance funded under the equivalence principle is denoted by P_x.

If a whole insurance is funded under the equivalence principle, then $\mathbb{E}L_x = 0$, that is, P_x is the annual benefit premium at which the insurer expects to break even.

Satz 5.2.

If a fully discrete whole insurance is funded using the equivalence principle, then

$$P_x = \frac{A_x}{\ddot{a}_x} = \frac{dA_x}{1 - A_x} = \frac{1}{\ddot{a}_x} - d$$

and

$$\mathbb{V}\mathrm{ar}(L_x) = \frac{{}^2A_x - (A_x)^2}{(1 - A_x)^2} = \frac{{}^2A_x - (A_x)^2}{(d\ddot{a}_x)^2}.$$

Satz 5.3.

Under constant force of mortality μ for life insurance funded through the equivalence principle, $P_x = vq_x$.

Example 5.4. Michael is 50 years old and purchases a whole life insurance policy with face value of $100,000$ payable at the end of the year of death. This policy will be paid by level benefit annual premiums at the beginning of each year while Michael is alive. Assume that $i = 6\%$ and death is modeled using De Moivre's model with terminal age 100.
(i) Find the net single premium for this policy.
(ii) Find the benefit annual premium for this policy.

(iii) Find the variance of the present value of the loss for this insurance contract.

Solution. (i) Under the De Moivre law, the probability to die in year k is $\frac{1}{\omega-x}$ for $1 \le k \le \omega - x$. Hence,

$$\begin{aligned}
A_x &= \frac{1}{\omega - x}\left(v + v^2 + \ldots + v^{\omega-x}\right) = \frac{1}{\omega-x}\frac{v}{1-v}\left(1 - v^{\omega-x}\right) \\
&= \frac{1}{\omega-x}\frac{1}{i}\left(1 - v^{\omega-x}\right) = \frac{a_{\overline{\omega-x}|i}}{\omega-x} \\
&= \frac{1}{50}\frac{1}{0.06}\left(1 - 1.06^{-50}\right) = 0.315237212728
\end{aligned}$$

and the net single premium is $\$31\,523.72$.

(ii) We have that

$$P_{50} = \frac{dA_{50}}{1 - A_{50}} = \frac{(0.06/1.06)(0.315237212728)}{1 - 0.315237212728} = 0.02605809799$$

and the benefit annual premium is $\$2\,605.81$.

(iii) We have that

$$^2A_{50} = \frac{1}{\omega - x}\frac{1}{i(2+i)}\left(1 - v^{2(\omega-x)}\right) = 0.1613354003,$$

Then,

$$\begin{aligned}
\mathbb{V}\text{ar}(L_{50}) &= 10^{10}\frac{^2A_{50} - (A_{50})^2}{(1 - A_{50})^2} = 10^{10}\frac{0.1613354003 - (0.315237212)^2}{(1 - 0.315237212)^2} \\
&= 1.32140947 \times 10^9
\end{aligned}$$

Example 5.5. Maria is 30 years old and purchases a whole life insurance policy with face value of $\$70,000$ payable at the end of the year of death. This policy will be paid by a level benefit annual premium at the beginning of each year while Maria is alive. Assume that $i = 6\%$ and death is modeled using the constant force of mortality $\mu = 0.02$.
(i) Find the net single premium for this policy.
(ii) Find the benefit annual premium for this policy.
(iii) Find the variance of the present value of the loss for this insurance contract.

Solution. (i)

$$A_{30} = \frac{q_x}{q_x + i} = \frac{1 - e^{-0.02}}{1 - e^{-0.02} + 0.06} = 0.2481328,$$

and the net single premium is $\$17\,369.30$.

(ii)

$$P_{50} = vq_x = (1.06)^{-1}(1 - e^{-0.02}) = 0.0186805,$$

and the benefit annual premium is $\$1\,307.63$.

(iii)

$$^2A_{30} = \frac{1 - e^{-0.02}}{1.06^2 - e^{-0.02}} = 0.138083288,$$

and

$$\begin{aligned}\mathbb{V}\text{ar}(L_{30}) &= (70\,000)^2 \frac{^2A_{50} - (A_{50})^2}{(1 - A_{50})^2} = (70\,000)^2 \frac{0.138083288 - (0.2481328007)^2}{(1 - 0.2481328007)^2} \\ &= 663\,210\,373\end{aligned}$$

Example 5.6. For a whole life insurance of $100,000$ on (x), you are given:

(i) Death benefits are payable at the moment of death.
(ii) Deaths are uniformly distributed over each year of age.
(iii) Premiums are payable monthly.
(iv) $i = 0.06$
(v) $\ddot{a}_x = 9.19$
Calculate the monthly net premium.

