Math 447 **Test 3** **April 28, Spring 2016**
Instructor: Vladislav Kargin

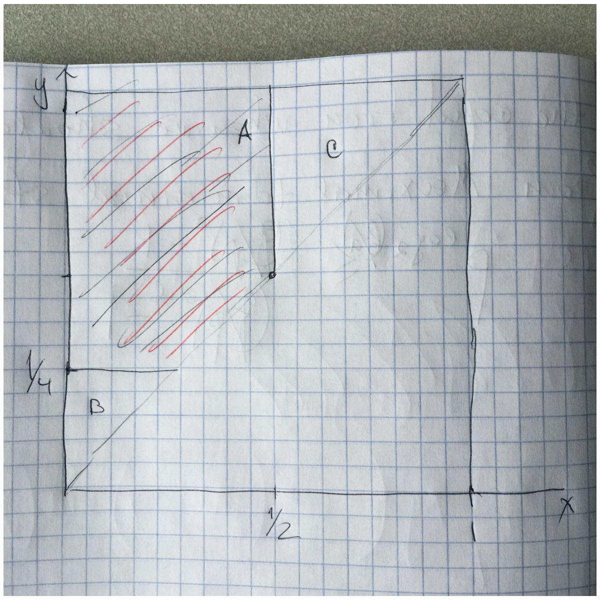
No books, no notes, only SOA-approved calculators. You must show work, unless the question is a true/false or fill-in-the-blank question.

Name: ____

Question	Points	Score
1	16	
2	6	
3	6	
4	6	
5	6	
6	11	
7	12	
8	9	
Total:	72	

1. Let X and Y represent the lifetimes in hours of two linked components in an electronic device. The joint density function for X and Y is uniform over the region defined by $0 < X < Y < 1$.

   (a) (2 points) What is probability that $X < 1/2$ and $Y > 1/4$? (Is it greater or less than 1/2? Make sure that it agrees with your intuition.)

   This probability is the area of the region A. So it is $1/2 - \text{area}(B) - \text{area}(C) = 1/2 - 1/8 - 1/32 = 11/32$. Well, something wrong here. The region A is clearly takes more than a half of the density support. OK, we should not forget to divide by the area of the support. $\mathbb{P} = \frac{11}{32} / \frac{1}{2} = \frac{11}{16}$.
   And we should remember that the density equals 2 on the support.

   (b) (3 points) Determine the expected value of the product XY.

$$\mathbb{E}(XY) = \int_0^1 \int_0^y xy2dxdy = \frac{1}{4}.$$

   (c) (3 points) Compute the marginal density of Y.

$$f_Y(y) = \int_0^y 2dx = 2y.$$

(d) (3 points) Compute the conditional density of Y given $X = x$. Conditional marginal density of Y is a positive constant on the interval $[x, 1]$. Hence it equals $\frac{1}{1-x}$ on this interval:

$$f(y|x) = \frac{1}{1-x} \text{ for } y \in [x, 1].$$

(e) (3 points) Compute the conditional expectation of Y given $X = 1/3$.

By using the previous formula we calculate

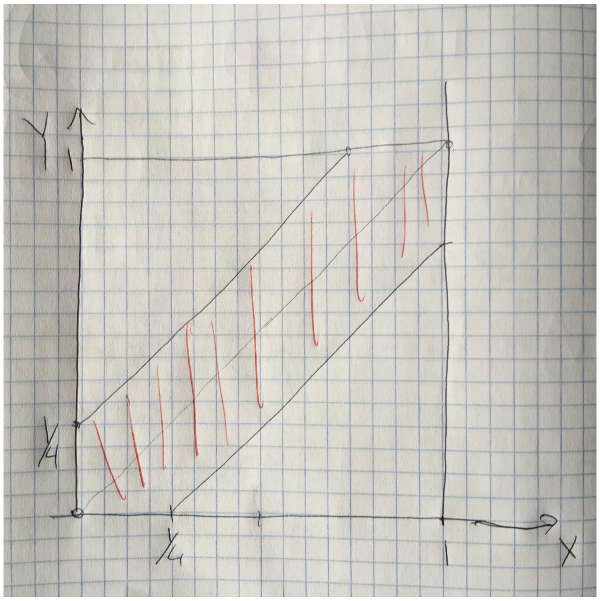
$$\mathbb{E}(Y|X = 1/3) = \frac{3}{2} \int_{1/3} 1y dy = 2/3.$$

Or we can just notice that Y is uniform on the interval $[1/3, 1]$ so its expectation is in the middle of the interval, at $2/3$.

(f) (2 points) Are variables X and Y independent? Why?

No, the support is not rectangular.

2. (6 points) Polina and Anton agree to meet at the Bolshoi Theatre at noon. By agreement, the first to arrive will wait 15 minutes for the second, after which he (or she) will leave. Each of them is busy and absent-minded, so the times of their arrivals are independent and uniformly distributed on the interval between 12 noon and 1PM. What is the probability that they actually meet?



This is the region of interest. It is easy to calculate that its area (and the corresponding probability) is 7/16.

3. (6 points) The random variables X_1, X_2, and X_3 are independent. The distribution of X_1 is exponential with parameter $\beta = 1$. The random variable X_2 is normal with mean 1 and variance 4. And X_3 is a beta-distributed random variable with parameters $\alpha = 2$ and $\beta = 3$. Define $Y_1 = X_1 + 2X_2$ and $Y_2 = 3X_1 + 4X_3 + 5$. What is the covariance of Y_1 and Y_2?

$$\mathrm{Cov}(X_1 + 2X_2, 3X_1 + 4X_3 + 5) = 3\mathrm{Cov}(X_1 X_1).$$

All other terms are zero because of independence of these random variables.

