Math 447 **Test 2** **March 17, Spring 2016**

No books, no notes, only SOA-approved calculators. You must show work, unless the question is a true/false or fill-in-the-blank question.

Name: ____

Question	Points	Score
1	6	
2	6	
3	9	
4	12	
5	4	
6	12	
7	6	
8	6	
9	10	
10	5	
Total:	76	

1. Let X has distribution function

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{x}{8}, & 0 \le x < 2, \\ \frac{x^2}{16}, & 2 \le x < 4, \\ 1, & x \ge 4. \end{cases}$$

   (a) (3 points) What is the density of X?
   Solution:

$$f(x) = \begin{cases} 0, & y \le 0, \\ \frac{1}{8}, & 0 < x < 2, \\ \frac{x}{8}, & 2 \le x < 4, \\ 0, & x \ge 4. \end{cases}$$

   (b) (3 points) Find the mean of X.
   Solution: This is exercise 4.25 from the text.

$$E[X] = \int_{-\infty}^{\infty} xf(x)dy = \int_0^2 \frac{x}{8}dx + \int_2^4 \frac{x^2}{8}dx.$$

   This integration yields $31/12 \approx 2.5833$ after some calculation.

2. Scores on an examination are assumed to be normally distributed with mean 78 and variance 36.
   (a) (3 points) Suppose that students scoring in the top 10% of this distribution are to receive an A grade. What is the minimum score a student must achieve to earn an A grade?
   Solution:

$$P(X > x) = P(Z > \frac{x-78}{6}) = 10\%.$$

   Hence

$$\frac{x-78}{6} = 1.28,$$

   and

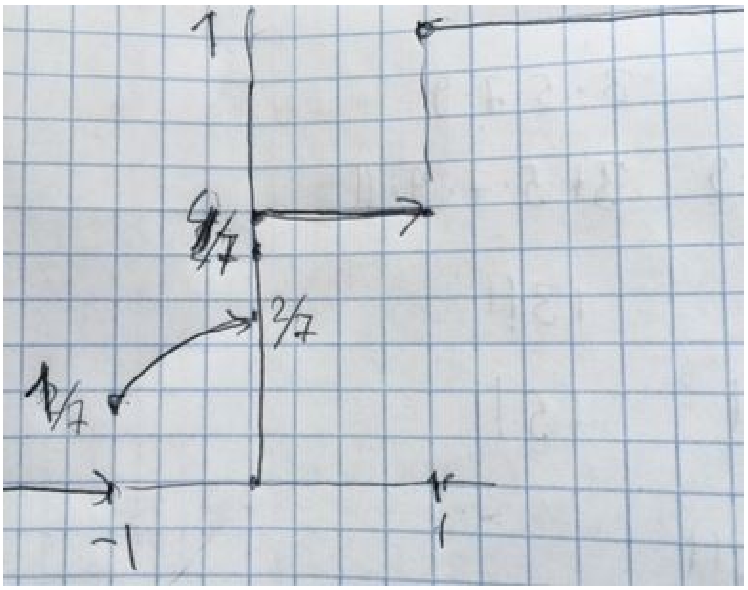
$$x = 6 \cdot 1.28 + 78 = 85.68.$$

   (b) (3 points) If it is known that a student's score exceeds 72, what is the probability that his or her score exceeds 84?

$$P(X > 84|X > 72) = \frac{P(X > 84)}{P(X > 72)} = \frac{P(Z > \frac{84-78}{6})}{P(Z > \frac{72-78}{6})} = \frac{P(Z > 1)}{1 - P(Z > 1)} = 18.85\%.$$

3. Let X have the cdf:

$$F(x) = \begin{cases} 0, & x < -1, \\ \frac{2-x^2}{7}, & -1 \le x < 0, \\ \frac{4}{7}, & 0 \le x < 1, \\ 1, & 1 \le x. \end{cases}$$



(a) (2 points) Find $P(X = 0)$.
Solution:

$$P(X = 0) = \frac{4}{7} - \frac{2}{7} = \frac{2}{7}.$$

(b) (2 points) Let A be the set $\{-2, -1/2, 0, 1, \pi/4\}$. Find $P(X \in A)$.
Solution: The only two points in the set that have positive probability mass are 0 and 1. Adding their masses together we get:

$$P(X \in A) = \frac{2}{7} + \frac{3}{7} = \frac{5}{7}.$$

(c) (2 points) Find $P(-1 \le X \le 0)$.