Solution. Let P denote the ***monthly*** net premium.

First, we calculate the EPV of premiums:

$$\begin{aligned}\text{EPV(premiums)} &= 12P\ddot{a}_x^{\mathbb{1}2} = 12P\Big[\alpha(12)\ddot{a}_x - \beta(12)\Big] \\ &= 12P\Big[1.00028 \times 9.19 - 0.46812] \\ &= 104.6934P\end{aligned}$$

Next, we calculate the EPV of benefits:

$$\begin{aligned}\text{EPV(benefits)} &= 100,000\overline{A}_x = 100,000\frac{i}{\delta}A_x \\ &= 100,000\frac{i}{\delta}(1 - d\ddot{a}_x) \\ &= 100,000\frac{0.06}{\log 1.06}\Big(1 - \frac{0.06}{1.06} \times 9.19\Big) \\ &= 49,406.59\end{aligned}$$

Finally,

$$P = \frac{49,406.59}{104.6934} = 471.92$$

section 5.3
Compensation for risk

If an insurer uses the equivalence principle, then it can expect to break even. However, an insurer should be compensated for facing losses. For this reason, the annual benefit premium in an insurance contract is usually bigger than the annual benefit premium obtained using the equivalence principle.

"You must realize that with your expected expiration date, your premiums are going to be astronomical."

The *risk charge* (or *security loading*) is the excess of the benefit annual premium over the benefit annual premium found using the equivalence principle.

The percentile annual premium can be found using either only one policy or an aggregate of policies.

Suppose that an insurer offers a whole life insurance paying $\$1$ at the end of the year of death to n lives aged x. The insurances are funded by a level annual benefit premium P collected at the beginning of each year.

Then the expected aggregate loss is

$$\mathbb{E}L_{Agg} = n(A_x - P\ddot{a}_x),$$

and the variance of the aggregate loss is

$$\mathbb{V}\mathrm{ar}(L_{Agg}) = n\Big(1 + \frac{P}{d}\Big)^2 \mathbb{V}\mathrm{ar}(Z_x).$$

Suppose that we aim to chose P so that the aggregate loss is non-positive (that is, aggregate profit is non-negative) with probability α, where α is large.

By the Central Limit Theorem, the aggregate loss is distributed approximately as a Gaussian random variable, and therefore, the ratio of expected profit to the standard deviation of the loss should equal z_α, the α-quantile of the standard normal distribution,

$$\frac{n(P\ddot{a}_x - A_x)}{\Big(1 + \frac{P}{d}\Big)\sqrt{n\mathbb{V}\mathrm{ar}(Z_x)}} = z_\alpha,$$

or

$$\sqrt{n}(P\ddot{a}_x - A_x) = z_\alpha\Big(1 + \frac{P}{d}\Big)\sqrt{{}^2A_x - A_x^2}.$$

This linear equation can be solved for the aggregate percentile annual premium P.

Example 5.7. An insurer offers a fully discrete whole life insurances of $\$10\,000$ on independent lives age 30. You are given:
(i) $i = 0.06$.
(ii) Mortality follows the life table in page 628.
(iii) The annual contract premium for each policy is $1.25P_x$.

Using the normal approximation, calculate the minimum number of policies the insurer must issue so that the probability that the aggregate loss for the issued policies is less than 0.05.

Solution. From the table we find:

$$1000A_{30} = 82.29543, \text{ and } 1000 \cdot {}^2A_{30} = 17.96859,$$

which implies

$$\mathbb{V}\mathrm{ar}(Z_{30}) = 0.01119605, \text{ and } \sqrt{\mathbb{V}\mathrm{ar}(Z_{30})} = 0.1058114.$$