So $\mathrm{Cov}(Y_1, Y_2) = 3\mathrm{Cov}(X_1 X_1) = 3\mathbb{V}(X_1) = 3$.

4. Three dies are tossed. The event A_{ij} occurs then the numbers on the die number i and the die number j are the same.

   (a) (3 points) Are events A_{12} and A_{13} independent. Why?

   Yes.

$$\mathbb{P}(A_{12} \cup A_{13}) = 6\frac{1}{6}\frac{1}{6}\frac{1}{6} = \frac{1}{36} = \mathbb{P}(A_{12})\mathbb{P}(A_{13}).$$

   Informally, the fact that we know that the numbers on dies 1 and 2 coincide does not help us to decide if the numbers on dies 1 and 3 coincide.

   (b) (3 points) Are events A_{12}, A_{23} and A_{13} jointly independent. Why? No. For example,

$$\mathbb{P}(A_{23}|A_{12} \cup A_{13}) = 1 \neq \mathbb{P}(A_{23}) = \frac{1}{6}.$$

   Informally, if we know that the numbers on dies 1 and 2 coincide and numbers on dies 1 and 3 coincide, then we know that the numbers on dies 2 and 3 coincide.

5. A traveler put his passport in one of the 3 drawers of his desk, but he forgot which one. Before starting a journey he nervously tries to find it. He tries the drawers at random, so he might try the same drawer several times. Even if he checks the drawer in which he had actually put the passport, he does not notice it with probability 1/2. It takes him 1 minute to search a drawer whether he finds his passport or not. If he does not find it, then he closes the drawer, and starts the search anew.

   (a) (3 points) At his first attempt, he checks a drawer and does not find the passport. What is the probability that the passport is actually in this drawer?

   Let F be event that he did not found his passport, and In be event that the passport is actually in the drawer. Then by the Bayes formula,

$$\mathbb{P}(In|F) = \frac{\mathbb{P}(F|In)\mathbb{P}(In)}{\mathbb{P}(F|In)\mathbb{P}(In) + \mathbb{P}(F|\overline{In})\mathbb{P}(\overline{In})} = \frac{\frac{1}{2}\frac{1}{3}}{\frac{1}{2}\frac{1}{3} + 1 \cdot \frac{2}{3}} = \frac{1}{5}.$$

   (b) (3 points) What is the expected time of the traveler's search ? The search time is geometric random variable with probability $p = \frac{1}{2}\frac{1}{3} = \frac{1}{6}$, so the expected search time is $1/p = 6$ minutes.

6. The random variables X and Y have the joint density e^{-y} for $0 < x < y$ and zero otherwise.

(a) (2 points) Are these variables independent? No, the support is not rectangular.

(b) (3 points) Find the marginal density of X.

$$\int_x^\infty e^{-y} dy = e^{-x} \text{ for } x > 0.$$

(c) (3 points) Find the conditional density of Y given $X = x$. (Don't forget to specify the support of the density.)

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = e^{x-y} \text{ for } y \in [x, \infty).$$

(d) (3 points) Find the conditional expectation of Y given $X = x$.

$$\mathbb{E}(Y|X = x) = \int_x^\infty y e^{x-y} dy = e^x \left(-ye^{-y} \Big|_x^\infty + \int_x^\infty e^{-y} dy \right) = x + 1.$$

7. Suppose that the probability that a head appears when a coin is tossed is p and the probability that a tail occurs is $q = 1 - p$. Person A tosses the coin until the first head appears and stops. Person B does likewise. Let Y_1 and Y_2 denote the number of times that persons A and B toss the coin, respectively. It is reasonable to assume that Y_1 and Y_2 are independent.

   (a) (3 points) What is the probability that A and B stop on exactly the same number toss?

   This problem was in homework and in a sample exam.

$$\mathbb{P}(Y_1 = Y_2) = \sum_{n=1}^{\infty} [q^{n-1}p]^2 = \ldots \text{(sum of a geometric progression)} \ldots = \frac{p}{2-p}.$$

   (b) (3 points) Find $\mathbb{E}(Y_1)$

   This is a geometric random variable, so the expectation is $1/p$.

   (c) (3 points) Find $\mathbb{V}(Y_1)$

   For the same reason, variance is $(1-p)/p^2$.

   (d) (3 points) Find $\mathbb{V}(Y_1 - Y_2)$

$$\mathbb{V}(Y_1 - Y_2) = \mathbb{V}(Y_1) + \mathbb{V}(Y_2) = \frac{2(1-p)}{p^2}.$$

8. Random variables X and Y are uniformly distributed on the interval $[0, 1]$. Let $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$.

(a) (3 points) Find $\mathbb{E}(U)$.

$\min\{x, y\} = x$ if $y > x$ and $= y$ otherwise. So we can write:

$$\mathbb{E} \min\{x, y\} = \int_0^1 \int_x^1 x dy dx + \int_0^1 \int_0^x y dy dx = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

(b) (3 points) Find $\mathbb{E}(V)$.

It is $2/3$ by the symmetry of the problem.

(c) (3 points) Find $\mathrm{Cov}(U, V)$.

Since $\min\{x, y\} \times \max\{x, y\} = xy$, we can write:

$$\mathbb{E}(UV) = \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) = \frac{1}{4},$$

and

$$\mathrm{Cov}(U, V) = \frac{1}{4} - \frac{1}{3}\frac{2}{3} = \frac{1}{36}.$$