$$P(-1 \le X \le 0) = P(X \le 0) - P(X < -1) = \frac{4}{7} - 0 = \frac{4}{7}.$$

(d) (3 points) Find $E(X)$.

$$d\mu = \frac{1}{7}\delta_{-1} + \frac{2}{7}\delta_0 + \frac{3}{7}\delta_1 + f(x)dx,$$

where $f(x) = -2x/7$ if $x \in [-1, 0]$ and 0, otherwise.
Hence

$$E(X) = \frac{1}{7} \cdot (-1) + \frac{2}{7} \cdot 0 + \frac{3}{7} \cdot 1 + \int_{-1}^{0} -\frac{2x^2}{7} dx = \frac{4}{21}$$

4. A continuous random variable X has pdf $f(x) = x + ax^2$ on $[0, 1]$ and 0 elsewhere.

(a) (3 points) Find a.

Solution: By a property of density funcion,

$$\int_0^1 (x + ax^2)dx = \Big|_0^2 \frac{x^2}{2} + a\frac{x^3}{3} = \frac{1}{2} + a\frac{1}{3} = 1.$$

Hence, $a = \frac{3}{2}$.

(b) (3 points) Find the CDF.

Solution: By integrating the pdf, we find that

$$F(x) = \begin{cases} \frac{1}{2}(x^2 + x^3), & \text{if } 0 \leq x < 1, \\ 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 1. \end{cases}$$

(c) (3 points) Find $P(0.5 < X < 1)$.

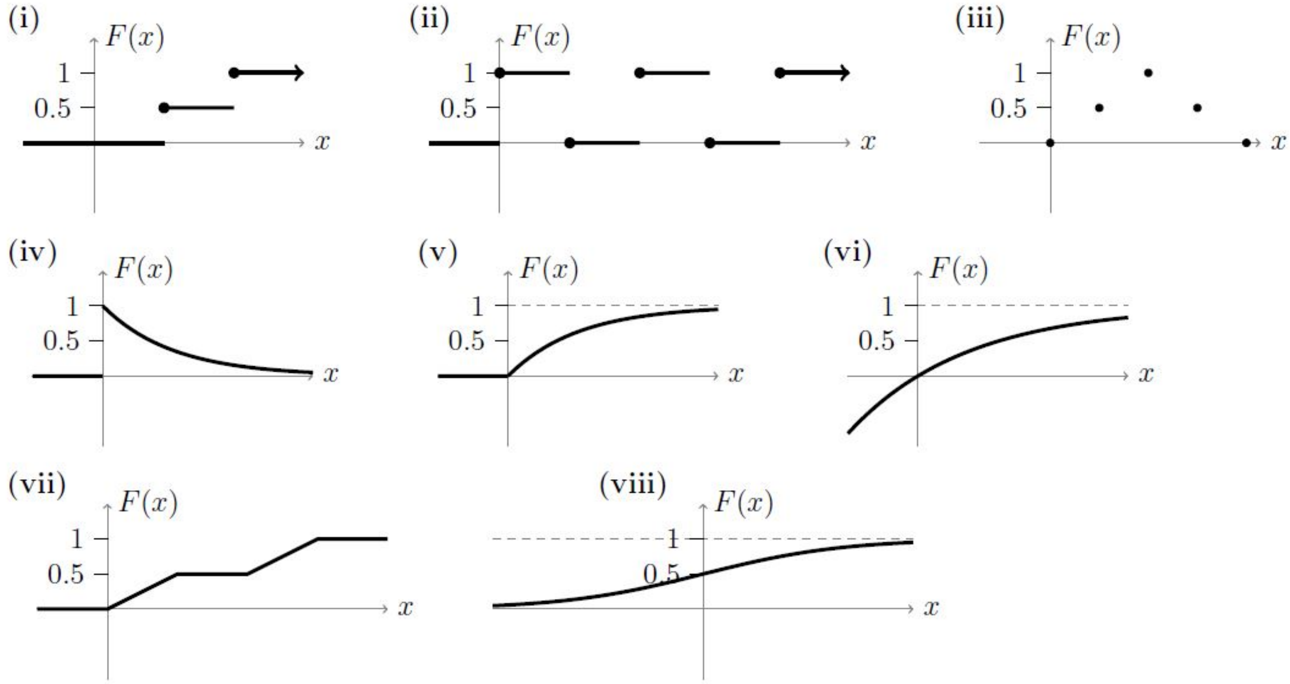
Solution: This is

$$F(1) - F(0.5) = 1 - \frac{(\frac{1}{2})^2 + (\frac{1}{2})^3}{2} = 1 - \frac{3}{16} = \frac{13}{16}.$$

(d) (3 points) Find the mean of X.