Also from the table:

$$\ddot{a}_{30} = 16.212781$$

and therefore

$$P_{30} = 10\,000\frac{A_{30}}{\ddot{a}_{30}} = \frac{822.9543}{16.212781} = 50.75960$$

and the annual contract premium is

$$P = 1.25P_{30} = 63.4495$$

Hence we should have

$$\sqrt{n}\Big(\frac{P}{B}\ddot{a}_x - A_x\Big) = z_\alpha\Big(1 + \frac{P}{Bd}\Big)\sqrt{{}^2A_x - A_x^2},$$

which we solve for $\sqrt{n}$,

$$\begin{aligned}\sqrt{n} &= z_\alpha\frac{\Big(1 + \frac{P}{Bd}\Big)\sqrt{{}^2A_x - A_x^2}}{\frac{P}{B}\ddot{a}_x - A_x} \\ &= z_\alpha\frac{\Big(B + \frac{P}{d}\Big)\sqrt{{}^2A_x - A_x^2}}{P\ddot{a}_x - BA_x}\end{aligned}$$

From the normal distribution table, $z_{0.95} = 1.64$, and

$$\begin{aligned}\sqrt{n} &= 1.64\frac{(10\,000 + \frac{63.4495}{0.06/1.06}) \cdot 0.1058114}{63.4495 \cdot 16.212781 - 822.9543} \\ &= 9.379986.\end{aligned}$$

Therefore, $n = \lceil 9.379986^2 \rceil = 88$.

Now, suppose that the funding scheme is limited to the first t years. From the equivalence principle we have

$$\mathbb{E}Z_x = P\mathbb{E}\ddot{Y}_{x:\overline{t}|}.$$

The benefit premium found in this way is denoted by ${}_tP_x$.

$${}_tP_x = \frac{A_x}{\ddot{a}_{x:\overline{t}|}}.$$

Example 5.8. Ethan is 30 years old and purchases a whole life insurance policy with face value of $\$50\,000$ payable at the end of the year of death.

This policy will be paid by a level benefit annual premium at the beginning of the next 30 years while Ethan is alive.

Assume that $\delta = 0.05$ and death is modeled using the constant force of mortality $\mu = 0.03$. Find the benefit annual premium for this policy.

section 5.4

Premiums paid m times a year


"I want the non-smokers rate, because I only smoke when I'm visiting my insurance agent."

Often insurance is funded several times a year. The computation of the premium when premiums are paid m times a year is similar to the annual case. The *total* amount of payments made a year is called the *annual funding rate* and it denoted by $P^{(m)}$. The funding payment paid m times a year is $\frac{P^{(m)}}{m}$.

Consider the case when the insurance is paid at the end of year of death and funded with level payments made at the beginning of the period of $\frac{1}{m}$ years while the individual is alive. We assume the equivalence principle is used.

For a whole life insurance to (x), the annual benefit premium is

$$P_x^{(m)} = \frac{A_x}{\ddot{a}_x^{(m)}}.$$

We can write

$$\ddot{a}_x^{(m)} = \frac{1 - A_x^{(m)}}{d^{(m)}},$$

where

$$d^{(m)} = m(1 - v^{1/m}).$$

And if we assume the uniform distribution of deaths, then we have

$$A_x^{(m)} = \frac{i}{i^{(m)}} A_x,$$

where

$$i^{(m)} = m\Big((1+i)^{1/m} - 1\Big).$$

Example 5.9. Consider the life table

x	80	81	82	83	84	85	86
l_x	250	217	161	107	62	28	0

Assume that $i = 6.5\%$ and that the distribution of deaths is uniform over each year of death.

Find $P_{80}^{(12)}$, using that $A_{80} = 0.8161901166$.

Solution. We calculate

$$A_{80}^{(12)} = \frac{i}{i^{(12)}} A_{80} = \frac{0.065}{12\Big((1.065)^{1/12} - 1\Big)} 0.8161901166 = 0.8402293,$$

$$\ddot{a}_{80}^{(12)} = \frac{1 - A_{80}^{(12)}}{d^{(12)}} = \frac{1 - 0.8402293}{12(1 - 1.065^{-1/12})} = 2.543720$$

$$P_{80}^{(12)} = \frac{A_{80}}{\ddot{a}_{80}^{(12)}} = \frac{0.8161901166}{2.543720} = 0.3208648.$$

section 5.5

Benefit Premiums - Adjusting for Expenses

5.5.1 Fully discrete insurance

When finding the annual premium expenses and commissions have to be taken into in account. Possible costs are underwriting (making the policy) and maintaining the policy.

The annual premium which an insurance company charges is called the *gross annual premium*. It is also often called the *contract premium*, the *loaded premium* and the *expense-augmented premium*.

