LaTeX submissions are mandatory. The template for this problem can be found on the Piazza resource page for this course.

**Problem 1**
Let

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix},$$

$A$ has exactly two distinct eigenvalues, which are $-2$, and 1.

If possible, construct matrices $P$ and $D$ such that $A = PDP^t$ , $P$ is a matrix with orthonormal columns, and $D$ is a diagonal matrix.

**Problem 2**

In each of the following cases determine whether the stochastic matrix $P$ is reversible:

1. $$\begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix};$$

   ($0 < p < 1$ and $0 < q < 1$.)

2. $$\begin{bmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{bmatrix};$$

   ($0 < p < 1$)

3. The state space is $\{0, 1, \ldots, N\}$ and $p_{ij} = 0$ if only if $|j - i| \geq 2$.

   Use the notation $p_i = p_{i,i+1} > 0$, $q_i = p_{i+1,i} > 0$, where $i = 0, N - 1$. Here is an example for $N = 2$:

   $$P = \begin{bmatrix} 1-p_0 & p_0 & 0 \\ q_0 & 1-p_0-q_0 & p_1 \\ 0 & q_1 & 1-p_1-q_1 \end{bmatrix}$$

**Problem 3**
True or False? If true, explain. If false, give a counterexample.

(a) Let A and B are two symmetric positive definite matrices. Then $A + B^{-1}$ is positive definite.

(b) Suppose $A$ is positive definite and $B$ is indefinite (i.e., there are $x \neq 0$ and $y \neq 0$ so that $x^tBx > 0$ and $y^tBy < 0$). Then $A + B$ is indefinite.

**Problem 4**

Recall that a symmetric matrix $A$ is called positive semi-definite (or non-negative definite) if $x^t Ax \geq 0$ for all $x$.

Find a $n \times n$ matrix $A$ such that $\det(A_k) \geq 0$ for all $k = 1, \ldots, n$ but the matrix $A$ is not positive semidefinite. ($A_k$ are the upper-left corner matrices of $A$.) [Hint: can you do it for $n = 2$?]

**Problem 5**

Let

$$A = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 7 & -5 \\ 2 & -5 & 8 \end{bmatrix}$$

Find a upper-triangular nonsingular matrix $C$, such that $D = C^t AC$ is diagonal, and find $sign(A)$, the signature of $A$.

**Problem 6**

Determine whether each of the following quadratic forms $Q$ is positive definite:

(a) $Q(x, y, z) = x^2 + 2y^2 - 4xz - 4yz + 7z^2$.

(b) $Q(x, y, z) = x^2 + y^2 + 2xz + 4yz + 3z^2$.