Solution:

$$E(X) = \int_0^1 x(x + \frac{3}{2}x^2)dx = \Big|_0^1 (\frac{x^3}{3} + \frac{3x^4}{8}) = \frac{1}{3} + \frac{3}{8} = \frac{17}{24}.$$



5.

[No work needed for this question.]

(a) (2 points) Which functions above are valid cdf's?

Solution: (i), (v), (vii), (viii).

(b) (2 points) Which functions are cdf's of continuous r.v.'s?

Solution: (v), (vii), (viii).

6. Let X be an exponential r.v. with parameter $\beta = 1$.

(a) (2 points) Find the median of this distribution.

Solution: Let the median be denoted m.

$$P(X \geq m) = e^{-m} = 0.5.$$

Hence $m = \ln 2 \approx 0.69$.

(b) (3 points) Find $P(|X - \mu| \geq 2\sigma)$, where μ and σ are the expectation and standard deviation of X.

Solution: $\mu = 1$, $\sigma = 1$, so we are interested in probability

$$P(|X - 1| \geq 2) = P(X \leq -1 \text{ or } X \geq 3) = P(X \geq 3) = e^{-3} \approx 5\%.$$

(c) (2 points) What is the estimate for this probability given by Chebyshev's inequality?

Solution: $\frac{1}{4} = 25\%$.

(d) (3 points) What is $P(X^2 + 2 \geq 3X)$?

$$P(X^2 + 2 \geq 3X) = P((X-1)(X-2) \geq 0) = P(X \leq 1) + P(X \geq 2) = 1 - e^{-1} + e^{-2} \approx 76.7\%.$$

(e) (2 points) What is $E(X^7)$?

$$E(X^7) = \int_0^\infty x^7 e^{-x} dx = \Gamma(8) = 7! = 5040.$$

This can also be done by differentiating mgf of X, $(1-t)^{-1}$, seven times and evaluating the result at $t = 0$.

7. Let X be an random variable with moment generating function $(1-2t)^{-2}$. Let Y be an random variable with moment generating function $1/(1-3.2t)$. Let Z be an random variable with moment generating function e^{5t+6t^2}.

[No work needed.]

(a) (2 points) What is the distribution of X?

Solution: The correct answer is (D). It is gamma with parameters $\alpha = 2$ and $\beta = 2$, so it is chi-square with 4 degrees of freedom

(A) normal

(B) beta

(C) uniform

(D) chi-square

(E) exponential

(F) none of the above

(b) (2 points) What is the distribution of Y?

Solution: The correct answer is (D). It is exponential with $\beta = 3.2$.

(A) uniform

(B) chi-square

(C) beta

(D) exponential

(E) normal

(F) none of the above

(c) (2 points) What is the distribution of Z?

Solution: The correct answer is (E). It is the mgf of normal distribution with $\mu = 5$ and $\sigma^2 = 6$.

(A) chi-square

(B) beta

(C) gamma

(D) uniform

(E) normal

(F) none of the above

8. The minimum force required to break a particular type of cable is normally distributed with mean $12,432$ and standard deviation 25. A random sample of 400 cables of this type is selected.

(a) (3 points) What is the probability that a randomly chosen cable will break under a force of $12,400$?

Solution: The correct answer is (C). Let X be the minimum force required to break a cable. Then

$$P(X \leq 12,400) = P(Z \leq \frac{12,400 - 12,432}{25}) = P(Z \geq 1.28) = 10\%.$$

(A) 2.5%
(B) 5%
(C) 10%
(D) 20%
(E) Other

(b) (3 points) Calculate the probability that at least 349 of the selected cables will not break under a force of $12,400$.
[Don't forget to show your work.]

Solution: The correct answer is (D). Let Y be the number of cables that will break. Then $Y \sim \text{Binom}(400, 0.1)$, where 0.1 is the probability from the previous part. This distribution can be approximated by the normal distribution $N(400 \cdot 0.1, 400 \cdot 0.1 \cdot 0.9) = N(40, 36)$. Then,

$$P(Y \leq 51) = P(Z \leq \frac{51 - 40}{6}) = 1 - P(Z > 1.83) \approx 97\%.$$

(A) 0.62
(B) 0.67
(C) 0.92
(D) 0.97
(E) 1.00

9. A man possesses five coins, two of which are double-headed, one is double-tailed, and two are normal. He shuts his eyes, picks a coin at random, and tosses it.