Usual types of expenses are:

(1) Issue cost.
(2) Percentage of annual benefit premium.
(3) Fixed amount per policy.
(4) Percentage of (face value) contract amount.

(5) Settlement cost.

Often the expenses related with the contract amount, are given as per thousand expenses, i.e. the per thousand expenses are the expenses made for each $\$1000$ of the face value of the insurance.

Example 5.10. Suppose that we have a whole life insurance on (x), with a death benefit of b paid at the end of the year of death. The fixed annual cost has an amount of e. In the the first year, there exists an additional cost of e_0^*. The percentage of the expense-augmented premium paid in expenses each year is r. During the first year, it is paid an additional percentage of the expense-augmented premium of r_0^*. The settlement cost is s. All cost except the settlement cost are paid at the beginning of the year. The insurance is funded by an expense-augmented premium of G paid at the beginning of the year while (x) is alive. If the equivalence principle is used, then

$$G = \frac{e_0^* + (b+s)A_x + e\ddot{a}_x}{(1-r)\ddot{a}_x - r_0^*}.$$

Indeed, the expected present value of the insurance liabilities and expenses is

$$bA_x + e\ddot{a}_x + e_0^* + rG\ddot{a}_x + r_0^* G + sA_x.$$

The expected present value (APV) of benefit premiums is $G\ddot{a}_x$. Using the equivalence principle,

$$G\ddot{a}_x = e_0^* + (b+s)A_x + e\ddot{a}_x + G(r\ddot{a}_x + r_0^*),$$

which implies that

$$G = \frac{e_0^* + (b+s)A_x + e\ddot{a}_x}{(1-r)\ddot{a}_x - r_0^*}.$$

Recall that the annual benefit payment of the unit insurance can be calculated as

$$P_x = \frac{A_x}{\ddot{a}_x},$$

and that

$$\ddot{a}_x = \mathbb{E}(1 + v + \ldots + v^{K_x}) = \frac{1 - \mathbb{E}(v^{K_x+1})}{1-v} = \frac{1-A_x}{d},$$

so that $A_x = 1 - d\ddot{a}_x$ and, therefore, $P_x + d = 1/\ddot{a}_x$.

By using these identities, we can see that the expence-augmented premium can be expressed in terms of the net premium as follows.

$$G = \frac{e_0^*(P_x + d) + (b + s)P_x + e}{(1 - r) - r_0^*(P_x + d)}.$$

Example 5.11. A fully discrete whole life insurance policy with face value of $\$50,000$ is made to (x). The following costs are incurred:

(1) $\$800$ for making the contract.
(2) Percent of expense-loaded premium expenses are 6% in the first year and 2% thereafter.
(3) Per thousand expenses are $\$2$ per year.
(4) $P_x = 0.11$.
(5) All expenses are paid at the beginning of the year.
(6) $d = 5\%$.

Calculate the expence-augmented annual premium using the equivalence principle.

Solution. We have that

$$G\ddot{a}_x = (50,000)A_x + 800 + (0.04)G + (0.02)G\ddot{a}_x + (2)(50)\ddot{a}_x.$$

Hence,

$$\begin{aligned} G &= \frac{(50,000)A_x + 800 + (2)(50)\ddot{a}_x}{(0.98)\ddot{a}_x - 0.04} \\ &= \frac{(50,000)P_x + 800(P_x + d) + (2)(50)}{(0.98) - (0.04)(P_x + d)} \\ &= 5,883.32. \end{aligned}$$

5.5.2 Fully continuous insurance

The fully continuous case is similar to the fully discrete case. Let b be the death benefit paid at the time of the death. The fixed issue cost is e_0^*. The percentage of the expense-augmented premium paid in expenses at issue is r_0^*. There is an annual rate of contract expenses of e paid continuously while (x) is alive. The percentage of the expense-augmented premium paid continuously in expenses while (x) is alive is r. The settlement cost is s.

Let G be the expense-augmented annual premium rate using the equivalence principle. Then, we have that

$$G\overline{a}_x = b\overline{A}_x + e_0^* + r_0^* G + e\overline{a}_x + rG\overline{a}_x + s\overline{A}_x,$$

which implies

$$G = \frac{e_0^* + (b+s)\overline{A}_x + e\overline{a}_x}{(1-r)\overline{a}_x - r_0^*}.$$