(a) (2 points) What is the probability that the lower face of the coin is a head?
Solution: Let HH be the event the selected coin is double headed, TT be the event that it is double-tailed and N that it is normal. Let H_l denote the event that the lower face of the coin is a head. Then,

$$P(H_l) = P(H_l|HH)P(HH) + P(H_l|TT)P(TT) + P(H_l|N)P(N) = 1 \cdot \frac{2}{5} + 0 \cdot \frac{1}{5} + \frac{1}{2} \cdot \frac{2}{5} = \frac{3}{5}.$$

(b) (2 points) He opens his eyes and sees that the coin is showing heads; what is the probability that the lower face is a head?
Solution: Let H_u be the event that the upper face of the coin is a head. Then,

$$P(H_l|H_u) = \frac{P(HH)}{P(H_u)}.$$

Since $P(H_u) = \frac{3}{5}$ (the calculation is the same as for $P(H_l)$), therefore

$$P(H_l|H_u) = \frac{2/5}{3/5} = \frac{2}{3}.$$

(c) (2 points) He shuts his eyes again, and tosses the coin again. What is the probability that the lower face is a head?
Solution: Let H_l^2 be the event that the coin's lower in the second toss is a head. Then,

$$P(H_l^2|H_u) = P(H_l^2|HH, H_u)P(HH|H_u) + P(H_l^2|N, H_u)P(N|H_u) = 1 \cdot P(HH|H_u) + \frac{1}{2} \cdot (1 - P(HH|H_u)).$$

Since $HH = H_l \cap H_u$, we can write this probability as

$$P(H_l^2|H_u) = 1 \cdot P(H_l|H_u) + \frac{1}{2} \cdot (1 - P(H_l|H_u)) = \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3} = \frac{5}{6}.$$

(d) (2 points) He opens his eyes and sees that the coin is showing heads; what is the probability that the lower face is a head?
Solution:

$$P(H_l^2|H_u^2, H_u) = \frac{P(HH)}{P(H_u^2 \cap H_u)} = \frac{P(HH)}{P(H_u^2|H_u)P(H_u)} = \frac{2/5}{\frac{5}{6} \cdot \frac{3}{5}} = \frac{4}{5},$$

where we used results of (a) and (c). (The probabilities $P(H_u^2|H_u)$ and $P(H_u)$ are the same as $P(H_l^2|H_u)$ and $P(H_l)$, respectively.

(e) (2 points) He discards this coin, picks another at random, and tosses it. What is the probability that it shows heads?
Solution: The probability that he discards a double-headed coin is $\frac{4}{5}$ by the previous part, the probability that he discards a normal coin is $\frac{1}{5}$. In the first case we have 1 double-headed coin, 1 double-tailed, and 2 normal coins. In the second case, we have 2 double-headed coins, 1 double-tailed and 1 normal. Hence, by conditioning on the discard we have:

$$P(H_u^3) = \frac{4}{5}(1 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{2}{4}) + \frac{1}{5}(1 \cdot \frac{2}{4} + \frac{1}{2} \cdot \frac{1}{4}) = \frac{21}{40}.$$

10. (5 points) An insurance policy covers losses incurred by a policyholder, subject to a deductible of $10,000$. Incurred losses follow a normal distribution with mean $12,000$ and standard deviation c. The probability that a loss is less than k is 0.9582, where k is a constant. Given that the loss exceeds the deductible, there is a probability of 0.9500 that it is less than k.

Calculate c.

Solution: The correct answer is (A). Let L be the (random) amount of loss. We write

$$P(L < k | L > 10^4) = \frac{P(10^4 < L < k)}{P(L > 10^4)} = \frac{P(L < k) - P(L < 10^4)}{P(L > 10^4)} = \frac{P(L < k) - (1 - P(L > 10^4))}{P(L > 10^4)},$$

and we know that this equals 0.9500. After some re-arrangement we get

$$P(L < k) = 1 - 0.05P(L > 10^4),$$

which equals 0.9582. Hence,

$$P(L > 10^4) = \frac{0.0418}{0.05} = 0.836.$$

After standatization, we get

$$P(Z > \frac{-2000}{\sigma}) = 1 - P(Z > \frac{2000}{\sigma}) = 0.836,$$

and

$$P(Z > \frac{2000}{\sigma}) = 0.164.$$

From the table we find

$$\frac{2000}{\sigma} = 0.98,$$

and

$$\sigma = \frac{2000}{0.98} = 2040.8.$$

(A) is the closest to this number.

(A) 2045

(B) 2267

(C) 2393

(D) 2505

(E) 2840